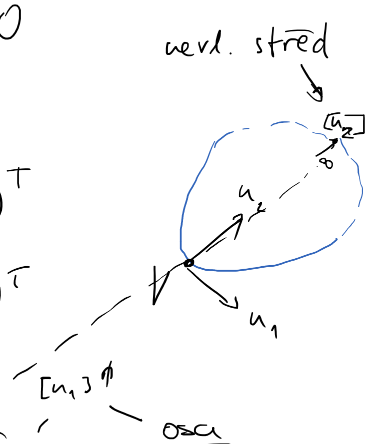
1 Cv. Najděte kanonickou rovnici kuželosečky

$$(x^1)^2 - 2x^1x^2 + (x^2)^2 - 4x^1 - 6x^2 + 3 = 0$$

$$\left(\begin{array}{c|cc} 3 & -2 & -3 \\ \hline -2 & 1 & -1 \\ -3 & -1 & 1 \end{array}\right) \qquad \text{char. poly: } \lambda(\lambda-2) = 0$$

$\lambda = 2: \quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \Rightarrow \quad u_1 = \frac{1}{\sqrt{2}}(0,1,-1)^T$

$\lambda = 0: \quad\quad u_2 = \frac{1}{\sqrt{2}}(0,1,1)^T$



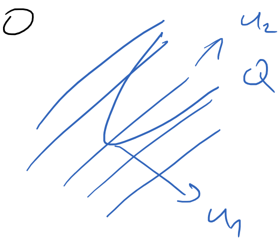
. střed $\left(\begin{array}{cc|c} 1 & -1 & 2 \\ -1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 5 \end{array}\right)$ nemá řešení

$\Rightarrow$ hledáme vrchol V

$$\left(\begin{array}{c|cc} 0 & 0 & * \\ \hline 0 & 2 & 0 \\ * & 0 & 0 \end{array}\right) \qquad \begin{array}{ll} f(V,V) = 0 & (V \in Q) \\ f(u_1,V) = 0 & (V \in [u_1]^\perp) \end{array}$$

$f(u_1,V) = 0$:

$$(0\ 1\ -1)\left(\begin{array}{c|cc} 3 & -2 & -3 \\ \hline -2 & 1 & -1 \\ -3 & -1 & 1 \end{array}\right)\begin{pmatrix} 1 \\ x^1 \\ x^2 \end{pmatrix} = (1\ \ 2\ \ -2)\begin{pmatrix} 1 \\ x^1 \\ x^2 \end{pmatrix} = 0$$



osa: $1 + 2x^1 - 2x^2 = 0$

param. popis řešení $(1,0,\frac{1}{2})^T + t\,(0,1,1)^T$

$x^1 = t \quad x^2 = \dots$

$f(V,V) = 0$:

$$\left((1,0,\tfrac{1}{2}) + t(0,1,1)\right)\begin{pmatrix} 3 & -2 & -3 \\ -2 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) =$$

$$= \left((1,0,\tfrac{1}{2}) + t(0,1,1)\right)\cdot\left(\begin{pmatrix} 3/2 \\ -5/2 \\ -5/2 \end{pmatrix} + t\begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix}\right) = \frac{1}{4} + t\cdot(-5) + t\cdot(-5) + t^2\cdot 0$$

$$= \frac{1}{4} - 10t = 0$$

$$\Rightarrow t = \frac{1}{40} \qquad \Rightarrow V = (1,0,\tfrac{1}{2})^T + t\,(0,1,1)^T = \left(1, \frac{1}{40}, \frac{21}{40}\right)^T$$

V bázi $(V, u_1, u_2) = \left(\begin{pmatrix} 1 \\ 1/40 \\ 21/40 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$ má Q matici

$$\left(\begin{array}{c|cc} 0 & 0 & -5/\sqrt{2} \\ \hline 0 & 2 & 0 \end{array}\right) \qquad f(u_2,V) = \underbrace{\frac{1}{\sqrt{2}}(0,1,1)\begin{pmatrix} 3 & -2 & -3 \\ -2 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 1/40 \\ 21/40 \end{pmatrix} = -\frac{5}{\sqrt{2}}$$

