Functors

Def^n) Let $\mathcal{A}, \mathcal{B}$ be categories. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

- a function $\mathrm{ob}\,\mathcal{A} \longrightarrow \mathrm{ob}\,\mathcal{B}$ which we write as $X \longmapsto FX$,
- For each pair $X, Y \in \mathcal{A}$ a function $\mathcal{A}(X,Y) \longrightarrow \mathcal{B}(FX, FY)$ written as $X \xrightarrow{\alpha} Y \longmapsto FX \xrightarrow{F\alpha} FY$.

This satisfies the axioms:

- given $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \in \mathcal{A}$, we have $F(\beta \circ \alpha) = F\beta \circ F\alpha$.
- given $X \in \mathcal{A}$ we have $F(1_X) = 1_{FX}$.

In other words, a functor sends obs to obs, arrows to arrows (respecting their domain & codomain) & preserves identities and composition.

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Examples

- Forgetful functors:

  Eg. $U : \mathrm{Grp} \longrightarrow \mathrm{Set}$

  $(X, \underbrace{m}_{\text{mult.}}, \underbrace{e}_{\text{unit}}) \longmapsto X$

  which sends a group to its underlying set, ie. forgets the group structure.

  Sim. it sends a group homomorphism to its underlying function:

  $f : (X, m_X, e_X) \to (Y, m_Y, e_Y) \longmapsto f : X \to Y$

  Clearly preserves composition & identities.

- Similarly, there are forgetful functors from cats of "sets with structure" to Set, eg

  $U : \mathrm{Top} \longrightarrow \mathrm{Set}$, $U : \mathbf{Grp} \longrightarrow \mathrm{Set}$

  Similarly, $U : \mathrm{Rng} \longrightarrow \mathrm{Mon}$

  $(R, +, \times, 0, 1) \longmapsto (R, \times, 1)$.

There is a functor $F : \mathrm{Set} \longrightarrow \mathrm{Mon}$
where $FX =$ "list/free" monoid on X

- Its elements are (possibly empty) lists $[x_1, x_2, \ldots, x_n]$ of elements of X with unit $[-]$ empty list.
- Composition of lists is "concatenation":
  $[x_1, \ldots, x_n] . [y_1, \ldots, y_m] = [x_1, \ldots, x_n, y_1, \ldots, y_m]$
- This makes FX a monoid.
- Given a function $f : X \longrightarrow Y$ obtain a monoid homomorphism

$$Ff : FX \longrightarrow FY$$
$$[x_1, \ldots, x_n] \longmapsto [Fx_1, \ldots, Fx_n]$$

& easy to check that this satisfies axioms for a functor.

Example) Let G be a group & consider its suspension ΣG as a groupoid with one object $\bullet$ & $\Sigma G(-,-) = G$.
Then a functor $\Sigma G \longrightarrow \mathrm{Set}$ is what?

$$\begin{tikzcd} \bullet \arrow[d, "g"'] \\ \bullet \end{tikzcd} \longmapsto \begin{tikzcd} X \arrow[d, "g.-"] \\ X \end{tikzcd} \quad \begin{tikzcd} x \arrow[d, mapsto] \\ g.x \end{tikzcd}$$

The functor axioms say $e.x = x$ & $h.(g.x) = (h.g).x$.

Therefore a functor $\Sigma G \longrightarrow \mathrm{Set}$ is exactly a <u>G-set</u>: a set X with an action of the group G.

What is a functor $\Sigma G \longrightarrow \mathrm{Vect}$?

Example Let Top_* be category of pointed spaces (X, a)

top space $\quad$ $a \in X$ basepoint

& continuous maps which preserve basepoint.

There is a functor $\Pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$ where $\Pi_1(X, a)$ = fundamental group of loops around a (up to homotopy)

eg.

$\Pi_1(S_1, *) = \mathbb{Z}$.

There are also functors $H_n : \mathrm{Top} \longrightarrow \mathrm{Ab}$ ~ abelian group sending a top. space to its n'th homotopy group.

