Exercises 7

- Let A be an Ω-algebra. A congruence on A is an equivalence relation E on A such that:

  (*) if $s \in \Omega_n$ & $x_1 E y_1, \ldots, x_n E y_n$ then
  $$s(x_1, \ldots, x_n) \, E \, s(y_1, \ldots, y_n).$$

(1) Explain what the condition (*) means in elementary terms if
- $\Omega = \{e, \cdot\}$ is signature for monoids
- $\Omega = \{e, \cdot, (-)^{-1}\}$ is sig. for groups.

(2) For a group G, show that if
- E is a congruence on G, then the set $N_E = \{x : x E e\}$ is a normal subgroup of G. Show that this describes a bijection

$$\begin{array}{ccc} \mathrm{Cong}(G) & \longrightarrow & \mathrm{NormalSubgroups}(G). \\ E & \longmapsto & N_E \end{array}$$

b) Show, moreover, that
$$G/E = G/N$$
quotient by cong. quotient by normal subgroup

(3) For a ring R, show that congruences on R $\sim$ ideals on R

(4) Let E be a congruence on A. Show that $E = \mathrm{Ker}(A \overset{P}{\to} A/E)$ where A/E is the quotient of A by E.