Algebra 3

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- lecture notes + 1S each week

- Course has 3 components:

Do together

Category theory

Universal algebra

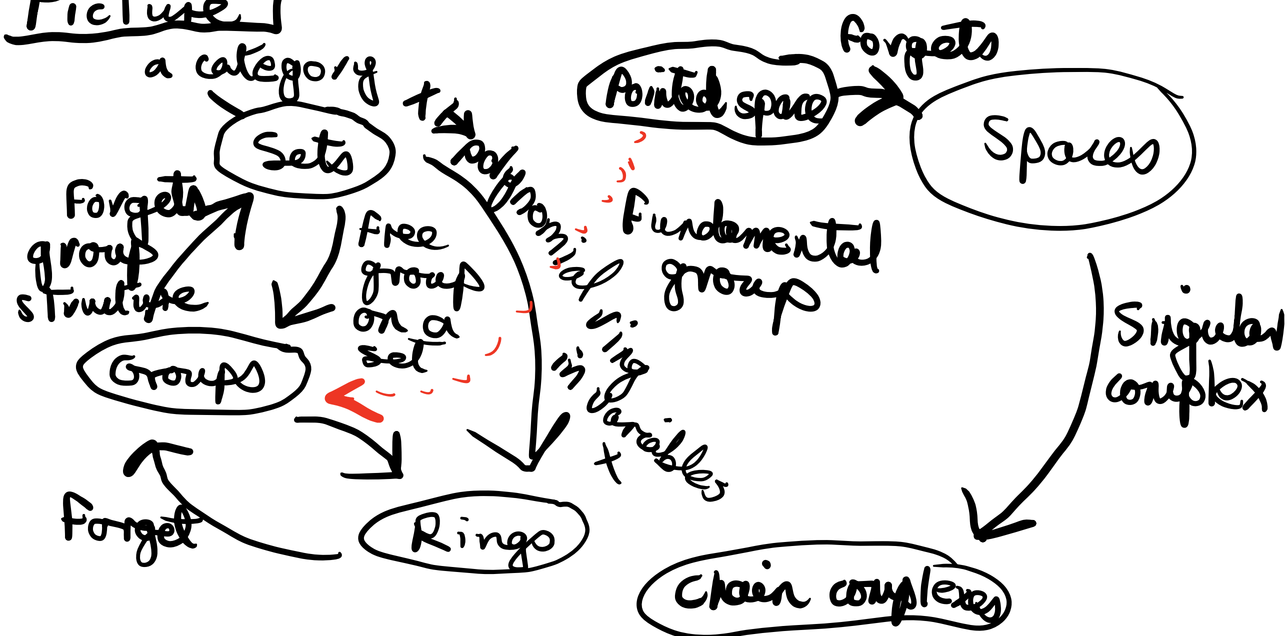
Module theory

Today: start with category theory.

What is category theory?

- In math, study structures like sets, groups, rings, topological spaces.
- Each forms category of "structures of that type".
- Cat. theory studies the relⁿ between these diff. areas of mathematics (or categories of mathematics)

Picture



- What do these different categories have in common?
- What structure is preserved when we move between them?
- Fundamental notion in category theory is an arrow/morphism:

$$A \longrightarrow B$$

captures relationship between two things.

Def) A category $\mathscr{C}$ consists of :

- a collection $\operatorname{ob}\mathscr{C}$ of objects $A, B, C,$
- & for each pair of objects $A, B \in \operatorname{ob}\mathscr{C}$ a collection $\mathscr{C}(A,B)$ of "arrows/morphisms" from A to B,

(depicted $A \xrightarrow{f} B$ or $f : A \to B$ to mean $f \in \mathscr{C}(A,B)$)

- For each $A, B, C \in \operatorname{ob}\mathscr{C}$ a function

$$\mathscr{C}(B,C) \times \mathscr{C}(A,B) \longrightarrow \mathscr{C}(A,C)$$

$$(B \xrightarrow{g} C, A \xrightarrow{f} B) \longmapsto A \xrightarrow{g \circ f} C$$

called composition.

- For each object $A \in \operatorname{ob}\mathscr{C}$ an arrow $1_A : A \longrightarrow A$ called the identity on A.
- These satisfy the following axioms:

Associativity

- Given $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

- Left & right unit laws
- Given $f : A \longrightarrow B$ we have

$$1_B \circ f = f = f \circ 1_A .$$

- Notation: often write $gf : A \to C$ instead of $g \circ f$.
- Associativity & unit laws imply that given

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$$

there is a <u>unique</u> way to compose the arrows $f_n f_{n-1} \cdots f_2 f_1$, indep. of brackets & identities.

Eg: When $n=4$,

$$((f_4 f_3) f_2) f_1 = (f_4 1_{A_3})((f_3 f_2) f_1).$$

- <u>Commutative diagrams</u>

Given a diagram such as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & & B \arrow[d, "g"] \\ C \arrow[r, "i"] & D \arrow[r, "j"] & E \end{tikzcd}$$

we say the <u>diagram commutes</u> if the paths agree: in this case, $gf = jih$.

- In a larger diagram, eg.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "="] & & B \arrow[r, "k"] \arrow[d, "g"] \arrow[dr, phantom, "="] & F \arrow[d, "l"] \\ C \arrow[r, "i"] & D \arrow[r, "j"] & E \arrow[r, "m"'] & G \end{tikzcd}$$

diag. commutes

the commutativity of subdiagrams implies the commutativity of the outer diagram:

This is <u>diagram chasing</u>.

Examples

- Set : objects are sets, morphisms $f: A \to B$ are functions.
- Given $A \xrightarrow{f} B$ & $B \xrightarrow{g} C$ the function $gf: A \to C$ is def. by $gf(x) = g(f(x))$.

