1 Cv. Popište v souřadnicích duální zobrazení k zobrazení $A: \mathbb{k}^n \longrightarrow \mathbb{k}^m$, tj. $A \in Mat_{m \times n}\, \mathbb{k}$.

$$x \longmapsto A \cdot x \qquad\qquad y \longmapsto y \cdot A$$

co splňuje duální zobrazení?

$$\underset{\underset{(\mathbb{k}^m)^*}{\cap}}{(y} \,,\, \underset{\underset{\mathbb{k}^m}{\cap}}{Ax})_{\mathbb{k}^m} = (A^* y \,,\, x)_{\mathbb{k}^n}$$

$$y \cdot (Ax) = (A^* y) \cdot x \implies \underline{\underline{A^* y = yA}}$$

$$\shortparallel$$

$$(yA) \cdot x$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^1 + 2x^2 + 3x^3 \\ x^2 + 2x^3 \end{pmatrix}$$

$$(\mathbb{k}^m)^* \to (\mathbb{k}^n)^*$$
$$y \longmapsto y \cdot A$$

$$(y^1, y^2) \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} = (y^1,\ 2y^1 + y^2,\ 3y^1 + 2y^2)$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}}_{A^T} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} y^1 \\ 2y^1 + y^2 \\ 3y^1 + 2y^2 \end{pmatrix}$$

$$\Downarrow$$

$$y^T \longmapsto A^T \cdot y^T$$
$$\mathbb{k}^m \longrightarrow \mathbb{k}^n$$

$$\mathbb{R}^n \cong \mathbb{R}^{n*}$$
$$v \longmapsto \underbrace{\langle v, - \rangle}_{v^T}$$

2 Cv. Dokažte: Nechť $U = V \oplus W$. Pak kompozice

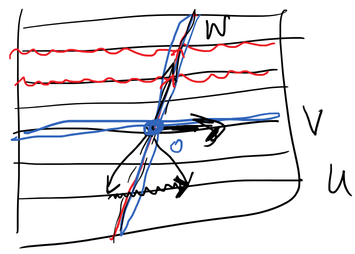
$$W \hookrightarrow U \twoheadrightarrow U/V$$
$$w \mapsto w \mapsto w + V$$

je izomorfismus.

$V = [u_1, \dots, u_k] \rightsquigarrow U = [u_1, \dots, u_k, u_{k+1}, \dots, u_n]$

$W = [u_{k+1}, \dots, u_n]$

- surj ?
- ker = 0 ?



$$U/V = \{u + V \mid u \in U\}$$

$G/H \;\cdots\; gH$

$u + V = u' + V$

$\Leftrightarrow u - u' \in V$

ex. $u + V \in U/V$ lze reprezentovat jako $w + V$, $w \in W$ $\Big\}$ $V + W = U$

$u = v + w \qquad u + V = v + w + V = w + V \qquad$ ex.

jedn. $w \longmapsto w + V = 0 + V \Rightarrow \underline{w \in V} \Rightarrow w = 0$. $\Big\}$ $V \cap W = \{0\}$

$\overset{?}{W}$

3

Cv. Popište implicitně podprostor

$$[(1,-1,0,0)^T, (1,0,-1,0)^T, (0,1,0,-1)^T] \subseteq \mathbb{K}^4$$

$V \longrightarrow V^\perp$ popsáno impl. $\quad (\underline{\quad})(\;) = 0$

$\downarrow$

$V^\perp$ popis param

$\downarrow$

$V = V^{\perp\perp}$ popsáno impl. param

v aplikacích: $\mathbb{Z}_2$ ... $(\;)^\perp$ lepší

$$V^\perp: \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$[(0\ 0\ 1\ 1), (1\ 1\ 0\ 0)]$$

$$V^\perp = [(1\ 1\ 1\ 1)]$$

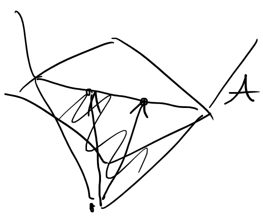
prostor řešení soustavy

$V^{\perp\perp}$: $(1\ 1\ 1\ 1)$ $\quad \underline{x^1 + x^2 + x^3 + x^4 = 0}$

Zajímavější: afinní podprostory ... afinní obal množ. bodů

... systém lin. rovnic

$\ell = [3,5]\ [1,2]$ napsat rovnici



$A_2: x^0 = 1$

$\mathbb{K}^3$

$$\rho = [(1,3,5), (1,1,2)]$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\rho^\perp = [(1\ 3\ -2)]$$

