Tutorial 2 and 3—Global Analysis

1. For $i = 1, ...n$ let $(M_i, \mathcal{A}_i)$ be a smooth manifolds. Suppose $M := M_1 \times ... \times M_n$ is endowed with the product topology. Then show that

$$\mathcal{A} := \{(U_1 \times ... \times U_n, u_1 \times ... \times u_n) : (U_i, u_i) \in \mathcal{A}_i\}$$

defines a smooth atlas on M and that the projections $\mathrm{pr}_i : M \to M_i$ are smooth. Moreover show that, if N is a smooth manifold and $f_i : N \to M_i$ smooth functions, then there exists a unique smooth function $f : N \to M$ such that $\mathrm{pr}_i \circ f = f_i$ and that this property characterizes the smooth manifold structure on M uniquely.

2. Suppose $(M_i, \mathcal{A}_i)$ are smooth manifolds for $i \in I$, where I is countable. Consider the disjoint union

$$M := \sqcup_{i \in I} M_i = \cup_{i \in I}\{(x, i) : x \in M_i\}$$

endowed with the disjoint union topology and denote by $\mathrm{inj}_i : M_i \hookrightarrow M$ the canonical injections ($\mathrm{inj}_i(x) = (x, i)$). Show that $\mathcal{A} := \cup_{i \in I}\mathcal{A}_i$ defines a smooth atlas on M and that the injections inj_i are smooth. Moreover, show that for any smooth manifold N and smooth functions $f_i : M_i \to N$, there exists a unique smooth function $f : M \to N$ such that $f \circ \mathrm{inj}_i = f_i$ and show that this property characterizes the smooth manifold structure on M uniquely.

3. Suppose $U \subset \mathbb{R}^m$ is open and $f : U \to \mathbb{R}^n$ a smooth map such that $D_x f : \mathbb{R}^m \to \mathbb{R}^n$ is of rank r for all $x \in U$.

   Show that for any $x_0 \in U$ there exists a diffeomorphism ϕ between an open neighbourhood of x_0 and an open neighbourhood of $0 \in \mathbb{R}^m$ and a diffeomorphism ψ between an open neighbourhood of $y_0 = f(x_0)$ and an open neighbourhood of 0 in $\mathbb{R}^n$ such that the locally defined map

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^r \times \mathbb{R}^{m-r} \to \mathbb{R}^r \times \mathbb{R}^{n-r}$$

   has the form $(x_1, ..., x_r, ..., x_m) \mapsto (x_1, ..., x_r, 0, ..., 0)$.

   Hint: The idea is that f locally around x_0 looks like $D_{x_0} f$, which is a linear map $\mathbb{R}^m \to \mathbb{R}^n$ of rank r, which up to a basis change has the form $(x_1, .., x_m) \mapsto (x_1, ..., x_r, 0, ..., 0)$.

(a) Set $E_2 := \ker(D_{x_0}f) \subset \mathbb{R}^m$ and $E_1 := E_2^\perp$, and $F_1 := \text{Im}(D_{x_0}f) \subset \mathbb{R}^n$ and $F_2 := F_1^\perp$. Decompose

$$\mathbb{R}^m = E_1 \oplus E_2 \qquad \text{and} \qquad \mathbb{R}^n = F_1 \oplus F_2,$$

and consider f as a map $f = (f_1, f_2) : E_1 \oplus E_2 \to F_1 \oplus F_2$ defined on $U \subset E_1 \oplus E_2 = \mathbb{R}^m$.

(b) Show that $\phi : E_1 \oplus E_2 \to F_1 \oplus E_2$ given by

$$\phi(x^1, x^2) = (f_1(x^1, x^2) - f_1(x_0^1, x_0^2), x^2 - x_0^2)$$

is a local diffeomorphism around $x_0 = (x_0^1, x_0^2)$ whose local inverse will be the required map.

(c) Show that $g := f \circ \phi^{-1} : F_1 \oplus E_2 \to F_1 \oplus F_2$ has the form

$$g(y^1, y^2) = (g_1((y^1, y^2), g_2((y^1, y^2)) = (y^1 + y_0^1, g_2(y^1, 0)).$$

Now ψ is easily seen to be...?

4. Suppose M and N are are manifolds of dimension m respectively n and let $f : M \to N$ be a smooth map of constant rank r. Deduce from (1) that for any fixed $y \in f(M)$ the preimage $f^{-1}(y) \subset M$ is a submanifold of dimension $m - r$ in M.

5. Consider the Grassmannian of r-planes in $\mathbb{R}^n$:

$$\text{Gr}(r, n) := \{E \subset \mathbb{R}^n : E \text{ is a r-dimensional subspace of } \mathbb{R}^n\}.$$

Denote by $\text{St}_r(\mathbb{R}^n)$ the set of r-tuples of linearly independent vectors in $\mathbb{R}^n$. Identifying an element $X \in \text{St}_r(\mathbb{R}^n)$ with a $n \times r$ matrix

$$X = (x^1, ..., x^r) \qquad x^i \in \mathbb{R}^n,$$

shows that $\text{St}_r(\mathbb{R}^n)$ equals the subset of rank r matrices in the vector space $\text{M}_{n\times r}(\mathbb{R})$, which we know from Tutorial 1 is an open subset. Write

$$\pi : \text{St}_r(\mathbb{R}^n) \to \text{Gr}(r, n)$$

for the natural projection given by $\pi(X) = \text{span}(x^1, ..., x^r)$ and equip $\text{Gr}(r, n)$ with the quotient topology with respect to π.

(a) Fix $E \in \text{Gr}(r, n)$ and let $F \subset \mathbb{R}^n$ be a subspace of dimension $n - r$ such that $\mathbb{R}^n = E \oplus F$. Show that

$$U_{(E,F)} = \{W \in \text{Gr}(r, n) : W \cap F = \{0\}\} \subset \text{Gr}(r, n)$$

is an open neighbourhood of E.

(b) Show that any element $W \in U_{(E,F)}$ determines a unique linear map

$$\widetilde{W} : E \to F$$

such that its graph equals W, i.e. $W = \{(x, \widetilde{W}x) : x \in E\}$.

(c) Show that the map $u_{E,F} : U_{(E,F)} \to \mathrm{Hom}(E,F)$ given by $u_{E,F}(W) = \widetilde{W}$ is a homeomorphism.

(d) Show that

$$\mathcal{A} := \{(U_{(E,F)}, u_{(E,F)}) : E, F \subset \mathbb{R}^n \text{ complimentary subspaces of dimension } r \text{ resp. } n{-}r\}$$

is a smooth atlas for $\mathrm{Gr}(r,n)$.

6. For a topological space M denote by $C^0(M)$ the vector space of continuous real-valued functions $f : M \to \mathbb{R}$. Any continuous map $F : M \to N$ between topological spaces M and N induces a map $F^* : C^0(N) \to C^0(M)$ given by $F^*(f) := f \circ F : M \to \mathbb{R}$.

(a) Show that F^* is linear.

(b) If M and N are (smooth) manifolds, show that $F : M \to N$ is smooth $\iff$ $F^*(C^\infty(N)) \subset C^\infty(M)$.

(c) If F is a homeomorphism between (smooth) manifolds, show that F is a diffeomorphism $\iff$ F^* is an isomorphism.