Tutorial 6-8—Global Analysis

1. Suppose α^i_j for $i = 1, ..., k$ and $j = 1, ..., n$ are smooth real-valued functions defined on some open set $U \subset \mathbb{R}^{n+k}$ satisfying

$$\frac{\partial \alpha^i_j}{\partial x^k} + \sum_{\ell=1}^{k} \alpha^\ell_k \frac{\partial \alpha^i_j}{\partial z^\ell} = \frac{\partial \alpha^i_k}{\partial x^j} + \sum_{\ell=1}^{k} \alpha^\ell_j \frac{\partial \alpha^i_k}{\partial z^\ell},$$

   where we write $(x, z) = (x^1, ..., x^n, z^1, ..., z^k)$ for a point in $\mathbb{R}^{n+k}$. Show that for any point $(x_0, z_0) \in U$ there exists an open neighbourhood V of x_0 in $\mathbb{R}^n$ and a unique C^∞-map $f : V \to \mathbb{R}^k$ such that

$$\frac{\partial f^i}{\partial x^j}(x^1, ..., x^n) = \alpha^i_j(x^1, ..., x^n, f^1(x), ..., f^k(x)) \quad \text{and} \quad f(x_0) = z_0.$$

   In the class/tutorial we proved this for $k = 1$ and $j = 2$.

2. Which of the following systems of PDEs have solutions $f(x, y)$ (resp. $f(x, y)$ and $g(x, y)$) in an open neighbourhood of the origin for positive values of $f(0, 0)$ (resp. $f(0, 0)$ and $g(0, 0)$)?
   (a) $\frac{\partial f}{\partial x} = f \cos y$ and $\frac{\partial f}{\partial y} = -f \log f \tan y$.
   (b) $\frac{\partial f}{\partial x} = e^{xf}$ and $\frac{\partial f}{\partial y} = xe^{yf}$.
   (c) $\frac{\partial f}{\partial x} = f$ and $\frac{\partial f}{\partial y} = g$; $\frac{\partial g}{\partial x} = g$ and $\frac{\partial g}{\partial y} = f$.

3. Suppose $E \to M$ is a (smooth) vector bundle of rank k over a manifold M. Then E is called *trivializable*, if it isomorphic to the trivial vector bundle $M \times \mathbb{R}^k \to M$.
   (a) Show that $E \to M$ is trivializable $\iff$ $E \to M$ admits a global frame, i.e. there exist (smooth) sections $s_1, ..., s_k$ of E such that $s_1(x)$, ..., $s_k(x)$ span E_x for any $x \in M$.
   (b) Show that the tangent bundle of any Lie group G is trivializable.
   (c) Recall that $\mathbb{R}^n$ has the structure of a (not necessarily associative) normed division algebra over $\mathbb{R}$ for $n = 1, 2, 4, 8$. Use this to show that the tangent bundle of the spheres $S^1 \subset \mathbb{R}^2$, $S^3 \subset \mathbb{R}^4$ and $S^7 \subset \mathbb{R}^8$ is trivializable.

4. Let V be a finite dimensional real vector space and consider the subspace of r-linear alternating maps $\Lambda^r V^* = L^r_{\text{alt}}(V, \mathbb{R})$ of the vector space of r-linear maps $L^r(V, \mathbb{R}) = (V^*)^{\otimes r}$. Show that for $\omega \in L^r(V, \mathbb{R})$ the following are equivalent:

(a) $\omega \in \Lambda^r V^*$

(b) For any vectors $v_1, ..., v_r \in V$ one has

$$\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k)$$

(c) ω is zero whenever one inserts a vector $v \in V$ twice.

(d) $\omega(v_1, ..., v_k) = 0$, whenever $v_1, ..., v_k \in V$ are linearly dependent vectors.

5. Let V be a finite dimensional real vector space. Show that the vector space $\Lambda^* V^* := \bigoplus_{r \geq 0} \Lambda^r V^*$ is an associative, unitial, graded-anticommutative algebra with respect to the wedge product $\wedge$, i.e. show that the following holds:

   (a) $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$ for all $\omega, \eta, \zeta \in \Lambda^* V^*$.

   (b) $1 \in \mathbb{R} = \Lambda^0 V^*$ satisfies $1 \wedge \omega = \omega \wedge 1 = 1$ for all $\omega \in \Lambda^* V^*$.

   (c) $\Lambda^r V^* \wedge \Lambda^s V^* \subset \Lambda^{r+s} V^*$.

   (d) $\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega$ for $\omega \in \Lambda^r V^*$ and $\eta \in \Lambda^s V^*$.

   Moreover, show that for any linear map $f : V \to W$ the linear map $f^* : \Lambda^* W^* \to \Lambda^* V^*$ is a morphism of graded unitlal algebras, i.e. $f^* 1 = 1$, $f^*(\Lambda^r W^*) \subset \Lambda^r V^*$ and $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

6. Let V be a finite dimensional real vector space. Show that:

   (a) If $\omega_1, ..., \omega_r \in V^*$ and $v_1, ..., v_r \in V$, then

   $$\omega_1 \wedge ... \wedge \omega_r(v_1, ..., v_r) = \det((\omega_i(v_j))_{1 \leq i,j \leq r}).$$

   In particular, $\omega_1, ..., \omega_r$ are linearly independent $\iff \omega_1 \wedge ... \wedge \omega_r \neq 0$.

   (b) If $\{\lambda_1, ..., \lambda_n\}$ is a basis of V^*, then

   $$\{\lambda_{i_1} \wedge ... \wedge \lambda_{i_r} : 1 \leq i_1 < ... < i_r \leq n\}$$

   is a basis of $\Lambda^r V^*$.

