Define

$$\delta_U : U \to \mathrm{Hom}(V, U \otimes V)$$
$$u \mapsto \delta_u(u)(-) = u \otimes -$$

i.e., $\delta_U(u)(v) := u \otimes v$, so $\delta_U(u)(-) := u \otimes -$

and

$$\varepsilon_W : \mathrm{Hom}(V, W) \otimes V \to W$$
$$f \otimes v \mapsto f(v)$$

i.e., $\varepsilon_W(f \otimes v) := f(v)$

Set $L := - \otimes V$ and $R := \mathrm{Hom}(V, -)$

Show that

$$\varepsilon_{\Bbbk \otimes V} \cdot L\delta_{\Bbbk} = \mathrm{id}_{\Bbbk \otimes V} \quad \text{and}$$
$$R\varepsilon_{\Bbbk} \cdot \delta_{V^*} = \mathrm{id}_{V^*}$$

Ans: First of all we understand what L and R do on morphisms.

given $f : A \to B$, $Lf : A \otimes V \to B \otimes V$
$$a \otimes v \mapsto f(a) \otimes v$$

and $Rf : \mathrm{Hom}(V, A) \to \mathrm{Hom}(V, B)$
$$V \xrightarrow{g} A \mapsto V \xrightarrow{g} A \xrightarrow{f} B$$

Now

$$\Bbbk \otimes V \xrightarrow{L\delta_{\Bbbk}} \mathrm{Hom}(V, V) \otimes V \xrightarrow{\varepsilon_{\Bbbk \otimes V}} V \cong \Bbbk \otimes V$$
$$k \otimes v \mapsto (k \cdot -) \otimes v \mapsto k \cdot v \cong k \otimes v$$

and

composition with $\varepsilon_{\Bbbk} : V^* \otimes V \to \Bbbk$

$$V^* \xrightarrow{\delta_{V^*}} \mathrm{Hom}(V, V^* \otimes V) \xrightarrow{R\varepsilon_{\Bbbk}} \mathrm{Hom}(V, \Bbbk) = V^*$$
$$\eta \mapsto \delta_{V^*}(\eta)(-) = \eta \otimes - \mapsto \eta$$

Claim: $R\varepsilon_{\Bbbk}(\eta \otimes -) = \eta$:

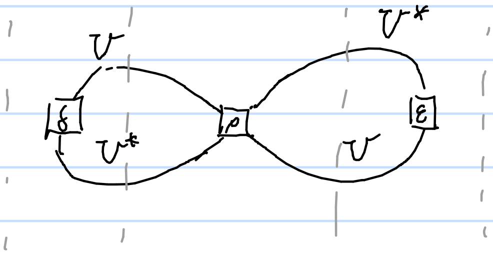
$$V \xrightarrow{\eta \otimes -} V^* \otimes V \xrightarrow{\varepsilon_{\Bbbk}} \Bbbk$$
$$v \mapsto \eta \otimes v \mapsto \eta(v)$$

7.7 Prove that $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ for $\psi: U \to V$, $\varphi: V \to W$.

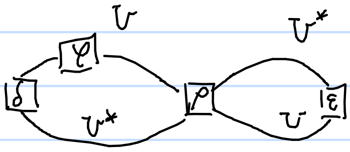
Pf. $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$:

Note that $[\, y, \varphi(\psi(x))\,] = [\, \varphi^* y, \psi x\,]$

$= [\, \psi^* \varphi^* y, x\,]$

7.8 Compute 

where $\rho: U \otimes U^* \xrightarrow{\cong} U^* \otimes U$

and more generally 

Ans:

$$\Bbbk \xrightarrow{\delta} U \otimes U^* \xrightarrow{\rho} U^* \otimes U \xrightarrow{\varepsilon} \Bbbk$$

$$1 \mapsto \sum_i e_i \otimes f^i \mapsto \sum_i f^i \otimes e_i \mapsto \sum_i f^i(e_i) = \dim U$$ } hint

So this map is the scalar multiple by $\dim U$.

As φ is linear, we can represent it by (a_{ij}).

So this is

$$\Bbbk \xrightarrow{\delta} U \otimes U^* \xrightarrow{\varphi \otimes 1} U \otimes U^* \xrightarrow{\rho} U^* \otimes U \xrightarrow{\varepsilon} \Bbbk$$

$$1 \mapsto \sum_i e_i \otimes f^i \mapsto \sum_{i,j} a_{ij} e_j \otimes f^i \mapsto \sum_{i,j} f^i \otimes a_{ij} e_j \mapsto \sum_i a_{ii}$$

So this map is the scalar multiple by $\sum_i a_{ii}$.

7.9 Let $\varphi : V \to V$, $\psi : W \to W$ be 2 linear maps.
Define
$$\varphi \otimes \psi : V \otimes W \to V \otimes W$$
$$v \otimes w \mapsto \varphi(v) \otimes \psi(w)$$
Prove that $\operatorname{tr}(\varphi \otimes \psi) = \operatorname{tr}\varphi \cdot \operatorname{tr}\psi$.

Pf. Write $\varphi : V \to V$, $e_i \mapsto \sum_j a_{ij} e_j$ and $\psi : W \to W$, $d_k \mapsto \sum_l b_{kl} d_l$

We know that
$$\operatorname{Tr}\varphi = \sum_i a_{ii} \quad \text{and} \quad \operatorname{Tr}\psi = \sum_k b_{kk}.$$
Now
$$\varphi \otimes \psi : V \otimes W \to V \otimes W$$
$$e_i \otimes d_k \mapsto \sum_{j,l} a_{ij} b_{kl} (e_j \otimes f_l)$$
$\therefore$ $\operatorname{Tr}(\varphi \otimes \psi)$
$$= \sum_{i,k} a_{ii} b_{kk} = \left(\sum_i a_{ii}\right)\left(\sum_k b_{kk}\right) = \operatorname{Tr}\varphi \cdot \operatorname{Tr}\psi$$