$\rho^{\perp\perp}$: $x^0 + 3x^1 - 2x^2 = 0$

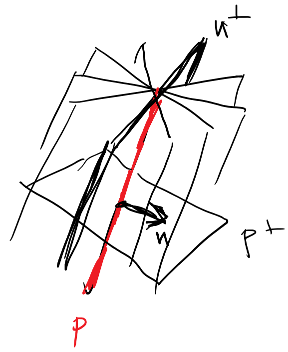
$\underline{1 + 3x^1 - 2x^2 = 0}$

4

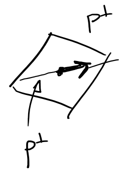
Cv. Popište všechny roviny procházející přímkou

$$p: \begin{array}{l} x^1 + x^2 - x^3 = 0 \\ x^1 - x^2 + x^3 = 0 \end{array}$$

param./impl.



Zase přes $(\ )^\perp$: $\quad \underset{\text{př.}}{p} \subseteq \underset{\text{rov.}}{\rho} \qquad \underset{\text{rov.}}{p^\perp} \supseteq \underset{\text{př.}}{\rho^\perp}$



p parametricky: $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \end{pmatrix}$

$$p: \begin{array}{l} x^1 + x^2 - x^3 = 0 \\ x^1 - x^2 + x^3 = 0 \end{array}$$

param $\downarrow$ param $\downarrow$ $p = [(0 \ 1 \ 1)^T]$

$p^\perp$ parametricky: $(0 \quad \underline{1} \quad 1)$

$$p^\perp = [(0 \ -1 \ 1), (1 \ 0 \ 0)]$$

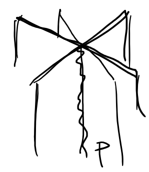
$$n = \alpha \cdot (0, -1, 1) + \beta \cdot (1, 0, 0)$$

$$\rho = n^\perp : \quad \beta x^1 - \alpha x^2 + \alpha x^3 = 0$$

$\forall \alpha, \beta$ lib., ne obě nulové

$(\alpha : \beta) \in \mathbb{P}_1$ parametrizuje roviny jednoznačně

NEBO



$$p^\perp = [(1 \ 1 \ -1), (1 \ -1 \ 1)]$$

(protože $p = [(\ ), (\ )]^\perp$)

$$\alpha(x^1 + x^2 - x^3) + \beta(x^1 - x^2 + x^3) = 0$$

5

Cv. Pomocí Motzkinovy eliminace rozhodněte o řešitelnosti

$$x, y, z \geq 0$$
$$4 \geq x+y+z \geq 2$$
$$3 \geq x+y \geq 1$$
$$2 \geq z$$

$\min\{cx \mid Ax \geq b\}$

$\{\delta \mid \underbrace{\exists x: Ax \geq b,\ cx = \delta}_{\text{elim } \exists x}\}$ = množ. hodn. funkce cx

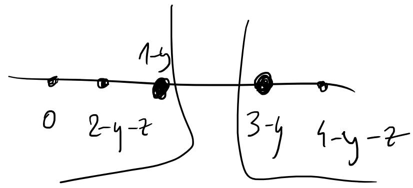
→ omezení pro δ

eliminace x:

$$\boxed{\begin{matrix} 4-y-z \\ 3-y \end{matrix}} \begin{matrix} \geq x \\ \geq x \end{matrix} \qquad \begin{matrix} x \geq \\ x \geq \\ x \geq \end{matrix} \boxed{\begin{matrix} 0 \\ 2-y-z \\ 1-y \end{matrix}} \qquad \begin{matrix} y \geq 0 \\ z \geq 0 \\ 2 \geq z \end{matrix}$$

2 (beside $3-y$), 0 (beside 0)

$y=1, z=1$



kdy takové x existuje? právě když

$4-y-z \geq 0$
$4-y-z \geq 2-y-z$
$4-y-z \geq 1-y$
$3-y \geq 0$
$3-y \geq 2-y-z$
$3-y \geq 1-y$
$y \geq 0$
$z \geq 0$
$2 \geq z$

