Tutorial 10—Global Analysis

1. Suppose $(M, g) \subset (\mathbb{R}^3, g) = (\mathbb{R}^3, g^{\text{euc}}) = (\mathbb{R}^3, <-,->)$ is a surface in Euclidean space. Let $u : U \to u(U)$ be a local chart for M with corresponding local parametrization
$$v = u^{-1} : u(U) \to U.$$
With respect to the frame $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ of $T\mathbb{R}^2$, we can write v^*g and v^*II as matrices
$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix},$$
where
$$E = g(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}) \circ v \quad F = g(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}) \circ v \quad G = g(\frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2}) \circ v,$$
and
$$\tilde{E} = II(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}) \circ v \quad \tilde{F} = II(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}) \circ v \quad \tilde{G} = II(\frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2}) \circ v.$$
Compute in terms of $E, F, G, \tilde{E}, \tilde{F}$ and $\tilde{G}$, the Weingarten map $L \circ v$, the Gauß curvature $K \circ v$, the mean curvature $H \circ v$, and the principal curvatures $\kappa_1 \circ v$ and $\kappa_2 \circ v$.

2. Let us write (x^1, x^2, x^3) for the coordinates in $\mathbb{R}^3$. Take a circle of radius $r > 0$ in the (x^1, x^3)-plane and rotate it around a circle of radius $R > r$ in the (x^1, x^2)-plane. The result is a 2-dimensional torus M in $\mathbb{R}^3$. If $I \subset \mathbb{R}$ is an open interval of length $< 2\pi$ the map $v : I \times I \to \mathbb{R}^3$ given by
$$v(\phi, \theta) = ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)$$
defines a local parametrization of M. With respect to v, compute, using the previous exercise, the metric g on M induced by the Euclidean metric on $\mathbb{R}^3$, the 2nd fundamental form, the Gauß and the mean curvature, the principal curvatures and the principal curvature directions of the surface (M, g) in $\mathbb{R}^3$.

Hint: Note that $\nu(\phi, \theta) = (\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta)$ defines a local unit normal vector field for M.

3. Suppose $(M, g) \subset (\mathbb{R}^{m+1}, g) = (\mathbb{R}^{m+1}, g^{\text{euc}})$ is a connected oriented hypersurface in Euclidean space. Show that all points in M are umbilic if and only if M is part of an affine hyperplane or a sphere.

   Hint: For $, \Longrightarrow'$ show the following:

   - Fix a global unit normal vector field $\nu : M \to \mathbb{R}^{m+1}$. Then, by assumption, for any $x \in M$ there exists $\lambda(x) \in \mathbb{R}$ such that
   $$L_x = \lambda(x)\text{Id}_{T_xM}.$$
   Since $\lambda = \frac{g(L(\xi),\xi)}{g(\xi,\xi)}$ for any local vector field ξ on M, $\lambda : M \to \mathbb{R}$ is smooth. Show that λ is constant, by, for instance, picking a chart and computing the left-hand-side of $[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] \cdot \nu = 0$.
   - If $\lambda = 0$, show that any curve in M is contained in an affine hyperplane with (constant) normal vector ν.
   - If $\lambda \neq 0$, show that $f : M \to \mathbb{R}^{m+1}$, given by $f(x) = x - \frac{1}{\lambda}\nu(x)$, is constant.