Tutorial 9—Global Analysis

1. Prove the **Poincaré Lemma**: Suppose $\omega \in \Omega^k(\mathbb{R}^m)$ is a closed k-form, where $k \geq 1$. Show that there exists $\tau \in \Omega^{k-1}(\mathbb{R}^m)$ such that $d\tau = \omega$.

   Hint: Show that for any k-form $\omega = \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ on $\mathbb{R}^m$,

   $$P(\omega) = \sum_{\alpha=1}^{k} \sum_{i_1<\ldots<i_k} (-1)^{\alpha-1} \left[\int_0^1 t^{k-1}\omega_{i_1\ldots i_k}(tx)dt\right] x^{i_\alpha} dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_\alpha}} \wedge \ldots \wedge dx^{i_k}.$$

   is a $(k-1)$-form on $\mathbb{R}^m$ satisfying

   $$\omega = d(P(\omega)) + P(d\omega).$$

   Here, $\widehat{dx^{i_\alpha}}$ means that this term is omitted.

2. Show that any manifold with a parallelizable tangent bundle is orientable.

3. Suppose $M \subset N$ is a submanifold of codimension 1 (i.e. $\dim M = \dim N - 1$) of an oriented manifold N. Suppose there exists a smooth vector field along M that is transverse everywhere to M, that is, a smooth map $\nu : M \to TN$ such that for all $x \in M$ one has

   (i) $\nu(x) \in T_xN$ and

   (ii) $\nu(x)$ and T_xM span T_xN.

   Show that M is orientable. Deduce that a hypersurface

   $$(M, g) \subset (\mathbb{R}^{m+1}, g) = (\mathbb{R}^{m+1}, g^{\text{euc}})$$

   in Euclidean space is orientable if and only if M admits a globally defined unit normal vector field.

4. Consider $S^m \subset \mathbb{R}^{m+1}$ the unit sphere and the global unit normal vector field $\nu(x) = \sum_{i=1}^{m+1} x^i \frac{\partial}{\partial x^i}$ for S^m. Show that for the nowhere vanishing $m+1$-form

   $$\Omega = dx^1 \wedge \ldots \wedge dx^{m+1}$$

   on $\mathbb{R}^{m+1}$,

   $$\omega(x) := (i_\nu\Omega)(x) = \Omega(x)(\nu(x), \_, \ldots, \_) \text{ for } x \in S^m$$

restricts to a nowhere vanishing m-form on S^m that satisfies

$$A^*\omega = (-1)^{m+1}\omega,$$

where $A : S^m \to S^m$ is the antipodal map $A(x) = -x$.

5. Show that n-dimensional projective space $\mathbb{R}P^m$ is orientable $\iff$ m is odd.

   Hint: For ,$\Longrightarrow'$ consider the natural projection $\pi : S^m \to \mathbb{R}P^m$, given by $\pi(x) = [x]$, and use the previous exercise. For ,$\Longleftarrow'$ construct an oriented atlas.

6. Suppose M and N are connected, compact, oriented manifolds of the same dimension m. Let $f_0, f_1 : M \to N$ be smooth maps that are homotopic to each other, i.e. there exists a smooth map $F : M \times [0,1] \to N$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Show that for any $\omega \in \Omega^m(N)$ one has

$$\int_M f_0^*\omega = \int_M f_1^*\omega.$$

   Hint: $M \times [0,1]$ is an oriented manifold with boundary $\partial M = -(M \times \{0\}) \cup M \times \{1\}$, where the minus indicates that the orientation on $M \times \{0\}$ is reversed. Use Stokes' Theorem.

7. Use the previous exercise to show that, if the antipodal map $A : S^m \to S^m$ on the sphere S^m is homotopic to the identity Id_{S^m} on S^m, then m is odd.

8. Show that on a sphere S^{2m} of even dimension any smooth vector field $\xi \in \mathfrak{X}(S^{2m})$ has a zero.
   Hint: Show that if $\xi \in \mathfrak{X}(S^{2m})$ is nowhere vanishing, then there exists a homotopy between the antipodal map and the identity.