L12 - Tensor products of R-modules

- Let R be a commutative ring, M, N, L R-modules.
- A bilinear map is a function
$$F : M \times N \longrightarrow L$$
  - such that for $m \in M$, $F(m,-) : N \longrightarrow L$ is linear,

    i.e. an R-module map
  - & for $n \in N$, $F(-,n) : M \longrightarrow L$ is linear.

(In other words, F is "linear in each variable".

- Explicitly,

$$\left\{\begin{array}{l} - F(m, a+b) = F(m,a) + F(m,b) \\ - F(m, ra) = r.F(m,a) \\ - F(a+b, n) = F(a,n) + F(b,n) \\ - F(ra, n) = r.F(a,n). \end{array}\right\}$$

Def^n) Let $M, N \in \mathrm{Mod}_R$. The tensor product $M \otimes N \in \mathrm{Mod}_R$ comes with a universal bilinear map

$$\theta : M \times N \longrightarrow M \otimes N :$$

that is, θ is bilinear & given $F : M \times N \longrightarrow L$ bilinear $\exists!$ linear map $\bar{F} : M \otimes N \longrightarrow L$ such that the triangle

$$\begin{tikzcd} M \times N \arrow[d, "\theta"'] \arrow[dr, "F"] & \\ M \otimes N \arrow[r, "\bar{F}"'] & L \end{tikzcd}$$

commutes.

Remarks

- Let $\mathrm{Bil}(M, N; L)$ denote the set of bilinear maps $M \times N \longrightarrow L$.
- The definition says that we have a bijection

$$\mathrm{Bil}(M, N; L) \cong \mathrm{Mod}_R(M \otimes N, L)$$

natural in L.

- We have not yet proved the existence of $M \otimes N$ (which we will do). It's main properties follow from its defining universal property.
- Let's prove them now.

Theorem

Let R be a comm. ring.

1. The tensor product $M \otimes N$ is unique up to isomorphism.
2. The tensor product extends to a functor $\mathrm{Mod}_R \times \mathrm{Mod}_R \xrightarrow{\otimes} \mathrm{Mod}_R$
3. We have (natural) isomorphisms of R-modules
$$M \otimes N \cong N \otimes M, \quad R \otimes M \cong M, \quad M \otimes R \cong M.$$

Remark: Also have isomorphisms $(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$ which I will discuss at end - a bit harder.

Proof

① Suppose $\theta : M \times N \longrightarrow M \otimes_1 N$ & $\varphi : M \times N \longrightarrow M \otimes_2 N$ are universal bilinear maps.

- By their univ. properties $\exists$ unique linear maps $M \otimes_1 N \xrightarrow{\bar{\varphi}} M \otimes_2 N$ & $M \otimes_2 N \xrightarrow{\bar{\theta}} M \otimes_1 N$ such that

$$\begin{tikzcd} M \otimes_1 N \arrow[r, "\bar{\varphi}"] & M \otimes_2 N \\ M \times N \arrow[u, "\theta"] \arrow[ur, "\varphi"'] \end{tikzcd} \quad \& \quad \begin{tikzcd} M \otimes_2 N \arrow[r, "\bar{\theta}"] & M \otimes_1 N \\ M \times N \arrow[u, "\varphi"] \arrow[ur, "\theta"'] \end{tikzcd}$$

Using the univ. props again, we see that $\bar{\varphi}$ & $\bar{\theta}$ are inverses.

(2) Firstly, by $Mod_R \times Mod_R$ I mean the cartesian product of categories: objects - pairs (A, B) of R-mods, arrows - $(f, g) : (A, B) \longrightarrow (C, D)$ are pairs $f : A \to C$ & $g : B \to D$.

- Given (f, g), as above, we obtain fn

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{\theta_{C,D}} C \otimes D$$

& this is bilinear : i.e.

- $\theta_{C,D} \circ (f \times g)\langle a, - \rangle = \theta_{C,D}(fa, -) \circ g$ is a comp of two linear maps & so linear

- Sim. linearity in second variable.

- Therefore by the u.p. of $A \otimes B$ we obtain a unique $A \otimes B \xrightarrow{f \otimes g} C \otimes D$ such that

$$\begin{tikzcd} A \times B \arrow[r, "f \times g"] \arrow[d, "\theta_{A,B}"'] & C \times D \arrow[d, "\theta_{C,D}"] \\ A \otimes B \arrow[r, "f \otimes g"] & C \otimes D \end{tikzcd}$$

commutes.