$4-y-z \geq 0$ ✓
$\cancel{4 \geq 2}$
$3-z \geq 0$ ✓
$3-y \geq 0$ ✓
$1+z \geq 0$ ✓
$\cancel{2 \geq 0}$
$y \geq 0$ ✓
$z \geq 0$ ✓
$2 \geq z$ ✓

elim y →

$4-z \geq y \qquad y \geq 0$
$3 \geq y$

$3-z \geq 0$
$1+z \geq 0$
$z \geq 0$
$2 \geq z$

⇕

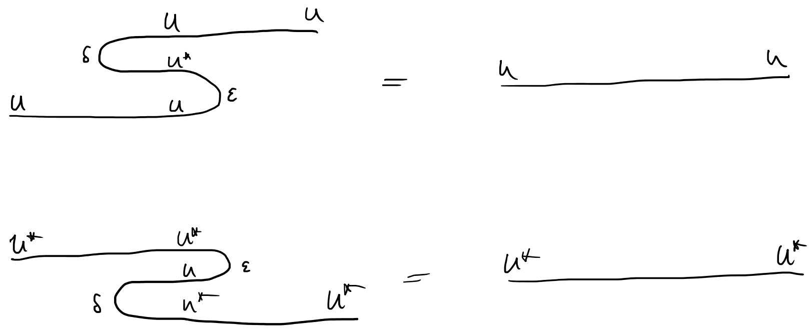
$4-z \geq 0$
$\cancel{3 \geq 0}$
$3-z \geq 0$
$1+z \geq 0$
$z \geq 0$
$2 \geq z$

$4 \geq z \qquad z \geq -1$
$3 \geq z \qquad \underline{\underline{z \geq 0}}$
$\underline{\underline{2 \geq z}}$

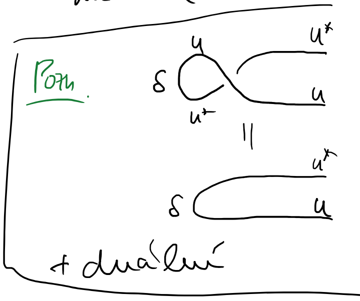
⇒ řešení ex. pro každé $z \in [0,2]$

$$\boxed{\begin{matrix} z = 1 \\ y = 1 \\ x = 1 \end{matrix}} \begin{matrix} \Rightarrow y \in [0,3] \\ \Rightarrow x \in [0,2] \end{matrix}$$

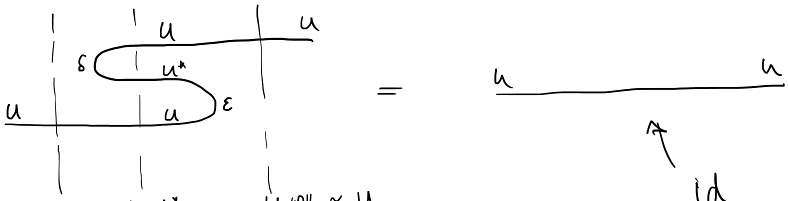
6 **Cv.** Dokažte … prvně co je $\delta : \Bbbk \to U\otimes U^*$, $\varepsilon : U^*\otimes U \to \Bbbk$?



"dualita"



Dokážeme:



id

$$U \cong \Bbbk\otimes U \xrightarrow{\delta\otimes id} U\otimes U^*\otimes U \xrightarrow{id\otimes\varepsilon} U\otimes\Bbbk \cong U$$

$\delta : \Bbbk \to U\otimes U^*$
$\qquad\qquad\quad \cong \operatorname{Hom}(U,U) \ni id$

$\varepsilon : U^*\otimes U \to \Bbbk$
$\qquad \eta\otimes u \mapsto \eta(u)$

① Co je $\delta : \Bbbk \to U\otimes U^*$ ⟵ prvky matice $(id)_{\alpha\alpha}$

$$1 \mapsto \sum a^i_j\, e_i\otimes f^j$$
$$= \sum_{ij} \delta^i_j\, e_i\otimes f^j$$
$$= \sum_i e_i\otimes f^i \quad (= e_1\otimes f^1 + \dots + e_n\otimes f^n)$$

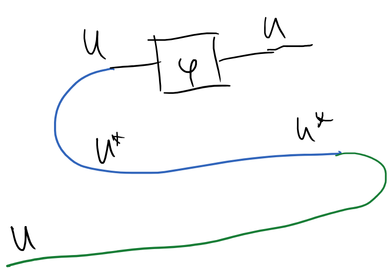
$u \mapsto 1\otimes u$
$U \cong \Bbbk\otimes U$
$\lambda\cdot u \leftarrow \lambda\otimes u$

$$U \cong \Bbbk\otimes U \xrightarrow{\delta\otimes id} U\otimes U^*\otimes U \xrightarrow{id\otimes\varepsilon} U\otimes\Bbbk \cong U$$

$$u \mapsto 1\otimes u \mapsto \sum e_i\otimes f^i\otimes u \mapsto \sum e_i\otimes f^i(u) \mapsto \sum_i e_i\cdot \underbrace{f^i(u)}_{i\text{-tá souřadnice } u} = u$$

