Tutorial 11—Global Analysis

1. Suppose ∇ is an affine connection on a manifold M.

   (a) Show that its curvature, given by,

   $$R(\xi,\eta)(\zeta) = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta,$$

   for vector fields $\xi, \eta, \zeta \in \mathfrak{X}(M)$ defines a $\begin{pmatrix}1\\3\end{pmatrix}$-tensor on M.

   (b) Show that, if ∇ is torsion-free, the Bianchi identity holds:

   $$R(\xi,\eta)(\zeta) + R(\eta,\zeta)(\xi) + R(\zeta,\xi)(\eta) = 0,$$

   for any $\xi, \eta, \zeta \in \mathfrak{X}(M)$.

2. Suppose $E \to M$ is a vector bundle over a manifold M equipped with a linear connection ∇, that is, a $\mathbb{R}$-bilinear map

   $$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(E) &\to \Gamma(E) \\ (\xi, s) &\mapsto \nabla_\xi s \end{aligned}$$

   such that for $\xi \in \Gamma(TM)$, $s \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$ one has

   - $\nabla_{f\xi} s = f \nabla_\xi s$
   - $\nabla_\xi f s = f \nabla_\xi s + (\xi \cdot f) s.$

   (a) Show that $\nabla : \Gamma(TM) \times \Gamma(E^*) \to \Gamma(E^*)$ (typically also denoted by ∇) given by

   $$(\nabla_\xi \mu)(s) = \xi \cdot \mu(s) - \mu(\nabla_\xi s), \text{ for } \mu \in \Gamma(E^*), \xi, \in \Gamma(TM), s \in \Gamma(E)$$

   defines a linear connection on the dual vector bundle $E^* \to M$.

   (b) Suppose $\tilde{E} \to M$ is another vector bundle equipped with a linear connection $\tilde{\nabla}$. Show the vector bundle $E \otimes \tilde{E} \to M$ admits a linear connection characterized by

   $$\nabla_\xi (s \otimes \tilde{s}) = \nabla_\xi s \otimes \tilde{s} + s \otimes \tilde{\nabla}_\xi \tilde{s}$$

   for $\xi \in \Gamma(TM)$, $s \in \Gamma(E)$ and $\tilde{s} \in \Gamma(\tilde{E})$.

3. Suppose ∇ is an affine connection on a manifold M. Then the previous exercise shows that ∇ induces a linear connection $\nabla : \Gamma(TM) \times \mathcal{T}^p_q(M) \to \mathcal{T}^p_q(M)$ on all tensor bundles. Show that it also induces a linear connection on the bundles $\Lambda^k T^*M$ for $k = 1, \ldots \dim(M)$ characterized by

$$\nabla_\xi(\omega \wedge \mu) = \nabla_\xi \omega \wedge \mu + \omega \wedge \nabla_\xi \mu$$

for $\omega \in \Gamma(\Lambda^k T^*M)$ and $\mu \in \Gamma(\Lambda^\ell T^*M)$ and give a formula.

4. Suppose (M, g) is a Riemannian manifold.

   (a) For vector fields $\xi, \eta \in \mathfrak{X}(M)$, let $\nabla_\xi \eta \in \mathfrak{X}(M)$ be the unique vector field such that

   $$g(\nabla_\xi \eta, \zeta) = \frac{1}{2}\Big(\xi \cdot g(\eta, \zeta) + \eta \cdot g(\zeta, \xi) - \zeta \cdot g(\xi, \eta) + g([\xi, \eta], \zeta) - g([\xi, \zeta], \eta) - g([\eta, \zeta], \xi)\Big)$$

   for all $\zeta \in \mathfrak{X}(M)$. Show that ∇ defines a torsion-free affine connection satisfying

   $$\xi \cdot g(\eta, \zeta) = g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta)$$

   for $\xi, \eta, \zeta \in \mathfrak{X}(M)$.

   (b) The connection ∇ in (a) is called the Levi-Civita connection of (M, g). Show that its curvature satisfies:

   - $g(R(\xi, \eta)(\zeta), \mu) = -g(R(\xi, \eta)(\mu), \zeta)$,
   - $g(R(\xi, \eta)(\zeta), \mu) = g(R(\zeta, \mu)(\xi), \eta)$,

   for $\xi, \eta, \zeta, \mu \in \mathfrak{X}(M)$.

   (c) Suppose (U, u) is a chart for M and let R be the Riemann curvature, i.e. the curvature of the Levi-Civita connection of (M, g). Compute

   $$R\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)\left(\frac{\partial}{\partial u^k}\right)$$

   in terms of the Christoffel symbols.

5. Suppose $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is the upper-half plane and equip it with the Riemannian metric

   $$g = \frac{1}{y^2} dx \otimes dx + \frac{1}{y^2} dy \otimes dy.$$

   (a) Compute the Christoffel symbols of g.

   (b) Compute the geodesics of g.

   (c) Compute the Riemann curvature.

6. Identifying $H = \{(x, y) \in \mathbb{R}^2 : y > 0\} = \{z = x + iy \in \mathbb{C} : y > 0\}$ in the previous example, we may write g as

   $$g = \frac{1}{\mathrm{Im}(z)^2} \mathrm{Re}(dz \otimes d\bar{z}) = \frac{4}{|z - \bar{z}|^2} \mathrm{Re}(dz \otimes d\bar{z}),$$

where $dz = dx + idy$, $d\bar{z} = dx - idy$, Im and Re denote imaginary and real part, and $|\_|$ is the absolute value of complex numbers.

Consider $\mathrm{SL}(2,\mathbb{R}) = \{A \in \mathrm{GL}(2,\mathbb{R}) : \det A = 1\}$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and $z \in \mathbb{C}$ let

$$f_A(z) = \frac{az+b}{cz+d}.$$

(a) Show that f_A is a diffeomorphism from H to itself for any $A \in \mathrm{SL}(2,\mathbb{R})$ and that $f_{AB} = f_A \circ f_B$.

(b) Show that f_A is an isometry of H.

(c) Show that for any two points $z, z' \in H$ there exists $A \in \mathrm{SL}(2,\mathbb{R})$ such that $f_A(z) = z'$.

(d) Characterize the elements $A \in \mathrm{SL}(2,\mathbb{R})$ such that $f_A(i) = i$.