- Uniqueness implies functoriality.

- Next, we construct iso
$$\bar{s}_{A,B} : A \otimes B \longrightarrow B \otimes A$$

For this, consider bilin. map
$$A \times B \xrightarrow{s_{A,B}} B \times A \xrightarrow{\otimes_{B,A}} B \otimes A$$
where $s_{A,B}(a,b) = (b,a)$.

This induces a unique lin map
$A \otimes B \xrightarrow{\bar{s}_{A,B}} B \otimes A$ such that

$$\begin{tikzcd}
A \times B \arrow[r, "s_{A,B}"] \arrow[d, "\otimes_{A,B}"'] & B \times A \arrow[d, "\otimes_{B,A}"] \\
A \otimes B \arrow[r, "\bar{s}_{A,B}"'] & B \otimes A
\end{tikzcd}$$

commutes.

By the universal property (ie. uniqueness) we see $\bar{s}_{B,A} \circ \bar{s}_{A,B} = id_{A \otimes B}$ so that $\bar{s}_{A,B}$ is iso.

— Next, let's show $R \otimes A \cong A$.

Consider the function

$$\begin{array}{ccc} R \times A & \xrightarrow{k} & A \\ (r,a) & \longmapsto & ra \end{array}$$

which is clearly bilinear (as A is R-module)

- Consider $R \times A \xrightarrow{F} B$ bilinear.

- Then each $F(-,a) : R \to B$ is linear so $F(r,a) = F(r.1,a) = r.F(1,a)$ & we obtain a unique factorisation

$$\begin{tikzcd} R \times A \arrow[rr, "F"] \arrow[dr, "k"'] & & B \\ & A \arrow[ur, "F(1,-)"'] & \end{tikzcd}$$

as

$$\begin{aligned} & F(1,-)\,k(r,a) \\ &= F(1,-)\,r.a \\ &= F(1,r.a) \\ &= r.F(1,a) \\ &= F(r,a). \end{aligned}$$

so that

$$R \times A \xrightarrow{k} A$$

is universal bilinear;

so $\exists!$ iso $R \otimes A \xrightarrow{\ell} A$ such that

$$\begin{tikzcd} R \times A \arrow[r, "\theta"] \arrow[dr, "k"'] & R \otimes A \arrow[d, "\ell", "\wr"'] \\ & A \end{tikzcd}$$

commutes.

- Finally, we have composite iso $A \otimes R \cong R \otimes A \cong A$. $\square$

Construction of the tensor product

- Write $\mathrm{Mod}_R \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Set}$.

- Given R-modules M, N consider the free R-module $F(UM \times UN)$, so have a function

$$\begin{array}{ccc} UM \times UN & \xrightarrow{\eta} & UF(UM \times UN) \\ (a,b) & \longmapsto & (a,b) \end{array}.$$

- This is not bilinear as we would require **precisely** the equations $E =$

$$\left\{ \begin{array}{rcl} (a, b+b') & = & (a,b) + (a,b') \\ (a, rb) & = & r.(a,b) \\ (ra, b) & = & r.(a,b) \\ (a+a', b) & = & (a,b) + (a',b) \end{array} \right\}$$

$$\{ \forall r \in R, a, a' \in M, b, b' \in N \}$$

& these do not hold in $F(UM \times UN)$.

- **Therefore we consider the congruence $\bar{E}$ on $F(UM \times UN)$ generated by E, & form the quotient $F(UM \times UN)/\bar{E}$ R-module; then the composite function**

$$UM \times UN \xrightarrow{\eta} UF(UM \times UN) \xrightarrow{p} U(F(UM \times UN)/\bar{E}) \text{ is bilinear}$$

since it forces the equations in E to hold.

Theorem

$$\begin{tikzcd} UM \times UN \arrow[r, "\eta"] \arrow[rr, bend right, "\theta"'] & UF(UM \times UN) \arrow[r, "p"] & U(F(UM \times UN)/\bar{E}) \end{tikzcd}$$

is a universal bilinear map.

~~Proof~~

- Consider a function $UM \times UN \xrightarrow{H} UA$ with A an R-mod. Then $\exists !$ lin map $F(UM \times UN) \xrightarrow{\bar{H}} A$ such that $U\bar{H} \circ \eta = H$.

i.e. $\bar{H}(a,b) = H(a,b)$ on generators.