  We have $1_A(x) = x$.

Categories of sets with structure

- Mon, the cat of monoids & monoid homorphisms:

  $f: (A, m_A, e_A) \longrightarrow (B, m_B, e_B)$

  $f(e_A) = e_A$,

  $f(m_A(x,y)) = m_B(fx, fy)$
- Grp, the category of groups & group homomorphisms.
- Rng ~ rings, ring hom...

  k-Vect ~ k-vector spaces, lin. transf.
- These are examples of algebraic categories.

- More generally, given

signature (Ω, E) set of equations

we can consider the cat. (Ω, E)-Alg. This captures all of the above examples. Return to this example later.

- Top is the category of topological spaces and continuous functions.

Def) A morphism $f : A \to B \in \mathcal{C}$ is an isomorphism if $\exists\, g : B \to A$ such that $gf = 1_A$ & $fg = 1_B$.

Remark) Can express this via the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "1_A"'] & B \arrow[d, "g"] \arrow[dr, "1_B"] & \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

We say that g is the inverse of f and write $g = f^{-1}$.
This is justified by:

Proposition) If $f : A \to B$ is an iso, then its inverse is unique.

Proof) Suppose $g, h : B \to A$ are inverses to f.

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "1_B"'] \arrow[drr, bend left=60, "g"] \arrow[drr, bend right=60, "h"'] & A \arrow[d, "f"] \arrow[dr, "1_A"] & \\ & B \arrow[r, "h"'] & A \end{tikzcd}$$

Therefore $g = h$.

In alg. notation, the corresponding proof is

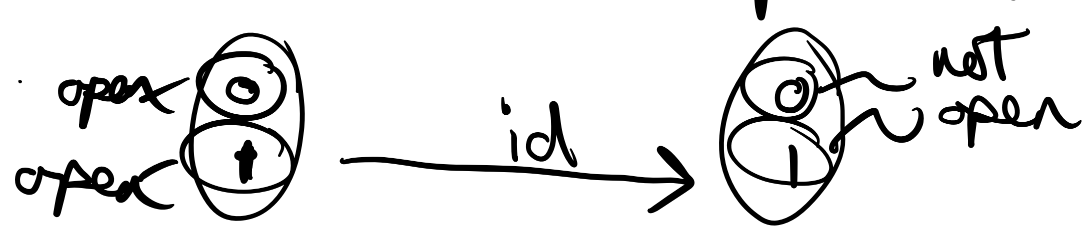
$$g = 1_A \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ 1_B = h.$$

Examples

- In Set, the isomorphisms $f : A \to B$ are the bijections.
- In other algebraic cats such as Grp, an isomorphism is sometimes defined as a bijective homomorphism: this coincides with the categorical notion, since if a homomorphism is bijective its inverse (as a function) is also a homomorphism.
- For Top, an isomorphism is a homeomorphism: a cts function with a cts inverse.

In Top, $\exists$ bijective cts maps which are not homeomorphisms

eg.



not a homeomorphism.

- In summary, the categorical defⁿ of isomorphism captures the correct notion in all of our examples.

Defⁿ) - A category $\mathcal{C}$ is said to be locally small if for all $A, B \in \mathcal{C}$ the collection $\mathcal{C}(A,B)$ is a set.

- If, furthermore, the collⁿ ob $\mathcal{C}$ is a set, we say that it is small.

Examples of locally small cats

- Set is not small: one cannot form a "set of all sets".
  However, it is locally small as the collection Set(A, B) of functions from A to B forms a set.
- All of the examples considered so far are locally small, but not small.

Examples of small categories

Ex 1) - $\emptyset$ empty cat

- $\mathbf{1} = \{\bullet\}$ 1 object, 1 identity morph
- plus identities

$$\boxed{0 \longrightarrow 1}\ ,\quad \boxed{\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[d, "g"] \\ & c \end{tikzcd}}\ , \ldots$$

$\boxed{\text{Ex. 2}}$ Preorders & posets

- Preorder $(X, \leq)$
  satisfying $x \leq x$
  $x \leq y$ & $y \leq z \Rightarrow x \leq z$
  Poset : also $x \leq y$ & $y \leq x \Rightarrow x = y$.
- If $(X, \leq)$ a preorder,
  can form a category X^*
  whose objects are elements of X
  & such that there exists a single
  morph. $x \to y \iff x \leq y$ &
  no morphisms from x to y
  otherwise.
- In fact,
  Preorders $\overset{\text{same as}}{\equiv}$ small cats with at most 1 morph. between any two objects.
  hopefully later in course
  (equivalence of cats)

$\boxed{\text{Example 3}}$

If $(M, \overset{\text{mult}}{\times}, \overset{\text{unit}}{e})$ is a monoid we can form a category ΣM w/ one object $\bullet$ & whose morphisms $\bullet \xrightarrow{m} \bullet$ are the elements of M.

We compose by

$$\begin{tikzcd} \bullet \arrow[r, "m"] \arrow[dr, "n \times m"'] & \bullet \arrow[d, "n"] \\ & \bullet \end{tikzcd} \qquad \bullet \xrightarrow[1]{e} \bullet$$

unit is $1_\bullet$

& assoc. & unit laws for monoid $\Rightarrow$ axioms for a category.

- If $(M, \times, e)$ is a group (all elts. have inverse) then ΣM is a groupoid: a category in which all morphisms are isomorphisms.

- Monoids $\equiv$ 1-object small cats
  $\cup$ $\quad$ $\cup$
  Groups $\equiv$ 1-ob. small groupoids

- **Natural**: e.g. symmetric group S_n consists of the bijections $f \circlearrowright \bar{n} = \{1, \dots, n\}$.

  Groups of transformations ~ symmetries of an object.