Lecture 11 - Modules

Def) Let R be a ring. A left R-module $(M, +, 0)$ is an abelian group M with a function

$$\cdot : R \times M \longrightarrow M$$

satisfying

1. $r.(a+b) = r.a + r.b$
2. $(r+s).a = r.a + s.a$ } bilinear
3. $(r.s).a = r.(s.a)$
4. $1.a = a$ } assoc. unital action

- A homomorphism $f: M \longrightarrow N$ of R-modules is a homomorphism of abelian groups such that $f(r.a) = r.f(a)$.

- There is a category Mod_R of left R-modules which has forgetful functors

$$Mod_R \longrightarrow Ab \longrightarrow Set, \qquad Mod_R \longrightarrow Set$$

Remark :- $Mod_R = (\Omega, E)\text{-}Alg$

(Exercise : describe the equations)

- From the previous chapter, it follows that Mod_R is complete & cocomplete, that forg. functors

$$\begin{array}{ccc} Mod_R & \longrightarrow & Ab \\ & \searrow & \downarrow \\ & & Set \end{array}$$

have left adjoints.

Examples

① When $R = k$ is a field, a k-module is a vector space over k.

② When $R = \mathbb{Z}$, a $\mathbb{Z}$-module is exactly an abelian group.

Indeed, if M is an ab. grp., we are forced to define $\cdot : \mathbb{Z} \times M \longrightarrow M$ as follows:

since bilinearity implies each

$$- \cdot m : \mathbb{Z} \longrightarrow M \text{ is homomorph.}$$

of abelian groups, and since $\mathbb{Z}$ is the free ab. group gen. by element 1,

we must def.

$$n \cdot m = \underbrace{(1 + \ldots + 1)}_{n \text{ times}} \cdot m = \underbrace{m + \ldots + m}_{n \text{ times}}$$

& sim. for negative n.

(Exercise: check remaining details)

③ G a group & R a ring.

Can form group ring $R[G] =$

$\{\lambda_1 g_1 + \cdots \lambda_1 g_k : \lambda_1, \ldots, \lambda_k, g_1, \ldots, g_k \in G\}$

the set of formal R-linear combinations of elements of G.

- This is an abelian group with obvious addition and has multiplication

$$\left(\sum_{i=1}^{k} \lambda_i g_i\right)\left(\sum_{j=1}^{\ell} \mu_j g_j\right) =$$

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell} (\lambda_i \mu_j)(g_i g_j)$$ obtained by extend the mult. of G bilinearly.

$R[G]$-modules are often called group representations:

they amount to R-module M with

$G \times M \xrightarrow{\cdot} M$ sat

$e . m = m$ &

Ex!

$$\left.\begin{array}{l}(g \cdot h) . m = g . (h . m) \\ g . - : M \to M \text{ a } R\text{-mod map}\end{array}\right\}$$

This is of most interest when $R = k$ is a field and the vect. space M is fin. dim., so $M = k^n$; then $k[G]$ module:

group hom $G \longrightarrow Gl(n,k)$.

Ex: Show that the group ring $G \longmapsto \mathbb{Z}[G]$ is the value of left adjoint to forgetful functor

$$\begin{array}{ccc} Rng & \xrightarrow{U} & Mon \\ R & \longmapsto & (R, \cdot, 1) \end{array}$$

Another perspective

Let A be an abelian group. The set of homomorphisms $Ab(A,A)$ is a ring:

componentwise structure

- $f+g\,(x) := fx+gx$, $0(x) := 0$
- $f.g\,(x) = f(g(x))$ — composition.

Proposition

There is a bij^n between
- R-module structures on A &
- ring homomorphisms $R \longrightarrow Ab(A,A)$.

Proof

- Given R-module $R\times A \xrightarrow{\cdot} A$
  define $R \xrightarrow{\varphi} Ab(A,A) : r \longmapsto r.- : A \to A$.