- Then H is bilinear $\iff$ $\bar{H}$ forces the equations in E to hold $\iff$ $\bar{H}$ factors through quotient map p.

- In particular, if H is bilinear, we obtain a unique R-module map $F(UM \times UN)/\bar{E} \xrightarrow{\bar{\bar{H}}} A$ such that

$$\begin{tikzcd} UM \times UN \arrow[rr, "\theta"] \arrow[dr, "H"'] & & U(F(UM \times UN)/\bar{E}) \arrow[dl, "U\bar{\bar{H}}"] \\ & UA & \end{tikzcd}$$

commutes

so that θ is universal bilinear $\square$

- Explictly $M \otimes N := F(UM \times UN)/E$ has generators $a \otimes b := [(a,b)]$ for $a \in M$, $b \in N$ subject to the relations

$$a \otimes (b+c) = a \otimes b + a \otimes c$$
$$a \otimes (rb) = r(a \otimes b)$$
$$(a+b) \otimes c = a \otimes c + b \otimes c$$
$$ra \otimes c = r(a \otimes c).$$

- This is all we can say in general.

– Not every element of $M \otimes N$ is of form $a \otimes b$ — have sums

$$a_1 \otimes b_1 + a_2 \otimes b_2 \ldots$$

– In any case, the tensor product exists!

Remark

- Can also construct $M \otimes N$ as quotient of $F(UM \times UN)$ by the submodule gen by the subset

$$\left\{ \begin{array}{l} (a, b+b') - ((a,b) + (a,b')) \\ (a, rb) - r.(a,b) \\ (ra, b) - r.(a,b) \\ (a+a', b) - ((a,b) + (a',b)) \end{array} \;\middle|\; \forall r \in R, a, a' \in M, b, b' \in N \right\}$$

- Equivalent construction.

The internal hom in Mod_R

Proposition

Let $M, N \in \mathrm{Mod}_R$. Then the set $\mathrm{Mod}_R(M,N)$ of linear maps is an R-module when we define $(f+g)(a) = f(a)+g(a)$ &

$$(r.f)(a) = r.f(a).$$

Proof

Let's check $f+g \in \mathrm{Mod}_R$:

- $(f+g)(a+b) = f(a+b) + g(a+b)$

$$= f(a)+f(b)+g(a)+g(b)$$

(as N abelian) $= f(a)+g(a)+f(b)+g(b)$

$$= (f+g)(a) + (f+g)(b).$$

- $(f+g)(ra) = f(ra) + g(ra)$

$$= rf(a) + rg(a)$$

$$= r((f+g)(a)).$$

So $f+g \in \mathrm{Mod}_R$.

- Sim $r.f \in \mathrm{Mod}_R$ so we have an R-module str. on $\mathrm{Mod}_R(M,N)$. □

- For fixed $M \in Mod_R$, we can define a functor

$$\begin{aligned} Mod_R(M,-) : Mod_R &\longrightarrow Mod_R \\ N &\longmapsto Mod_R(M,N) \end{aligned}$$

&

$$N \xrightarrow{f} L \longmapsto Mod_R(M,N) \to Mod_R(M,L)$$
$$M \xrightarrow{g} N \longmapsto M \xrightarrow{fg} L$$

& it is easy to see this is a functor (follows from category axioms for Mod_R)

- We know $\otimes : Mod_R \times Mod_R \longrightarrow Mod_R$ is a functor & we obtain functors

- $M \otimes - : Mod_R \to Mod_R : N \mapsto M \otimes N$
$$N \xrightarrow{f} L \mapsto M \otimes N \xrightarrow{1 \otimes f} M \otimes L$$

& similarly

- $- \otimes M : Mod_R \to Mod_R$ sending
$$N \longmapsto N \otimes M$$
$$N \xrightarrow{f} L \longmapsto N \otimes M \xrightarrow{f \otimes 1} L \otimes M.$$

Proposition

We have adjunctions

$$- \otimes B \dashv \mathrm{Mod}_R(B,-) \quad \&$$

$$B \otimes - \dashv \mathrm{Mod}_R(B,-).$$

Proof

- For $- \otimes B \dashv \mathrm{Mod}_R(B,-)$ to be an adjunction we need a natural bijection

$$\mathrm{Mod}_R(A \otimes B, C) \cong \mathrm{Mod}_R(A, \mathrm{Mod}_R(B,C))$$

This is a composite bijection

$\mathrm{Mod}_R(A \otimes B, C) \cong$ (already described)

$\mathrm{Bilin}(A, B; C) \cong$

$\mathrm{Mod}_R(A, \mathrm{Mod}_R(B,C))$.