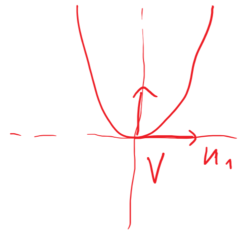
$$\left(\begin{array}{c|cc} 0 & 0 & -5/\sqrt{2} \\ \hline 0 & 2 & 0 \\ -5/\sqrt{2} & 0 & 0 \end{array}\right)$$

$$t(u_2, V) = \frac{1}{\sqrt{2}} \underbrace{(0,1,1) \begin{pmatrix} -2 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix}} \begin{pmatrix} 1/40 \\ 21/40 \end{pmatrix} = - \frac{\;}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} (-5, 0, 0) \begin{pmatrix} 1 \\ 1/40 \\ 21/40 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot (-5) \quad \checkmark$$

a tedy rovnici

$$2(x^1)^2 - 5\sqrt{2} \cdot x^2 = 0$$



2 Cv. Najděte kanonickou rovnici kvadriky

$$5(x^1)^2 + 8(x^2)^2 + 5(x^3)^2 + 4x^1x^2 - 8x^1x^3 + 4x^2x^3 + 6x^1 + 6x^2 + 6x^3 - 27 = 0$$

$\lambda = 9, 9, 0$

$$A = \left(\begin{array}{c|ccc} -27 & 3 & 3 & 3 \\ \hline 3 & 5 & 2 & -4 \\ 3 & 2 & 8 & 2 \\ 3 & -4 & 2 & 5 \end{array}\right) \qquad \overline{A} = \begin{pmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{pmatrix}$$

$$\det(\overline{A} - \lambda E) = -(\lambda - 9)^2 \lambda$$

$\lambda_3 = 0$:

$$\begin{pmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1 \\ 0 & 18 & 9 \\ 0 & -18 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$u_3 = \frac{1}{3}(0 \quad 2 \quad -1 \quad 2)^T$$

$\lambda_{1,2} = 9$: u_1, u_2

u_1 lib. kolmý na u_3

u_2 lib. kolmý na $\underline{u_3, u_1}$

(u_1, u_2, u_3) ON báze

$$u_1 = \frac{1}{\sqrt{5}}(0 \quad 0 \quad 2 \quad 1)^T$$

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{45}}(0 \quad -5 \quad -2 \quad 4)^T$$

V bázi (V, u_1, u_2, u_3) má Q matici

$$\left(\begin{array}{c|ccc} 0 & 0 & 0 & * \\ \hline 0 & 9 & & \\ 0 & & 9 & \\ * & & & 0 \end{array}\right)$$

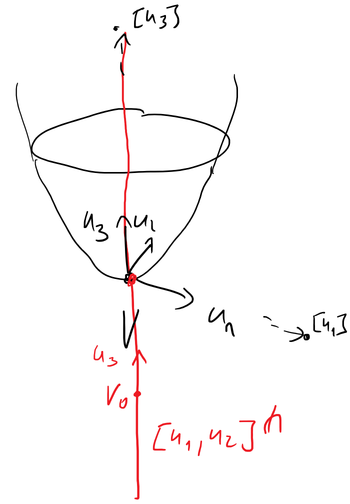
V vrchol $\quad f(V,V) = 0 \quad V \in Q$

$f(\sqrt{5} \cdot u_1, V) \iff f(u_1, V) = 0$

$\quad = \sqrt{5} \cdot f(u_1, V)$

$f(u_2, V) = 0$

$\Big\} \; f \in [u_1, u_2]^{\perp}$



$$\begin{matrix} u_1^T \\ u_2^T \end{matrix} \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & -5 & -2 & 4 \end{pmatrix} \left(\begin{array}{c|ccc} -27 & 3 & 3 & 3 \\ \hline 3 & 5 & 2 & -4 \\ 3 & 2 & 8 & 2 \\ 3 & -4 & 2 & 5 \end{array}\right) \begin{pmatrix} 1 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = 0$$