- Then $r.- : A \to A$ is homomorphism as $r.-$ linear.
- $\varphi.(r+s) = \varphi.r + \varphi.s$ as $(r+s).a = ra + sa$.
- Sim φ preserves multiplication.

- Conversely, given $\varphi : R \longrightarrow Ab(A,A)$, define $R\times A \longrightarrow A : (r,a) \longmapsto \varphi(r)(a)$.
- Clearly inverse parts of a bijection. $\square$

Kernels & quotients

- As for abelian groups, the notions of kernels & quotients for modules are very simple.

Def^n • Given a homomorphism $f: M \to N$ of modules, its <u>kernel</u> $\ker(f) \hookrightarrow M$ is the submodule
$$\ker f = \{ x \in M : fx = 0 \}$$

• Given a submodule $N \subseteq M$, the <u>quotient</u> $M/N = \{ m+N : m \in M \}$ is the quotient group (set of cosets) with $r.(m+N) = rm+N$. We then have $M \xrightarrow{p^N} M/N$ module homomorphism.

Exercise: Express $\ker f$ as an equaliser in $\mathrm{Mod}\,R$, & M/N as a coequaliser.

- The first isomorphism theorem in this setting says:

Theorem:

- If $f: M \longrightarrow N \in \text{Mod}\, R$, then $\text{im} f \cong M/\ker f$. In particular, if f is surj., then $N \cong M/\ker f$.
- If $N \subseteq M$ a submodule, then $N = \ker(M \longrightarrow M/N)$.

- This is all routine. What is interesting about Mod R is what happens for products and coproducts.

Products & coproducts

Proposition

- $0 = \{0\}$ is both the terminal & initial object in R-Mod.

Proof

Clearly terminal. 0 is initial as must define $f: 0 \longrightarrow M$ by $f(0) = 0$. □

Remark: The zero homomorphism $0: M \to N$ is the composite $M \longrightarrow 0 \longrightarrow N$.

- Given a set $(M_i)_{i \in I}$ of R-modules, as for any algebraic category, the product $\prod_{i \in I} M_i \xrightarrow{p_i} M_i$ consists of sequences $\{(a_i)_{i \in I} : a_i \in M_i\}$ with component-wise module structure.

Definition

The direct sum $\bigoplus_{i \in I} M_i \hookrightarrow \prod_{i \in I} M_i$ is the submodule consisting of those $(a_i)_{i \in I}$ for which $a_i \neq 0$ only for finitely many $i \in I$.

Remark :- In particular, if I is a finite set, then $\bigoplus_{i\in I} M_i = \prod_{i\in I} M_i$.

- So for instance,

$$A \oplus B = A \times B.$$

- Observe that there are R-module homomorphisms

$$M_i \xrightarrow{q_i} \bigoplus_{i\in I} M_i$$

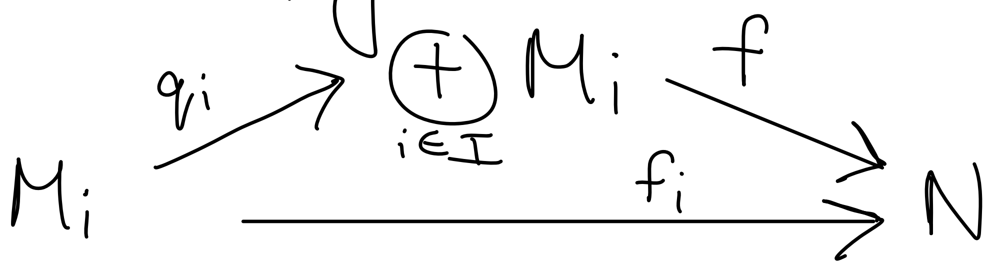
$$a \longmapsto q_i(a) \text{ seq. w/ } (q_i(a))_i = a, \ (q_i(a))_j = 0 \text{ otherwise.}$$

Theorem The maps $M_i \xrightarrow{q_i} \bigoplus_{i\in I} M_i$ exhibit $\bigoplus_{i\in I} M_i$ as the coproduct.