- Indeed given a bilin map $A \times B \xrightarrow{F} C$ the lin. map $\bar{F} : A \longrightarrow \mathrm{Mod}_R(B,C)$

$$a \longmapsto F_a : b \longmapsto F(a,b)$$

- Linearity of $F(a,-)$ corresponds to each F_a being linear.
- Linearity of $F(-,b)$ $\forall b$ corresp to $\bar{F}$ being linear.

Clear that this describes a bij^{n}.

Finally, for $B\otimes - \dashv \mathrm{Mod}_R(B,-)$ we use

$$\mathrm{Mod}_R(B\otimes A, C) \underset{\substack{\text{symmetry}\\ \text{iso}}}{\cong} \mathrm{Mod}_R(A\otimes B, C) \cong \mathrm{Mod}_R(A, \mathrm{Mod}_R(B,C))$$

Corollary

Both $A\otimes -$, $-\otimes A : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ preserve colimits (in particular direct sums).

$\boxed{\text{Proof}}$ By prev. prop., they are left adjoints & left adjoints preserve colimits.

The above says

$$A \otimes \left(\bigoplus_{i \in I} B_i \right) \cong \bigoplus_{i \in I} (A \otimes B_i)$$

&

$$\left(\bigoplus_{i \in I} B_i \right) \otimes A \cong \bigoplus_{i \in I} (B_i \otimes A).$$

Corollary

$$R^m \otimes R^n \cong R^{mn}$$

Proof

$$R^m \otimes R^n \cong R^m \otimes (\overbrace{R \oplus \cdots \oplus R}^{n \text{ times}})$$

$$\cong \underbrace{(R^m \otimes R) \oplus \cdots \oplus (R^m \otimes R)}_{n \text{ Times}}$$ by the above

$$\cong \underbrace{R^m \oplus \cdots \oplus R^m}_{n \text{ Times}}$$ as $A \otimes R \cong A$

$$\cong R^{mn}$$ as $R^m \cong \underbrace{R \oplus \cdots \oplus R}_{m \text{ Times}}$.

$\square$

Associativity of the tensor product

- The associativity iso $(M\otimes N)\otimes L \cong M\otimes(N\otimes L)$ is a little more subtle.
- The nicest proof (in my opinion) is as follows (I will only outline it).
- Let $\mathrm{Trilin}(M,N,L;A)$ be the set of trilinear functions

$$M\times N\times L \xrightarrow{F} A.$$

- We will show there are natural bijections

① $\mathrm{Trilin}(M,N,L;A) \cong \mathrm{Mod}_R((M\otimes N)\otimes L, A)$

& ② $\mathrm{Trilin}(M,N,L;A) \cong \mathrm{Mod}_R(M\otimes(N\otimes L), A)$

from which we obtain

$$\mathrm{Mod}_R((M\otimes N)\otimes L, A) \cong \mathrm{Mod}_R(M\otimes(N\otimes L), A)$$

naturally in A.

This implies $(M\otimes N)\otimes L \cong M\otimes(N\otimes L)$.

Actually I will just show ① since ② is similar.

We have

$\mathrm{Trilin}(M, N, L; A) \cong$ (exercise!)

$\mathrm{Bilin}(M, N; \mathrm{Mod}_R(L, A)) \cong$ (u.p. of $\otimes$)

$\mathrm{Mod}_R(M \otimes N, \mathrm{Mod}_R(L, A)) \cong$ $(- \otimes L \dashv \mathrm{Mod}_R(L, -))$.

$\mathrm{Mod}_R((M \otimes N) \otimes L, A)$

Remark

- A cat $\mathcal{C}$ with a "Tensor prod" functor
  $$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C} \text{ \& unit } I \in \mathcal{C}$$
  + natural iso
  $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, $I \otimes A \cong A$, $A \cong A \otimes I$
  & $A \otimes B \cong B \otimes A$ sat some axioms
  is called a symmetric monoidal cat.
- It is closed if $- \otimes A$, $A \otimes -$ have right adjoints $[A, -]$.
- We have (mostly) shown that $(\mathrm{Mod}_R, \otimes, R)$ is a closed symm. mon. cat.