$f(u_1, V) = 0$

$f(u_2, V) = 0$

$$\begin{pmatrix} 9 & 0 & 18 & 9 \\ -9 & -45 & -18 & 36 \end{pmatrix} \begin{pmatrix} 1 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = 0 \quad \leadsto \quad \left(\begin{array}{ccc|c} -45 & -18 & 36 & 9 \\ 0 & 18 & 9 & -9 \end{array}\right)$$

$$0 \quad -\tfrac{1}{2} \quad 0$$

$$V = (1, 0, -\tfrac{1}{2}, 0)^T + t(0, 2, -1, 2)^T$$

$f(V, V) = 0$

$V = (1, 0, -1/2, 0)^T + t\,(0, 2, -1, 2)^T$

$f(V,V) = 0$

$$\big((1,0,-1/2,0) + t(0,2,-1,2)\big)\begin{pmatrix} -27 & 3 & 3 & 3 \\ 3 & 5 & 2 & -4 \\ 3 & 2 & 8 & 2 \\ 3 & -4 & 2 & 5 \end{pmatrix}\left(\begin{pmatrix}1\\0\\-1/2\\0\end{pmatrix} + t\begin{pmatrix}0\\2\\-1\\2\end{pmatrix}\right)$$

$$= \big((1,0,-1/2,0) + t(0,2,-1,2)\big)\left(\begin{pmatrix}-57/2\\2\\-1\\2\end{pmatrix} + t\begin{pmatrix}9\\0\\0\\0\end{pmatrix}\right)$$

$$= -28 + t\cdot 9 + t\cdot 9 + t^2\cdot 0$$

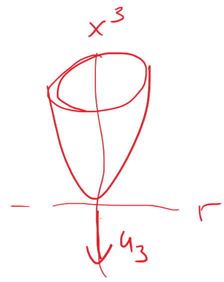
$$\Rightarrow \quad t = \frac{28}{18} = \frac{14}{9}$$

$$\Rightarrow \quad V = (1, 0, -1/2, 0)^T + t\,(0, 2, -1, 2)^T = \left(1, \frac{28}{9}, -\frac{37}{18}, \frac{28}{9}\right)^T$$

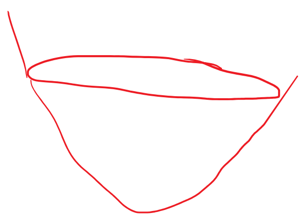
V bázi (V, u_1, u_2, u_3) má Q matici:

$$\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$

$$f(u_3, V) = \frac{1}{3}\,(0\ 2\ -1\ 2)\ A\cdot V = \frac{1}{3}\cdot 9 = 3$$



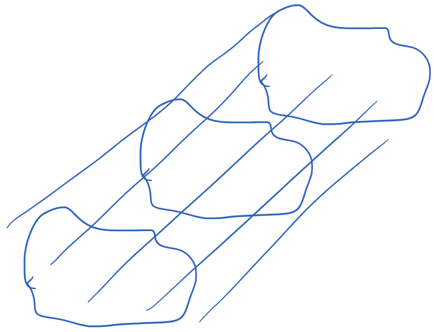
a tedy rovnici $9(x^1)^2 + 9(x^2)^2 + 6x^3 = 0$

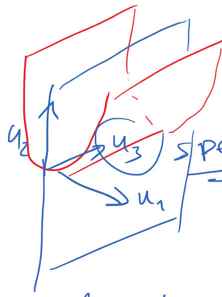


3 Cv. Najděte kanonický tvar kvadriky

$$(x^1)^2+(x^2)^2-2x^1x^2+2x^1+2x^2-2\sqrt{2}\,x^3-8=0$$

$\lambda = 2, 0, 0$





speciální

vl. podpr. přísl 0

4

Cv. Najděte bázi α prostoru $\mathbb{R}^3$ tak, aby k ní byla daná báze α^* prostoru $(\mathbb{R}^3)^*$ duální.

