Tutorial 5—Global Analysis

1. Suppose $M = \mathbb{R}^3$ with standard coordinates (x, y, z). Consider the vector field
$$\xi(x, y, z) = 2\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}.$$
How does this vector field look like in terms of the coordinate vector fields associated to the cylindrical coordinates (r, ϕ, z), where $x = r\cos\phi$, $y = r\sin\phi$ and $z = z$? Or with respect to the spherical coordinates (r, ϕ, θ), where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\cos\phi$ and $z = r\cos\theta$?

2. Consider $\mathbb{R}^3$ with coordinates (x, y, z) and the vector fields
$$\xi(x, y, z) = (x^2 - 1)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}$$
$$\eta(x, y, z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2xz^2\frac{\partial}{\partial z}.$$
Are they tangent to the cylinder $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$ with radius 1 (i.e. do they restrict to vector fields on M)?

3. Suppose $M = \mathbb{R}^2$ with coordinates (x, y). Consider the vector fields $\xi(x, y) = y\frac{\partial}{\partial x}$ and $\eta(x, y) = \frac{x^2}{2}\frac{\partial}{\partial y}$ on M. We computed in class their flows and saw that they are complete. Compute $[\xi, \eta]$ and its flow? Is $[\xi, \eta]$ complete?

4. Let M be a (smooth) manifold and $\xi, \eta \in \mathfrak{X}(M)$ two vector fields on M. Show that
   (a) $[\xi, \eta] = 0 \iff (\mathrm{Fl}^{\xi}_t)^*\eta = \eta$, whenever defined $\iff \mathrm{Fl}^{\xi}_t \circ \mathrm{Fl}^{\eta}_s = \mathrm{Fl}^{\eta}_s \circ \mathrm{Fl}^{\xi}_t$, whenever defined.
   (b) If N is another manifold, $f : M \to N$ a smooth map, and ξ and η are f-related to vector fields $\tilde{\xi}$ resp. $\tilde{\eta}$ on N, then $[\xi, \eta]$ is f-related to $[\tilde{\xi}, \tilde{\eta}]$.

5. Consider the general linear group GL$(n, \mathbb{R})$. For $A \in$ GL$(n, \mathbb{R})$ denote by
$$\lambda_A : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \qquad \lambda_A(B) = AB$$
$$\rho_A : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \qquad \rho_A(B) = BA$$
left respectively right multiplication by A, and by $\mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ the multiplication map.

(a) Show that λ_A and ρ_A are diffeomorphisms for any $A \in \mathrm{GL}(n, \mathbb{R})$ and that

$$T_B\lambda_A(B, X) = (AB, AX) \qquad T_B\rho_A(B, X) = (BA, XA),$$

where $(B, X) \in T_B\mathrm{GL}(n, \mathbb{R}) = \{(B, X) : X \in M_n(\mathbb{R})\}$.

(b) Show that

$$T_{(A,B)}\mu((A, B), (X, Y)) = T_B\lambda_A Y + T_A\rho^B X = (AB, AY + XB)$$

where $(A, B) \in \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ and $(X, Y) \in M_n(\mathbb{R}) \times M_n(\mathbb{R})$.

(c) For any $X \in M_n(\mathbb{R}) \cong T_{Id}\mathrm{GL}(n, \mathbb{R})$ consider the maps

$$L_X : \mathrm{GL}(n, \mathbb{R}) \to T\mathrm{GL}(n, \mathbb{R}) \qquad L_X(B) = T_{Id}\lambda_B(Id, X) = (B, BX).$$

$$R_X : \mathrm{GL}(n, \mathbb{R}) \to T\mathrm{GL}(n, \mathbb{R}) \qquad R_X(B) = T_{Id}\rho_B(Id, X) = (B, XB).$$

Show that L_X and R_X are smooth vector field and that $\lambda_A^* L_X = L_X$ and $\rho_A^* R_X = R_X$ for any $A \in \mathrm{GL}(n, \mathbb{R})$. What are their flows? Are these vector fields complete?

(d) Show that $[L_X, R_Y] = 0$ for any $X, Y \in M_n(\mathbb{R})$.