$u = \sum u^j e_j$

$\sum_i e_i\cdot f^i\left(\sum_j u^j e_j\right)$
$= \sum_{ij} e_i\cdot \cancel{f^i(e_j)}\cdot u^j$
$= \sum_i e_i u^i = u$

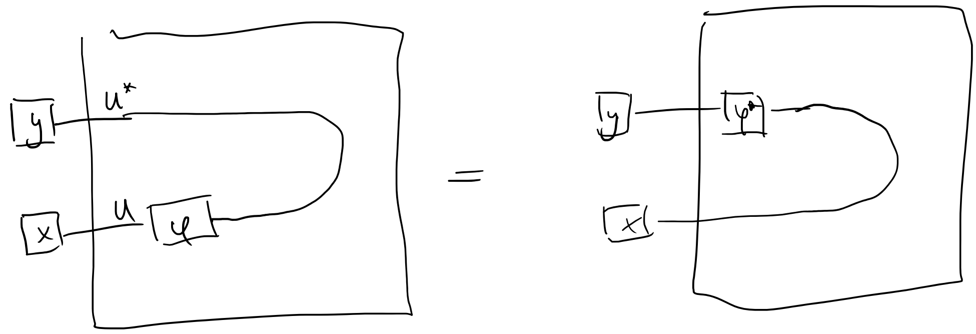


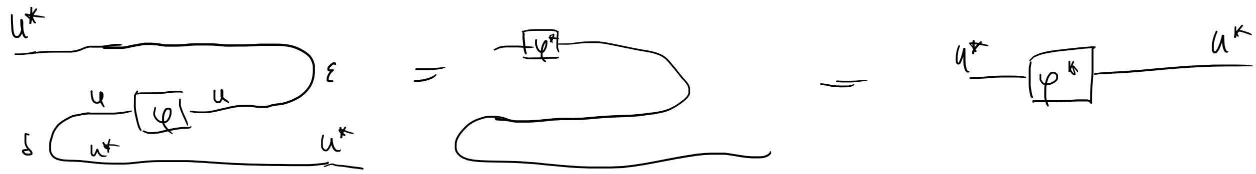
$U \to U$
$\downarrow$
$\Bbbk \to U\otimes U^*$
$\downarrow$
$U \to U$

7

Cv. Popište duální zobrazení a dokažte $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$

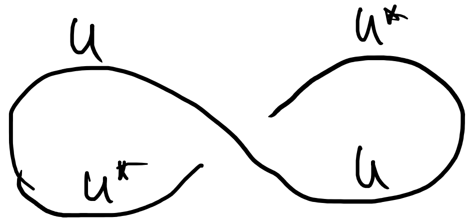
$$\underline{(y, \varphi x)} = \underline{(\varphi^* y, x)}$$



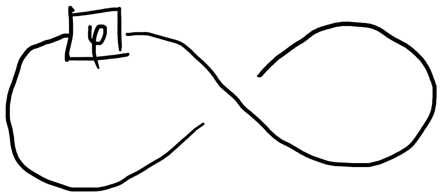


8

Cv. Spočtěte



a obecněji



9 Cv. Jaká je stopa tenzorového součinu? Stačí pro diagonalizovatelné, ale pomocí grafického zobr. to jde snadno i obecně.

10

Cv.

$$(\bar{e}_1\ \bar{e}_2\ \bar{e}_3) = (e_1\ \ e_2\ \ e_3)\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

Určete $\bar{t}^{12}_{\ \ 31}$ pro tenzor $t = e_3 \otimes e_1 \otimes f^2 \otimes f^1 + e_1 \otimes e_2 \otimes f^3 \otimes f^3$

Cv.

$$(\bar{e}_1 \ \ \bar{e}_2 \ \ \bar{e}_3) = (e_1 \ \ e_2 \ \ e_3) \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Určete $\bar{t}^{12}_{123}$ pro tenzor t se všemi $t^{i_1 i_2}_{j_1 j_2 j_3} = 1$.