Example : CAT

- Given functors
$$\begin{tikzcd} A \arrow[r, "F"] \arrow[rr, bend right, "G\circ F"'] & B \arrow[r, "G"] & C \end{tikzcd}$$
we can compose them in obvious way :
$$x \longmapsto G(F(x)) = GFX$$
$$X \xrightarrow{\alpha} Y \longmapsto GFX \xrightarrow{GF\alpha} GFY$$
- Likewise, we have identity functor $1_A : A \to A : x \mapsto x$

- In this way, we obtain a (large) category CAT of categories & functors. In fact, this is a <u>2-category</u>!

Limits & Colimits

Terminal & Initial Objects

Def^n) An object $T \in \mathcal{C}$ is <u>terminal</u> if given any other $X \in \mathcal{C}$ there exists a unique morphism $X \xrightarrow{f} T$.

$\boxed{\text{Theorem}}$ Terminal objects are unique up to unique isomorphism.

Proof) Let S, T be terminal.

As T is terminal, $\exists!\, S \xrightarrow{f} T$.

As S - - - - - - -, $\exists!\, T \xrightarrow{g} S$.

As T is terminal, the two maps

$$T \underset{1_T}{\overset{fg}{\rightrightarrows}} T \text{ are equal.}$$

As S is terminal, similarly $gf = 1_S$, so that $f : S \longrightarrow T$ is an isomorphism.

Example)- In Set, a set is terminal precisely if it has one element, eg. $\{*\}$

- Similarly, in Grp, Mon, Rng, Top ... the 1-elt set has the unique str of a group, monoid, ring, space... & this makes it the terminal object in the corresponding category.

Def^n) An object $I \in \mathcal{C}$ is <u>initial</u> if given $X \in \mathcal{C}$ there exists a unique morphism $I \xrightarrow{f} X$.

Remark) Def^n of initial object is <u>dual</u> to that of terminal object - all arrows reversed.

One can prove that initial objects are unique up to iso. by reversing arrows in proof for terminal objects.

Another approach - use <u>dual/opposite category $\mathcal{C}^{op}$</u>.

Def^n) $\mathcal{C}^{op}$ has same obs as $\mathcal{C}$ & $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$.

Comp in $\mathcal{C}$:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Comp in $\mathcal{C}^{op}$:
$$\begin{tikzcd} X & Y \arrow[l, "f"'] \\ & Z \arrow[u, "g"'] \arrow[ul, "g \circ f"] \end{tikzcd}$$

$$f \circ_{op} g = g \circ f.$$

Identities in $\mathcal{C}^{op}$ are as in $\mathcal{C}$.

Def of initial object 2

An ob $I \in \mathcal{C}$ is initial if it is terminal in $\mathcal{C}^{op}$.

- In this way, we see that init. obs are unique up to iso follows (indeed, is equivalent to) the corr. statement for terminal objects, since init objects are term. obs in $\mathcal{C}^{op}$.

Principle of duality: Every categorical def^n, theorem & proof has a dual, obtained by reversing the arrows. More formally, by interpreting original def^n, thm. or proof in $\mathcal{C}^{op}$ instead of $\mathcal{C}$.

Examples

In **Set**, the init. object is the empty set $\emptyset$ since $\exists! \; \emptyset \longrightarrow X$.

Exercises: What is the initial object in Grp? In Rng?

Products & coproducts

Def^n) Let $A, B \in \mathcal{C}$. We say that a diagram

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[dr, "q"'] & A \\ & B \end{tikzcd}$$

is a <u>product</u> diagram

if it has the "universal property" that:

given any other diagram $\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & A \\ & B \end{tikzcd}$

$\exists!\ X \xrightarrow{h} P$ s.t.

$$\begin{tikzcd} & & A \\ X \arrow[r, "h"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & P \arrow[ur, "p"] \arrow[dr, "q"'] \arrow[u, phantom, "="] \arrow[d, phantom, "="] & \\ & & B \end{tikzcd} \qquad *$$

- In this case, we call the object P the product of A & B, but this is abuse of notation as it is essential to consider the maps p, q.