<u>Proof</u>

- Consider $(f_i : M_i \longrightarrow N)_{i\in I}$ homomorphisms in ModR.

- We must show that $\exists!$ hom f making



commute.

- Given $a = (a_i)_{i \in I} \in \bigoplus M_i$, we have $a = q_{i_1}(a_{i_1}) + \dots + q_{i_n}(a_{i_n})$ where $i_1, \dots, i_n \in I$ are the values @ which a is non-zero.

- Then for f to be a homomorph. sat. $f \circ q_i = f_i$, we must define $f(a) = f_{i_1}(a_{i_1}) + \dots + f_{i_n}(a_{i_n})$.

- It is straightforward, using that N is commutative, to see that f is a homomorphism.

- Indeed: given a, b, let $i_1, \dots, i_n \in I$ be those @ which a or b non-zero.

Then $F(a+b) :=$

$f_{i_1}(a_{i_1} + b_{i_1}) + \dots + f_{i_n}(a_{i_n} + b_{i_n})$

$= f_{i_1}a_{i_1} + f_{i_1}b_{i_1} + \dots + f_{i_n}a_{i_n} + f_{i_n}b_{i_n}$

$= \underbrace{f_{i_1}a_{i_1} + \dots + f_{i_n}a_{i_n}} + \underbrace{f_{i_1}b_{i_1} + \dots f_{i_n}b_{i_n}}$

$= Fa + Fb.$ using commutativity. □

- In particular, $A \oplus B = A \times B$ is both product & coproduct.

- Therefore R^n is both n-fold product & n-fold coproduct in R-Mod.

· This corresponds to the fact that **homomorphisms** $R^m \xrightarrow{f} R^n$ correspond to **matrices**: we have a bijection between homomorphisms $R^m \xrightarrow{f} R^n$

$R \xrightarrow{f_i} R^n$ $i \in \{1, \dots, m\}$ as R^m is **m-fold coprod**

$R \xrightarrow{f_{ij}} R$ $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ as R^n is n-fold product

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Elts $(A_{ij})_{i \in \{1,\dots,m\}, j \in \{1,\dots,n\}}$, since a homomorphism $R \longrightarrow R$ is uniquely determined by value at 1 (R **is free R-mod on 1!**)

i.e. **(m×n)-matrices**

Under this correspondence, **composition** of **homomorphisms** corresponds to **matrix multiplication**!

Free modules

- The forgetful functor $U: Mod_R \longrightarrow Set$ has a left adjoint (as for any category of the form (Ω, E)-Alg), which assigns to a set X the free R-module FX.
- FX has a simple description: elements of FX are "formal linear combinations"
$$r_1 x_1 + \dots + r_n x_n \text{ where } r_i \in R, x_i \in X$$
with obvious R-module structure.
- The unit $X \xrightarrow{\eta_X} UFX : x \longmapsto x$ & given $X \xrightarrow{f} UY$ there exists a unique R-mod. map $\bar{f}: FX \longrightarrow Y$

such that
$$\begin{array}{ccc} UFX & & \\ \eta_X \uparrow & \searrow U\bar{f} & \\ X & \xrightarrow[f]{} & UY \end{array}.$$

In other words, $\bar{f}(x) = f(x)$.

Indeed, in order for $\bar{f}$ to be R-mod map we must define $\bar{f}(r_1 x_1 + \dots + r_n x_n)$
$$= r_1 f(x_1) + \dots + r_n (x_n)$$
and it is easy to see that $\bar{f}$ is R-module homomorphism.

Remark

Formal linear combination $r_1 x_1 + \dots + r_n x_n$ just means

$$((r_1, x_1), \dots, (r_n, x_n)).$$

Exercise

Check that $FX \cong \bigoplus_{X} R$

so $F1 \cong R$.