$$\alpha^* = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix} \qquad \begin{aligned} f^1(x^1,x^2,x^3) &= 2x^1 - x^2 &\longrightarrow (2\ -1\ 0) = f^1 \\ f^2(x^1,x^2,x^3) &= x^2 - x^3 &\longrightarrow (0\ 1\ -1) = f^2 \\ f^3(x^1,x^2,x^3) &= x^1 + x^2 + x^3 &\longrightarrow (1\ 1\ 1) = f^3 \end{aligned}$$

$e_1 = ?$ $\quad e_2 = ?$ $\quad e_3 = ?$

$f^1(e_1) = 1, \quad f^2(e_1) = 0, \quad f^3(e_1) = 0$ $\leftarrow$ soustava, jejíž jediným řešením je $\underline{\underline{e_1}}$

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}}^{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

dualita: $f^i(e_j) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & -1/5 \\ 0 & 0 & 1 & -1/5 \end{array}\right)$$

$$e_1 = \frac{1}{5}(2\ -1\ -1)^T$$

$e_2 = ?$ $\quad \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$e_3 = ?$ $\quad \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\boxed{\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & e_1 & e_2 & e_3 \\ 0 & 0 & 1 & & & \end{array}\right)}$$

Tj.: $(e_1\ e_2\ e_3) = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}^{-1}$ bylo na předn.

Dk: $\begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix} (e_1\ e_2\ e_3) = \begin{pmatrix} f^1 \cdot e_1 & f^1 \cdot e_2 & f^1 \cdot e_3 \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5 Cv. Najdéte duální bázi α^* k bázi α prostoru $\mathbb{R}^3$

$\alpha = (e_1, e_2, e_3)$ $\quad e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; e_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \; e_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Postup.

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix} = (e_1 \; e_2 \; e_3)$$

$$\eta \cdot v = (- - -)\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$
$$\eta \cdot v = \eta(v)$$

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}$$

celočíselné hodnoty

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 0 & 0 & \\ 0 & 4 & 0 & ? \\ 0 & 0 & 4 & \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 4 & 4 & 0 & 0 & -6 & 4 \\ 0 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & -2 & -5 & 4 \\ 0 & 4 & 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 0 & 2 & 0 \end{array}\right)$$

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} -2 & -5 & 4 \\ 2 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$f^1 = \tfrac{1}{4}(-2 \;\; -5 \;\; 4)$$
$$f^2 = \tfrac{1}{4}(2 \;\; -1 \;\; 0)$$
$$f^3 = \tfrac{1}{4}(0 \;\; 2 \;\; 0)$$

6 Cv. Popište v souřadnicích duální zobrazení k zobrazení $A: \mathbb{k}^n \to \mathbb{k}^m$, tj. $A \in Mat_{m \times n} \mathbb{k}$

7 Cv. Dokažte: Nechť $U = V \oplus W$. Pak kompozice

$$W \hookrightarrow U \twoheadrightarrow U/V$$

je izomorfismus.

8 Cv. Popište implicitně podprostor

$$[(1,-1,0,0)^T, (1,0,-1,0)^T, (0,1,0,-1)^T] \subseteq \mathbb{K}^4$$

9 Cv. Popište všechny roviny procházející přímkou

$$p: \begin{array}{l} x^1 + x^2 - x^3 = 0 \\ x^1 - x^2 + x^3 = 0 \end{array}$$

param./ impl.

10 Cv. Pomocí Motzkinovy eliminace rozhodněte o řešitelnosti

$$x, y, z \geq 0$$
$$4 \geq x + y + z \geq 2$$
$$3 \geq x + y \geq 1$$
$$2 \geq z$$