Exercise: The product of A & B is <u>unique</u> up to <u>unique isomorphism</u>. (Similar to case of terminal objects.)

For this reason, often write product as $\begin{tikzcd} A\times B \arrow[r, "p"] \arrow[dr, "q"'] & A \\ & B \end{tikzcd}$ & call it "the product".

Examples: In Set, the product is the <u>cartesian product</u>

$$A\times B = \{(a,b) : a\in A, b\in B\}$$

with

$$\begin{tikzcd}[column sep=small]
 & A & & & a \\
A\times B \arrow[ur, "p"] \arrow[dr, "q"'] & & & (a,b) \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & B & & & b
\end{tikzcd}$$

Given $\begin{tikzcd}[column sep=small] & A \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$ if $\begin{tikzcd} & & A \\ X \arrow[r, "h"'] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A\times B \arrow[ur, "p"'] \arrow[dr, "q"] \arrow[u, phantom, "="] \arrow[d, phantom, "="] & \\ & & B \end{tikzcd}$

then $phx = fx$ & $qhx = gx$ so
$hx = (fx, gx)$. In partic,
$h : X \longrightarrow A\times B : x \longmapsto (fx, gx)$ is the
unique function s.t. $ph = f$ & $qh = g$
as in $*$.

Similarly, in Grp, Rng etc, products
are the usual cartesian products.
(Also in Top.)

- The dual notion is that of coproduct: coproducts in $\mathcal{C}$ are products in $\mathcal{C}^{op}$.
- In elementary terms, the coproduct of A & B is a diagram

$$\begin{tikzcd} A \arrow[dr, "i"] & \\ & C \\ B \arrow[ur, "j"'] & \end{tikzcd}$$

with the universal property (u.p.) that given

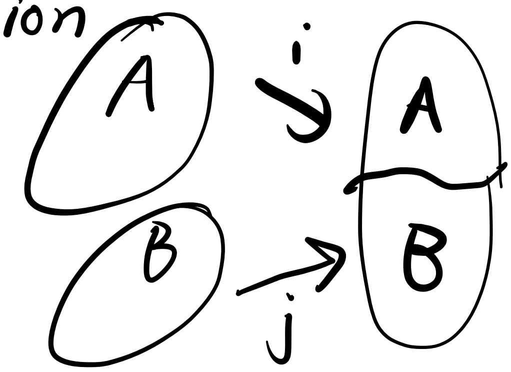
$$\begin{tikzcd} A \arrow[dr, "f"] & \\ & X \\ B \arrow[ur, "g"'] & \end{tikzcd} \quad \exists!\, C \xrightarrow{h} X \text{ s.t. } \begin{tikzcd} A \arrow[dr, "i"'] \arrow[drr, bend left, "f"] & & \\ & C \arrow[r, "h"] & X. \\ B \arrow[ur, "j"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

- Again, the coproduct is unique up to unique isomorphism. We often write it as

$$\begin{tikzcd} A \arrow[dr, "i"] & \\ & A+B \\ B \arrow[ur, "j"'] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} A \arrow[dr, "i"] & \\ & A \cup B \\ B \arrow[ur, "j"'] & \end{tikzcd}.$$

Examples

- In Set, the coproduct is the disjoint union



$A+B = \{(a,1) : a \in A\} \cup \{(b,2) : b \in B\}$ or various other ways.

- In Grp, the coproduct is the free product: elts of $G+H$ are "alternating words" in G & H.. eg. $g_1h_1g_2h_2g_3$.

Exercise: Coproducts & products are dual concepts. Thinking about Set, can you see a sense in which addition and multiplication of numbers are dual?

Exercise: What are coproducts of vector spaces, abelian groups or Top. space?

we will look at coproducts in this setting when we study modules.