7. Let V be a finite dimensional real vector space. An element $\mu \in L^r(V, \mathbb{R})$ is called *symmetric*, if $\mu(v_1, ..., v_r) = \mu(v_{\sigma(1)}, ..., v_{\sigma(r)})$ for any vectors $v_1, ..., v_r \in V$ and any permutation $\sigma \in S^r$. Denote by $S^r V^* \subset \mu \in L^r(V, \mathbb{R})$ the subspace of symmetric elements in the vector space $L^r(V, \mathbb{R})$.

   (a) For $\mu \in L^r(V, \mathbb{R})$ show that

   $$\mu \in S^r V^* \iff \mu(v_1, ..., v_i, ..., v_j, ..., v_k) = \mu(v_1, ..., v_j, ..., v_i, ..., v_k),$$

   for any vectors $v_1, ..., v_r \in V$.

(b) Consider the map Sym : $L^r(V,\mathbb{R}) \to L^r(V,\mathbb{R})$ given by

$$\mathrm{Sym}(\mu)(v_1,...,v_r) = \frac{1}{r!}\sum_{\sigma\in S^r}\mu(v_{\sigma(1)},...,v_{\sigma(r)}).$$

Show that Image(Sym) $= S^rV^*$ and that $\mu \in S^rV^* \iff \mathrm{Sym}(\mu) = \mu$.

8. Let V be a finite dimensional real vector space and set $S(V^*) := \oplus_{r=0}^{\infty} S^rV^*$ with the convention $S^0V^* = \mathbb{R}$ and $S^1V^* = V^*$. For $\mu \in S^rV^*$ and $\nu \in S^tV^*$ define their symmetric product by

$$\mu \odot \nu := \mathrm{Sym}(\mu \otimes \nu) \in S^{r+t}V^*.$$

By blinearity, we extend this to a $\mathbb{R}$-bilinear map $\odot : S(V^*) \times S(V^*) \to S(V^*)$. Show that $S(V^*)$ is an unitial, associative, commutative, graded algebra with respect to the symmetric product $\odot$.

9. Suppose $p : E \to M$ and $q : F \to M$ are vector bundles over M. Show that their direct sum $E \oplus F := \sqcup_{x\in M} E_x \oplus F_x \to M$ and their tensor product $E \otimes F := \sqcup_{x\in M} E_x \otimes F_x \to M$ are again vector bundles over M.

10. Suppose $E \subset TM$ is a smooth distribution of rank k on a manifold M of dimension n and denote by $\Omega(M)$ the vector space of differential forms on M.

(a) Show that locally around any point $x \in M$ there exists (local) 1-forms $\omega^1, ..., \omega^{n-k}$ such that for any (local) vector field ξ one has: ξ is a (local) section of $E \iff \omega_i(\xi) = 0$ for all $i = 1, ..., n-k$.

(b) Show that E is involutive $\iff$ whenever $\omega^1, ..., \omega^{n-k}$ are local 1-forms as in (a) then there exists local 1-forms $\mu^{i,j}$ for $i, j = 1, ..., n-k$ such that

$$d\omega^i = \sum_{j=1}^{n-k} \mu^{i,j} \wedge \omega^j.$$

(c) Show

$$\Omega_E(M) := \{\omega \in \Omega(M) : \omega|_E = 0\} \subset \Omega(M)$$

is an ideal of the algebra $(\Omega(M), \wedge)$. Here, $\omega|_E = 0$ for a ℓ-form ω means that $\omega(\xi_1, ..., \xi_\ell) = 0$ for any sections $\xi_1, ...\xi_\ell$ of E.

(d) An ideal $\mathcal{J}$ of $(\Omega(M), \wedge)$ is called differential ideal, if $d(\mathcal{J}) \subset \mathcal{J}$. Show that $\Omega_E(M)$ is a differential ideal $\iff$ E is involutive.

11. Suppose M is a manifold and $D_i : \Omega^k(M) \to \Omega^{k+r_i}(M)$ for $i = 1, 2$ a graded derivation of degree r_i of $(\Omega(M), \wedge)$.

(a) Show that

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$$

is a graded derivation of degree $r_1 + r_2$.

(b) Suppose D is a graded derivation of $(\Omega(M), \wedge)$. Let $\omega \in \Omega^k(M)$ be a differential form and $U \subset M$ an open subset. Show that $\omega|_U = 0$ implies $D(\omega)|_U = 0$.

Hint: Think about writing 0 as $f\omega$ for some smooth function f and use the defining properties of a graded derivation.

(c) Suppose D and $\tilde{D}$ are two graded derivations such that $D(f) = \tilde{D}(f)$ and $D(df) = \tilde{D}(df)$ for all $f \in C^\infty(M, \mathbb{R})$. Show that $D = \tilde{D}$.

12. Suppose M is a manifold and $\xi, \eta \in \Gamma(TM)$ vector fields.

(a) Show that the insertion operator $i_\xi : \Omega^k(M) \to \Omega^{k-1}(M)$ is a graded derivation of degree -1 of $(\Omega(M), \wedge)$.

(b) Recall from class that $[d, d] = 0$. Verify (the remaining) graded-commutator relations between $d, \mathcal{L}_\xi, i_\eta$:

(i) $[d, \mathcal{L}_\xi] = 0$.

(ii) $[d, i_\xi] = d \circ i_\xi + i_\xi \circ d = \mathcal{L}_\xi$.

(iii) $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]}$.

(iv) $[\mathcal{L}_\xi, i_\eta] = i_{[\xi,\eta]}$.

(v) $[i_\xi, i_\eta] = 0$.

Hint: Use (c) from 11.