lecture 9 - End of universal algebra

This time we will:

- show that (Ω, E)-Alg has all colimits (as well as limits which we already established)
- prove Birkhoff's variety theorem
- show that algebraic functors have left adjoints, if time is left.

• Firstly, we know that

$(\Omega, E)\text{-Alg} \longrightarrow \Omega\text{-Alg}$ has a left adjoint (it is a reflective subcat)

& that Ω-Alg is cocomplete (has colimits)

So it suffices to prove:

Proposition

Let $i : \mathcal{A} \hookrightarrow \mathcal{B}$ be fully faithful & have a left adjoint.
If $\mathcal{B}$ is cocomplete, so is $\mathcal{A}$.

Proof

- Consider $D : J \to \mathcal{A}$ & form $\mathrm{col}(iD) \in \mathcal{B}$ where $iD : J \to \mathcal{A} \to \mathcal{B}$.

- There is a bijection (nat. in X) between

① cocones $(D_j \to X)_{j \in J}$ (as i is fully faithful)

② cocones $(iD_j \to iX)_{j \in J}$ (by univ. prop. of $\mathrm{col}(iD)$)

③ morphisms $\mathrm{col}(iD) \to iX$

④ morphisms $R\,\mathrm{col}(iD) \to X$ (as $R \dashv i$)

Note: starting at ④ with $R\mathrm{col}(iD) \xrightarrow{1} R\mathrm{col}(iD)$) gives in ① cocone $(D_j \to R\mathrm{col}(iD))_{j \in J}$ through which each other cocone factors uniquely (by nat. in X); this shows it is the universal cocone. $\square$

- Birkhoff's theorem & related topics

Question) Which full subcategories $\mathcal{C} \hookrightarrow \Omega\text{-}Alg$ are of the form $(\Omega, E)\text{-}Alg \hookrightarrow \Omega\text{-}Alg$ for a set of equations E?

Answer) Well, full subcats $(\Omega, E)\text{-}Alg$ are closed under homomorphic images, products and subalgebras.

- in fact, we will see that $\mathcal{C} = (\Omega, E)\text{-}Alg$ $\Longleftrightarrow$ it has these closure properties.

- This is Birkhoff's theorem.

Recall an equation is a pair $s, t \in T_\Omega X$ for some X.

In fact, it suffices to consider equations in a fixed countable set $\omega = \{x_1, x_2, \ldots\}$ of variables.

$\boxed{\text{lemma}}$ let $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$. TFAE:

① $\mathcal{C} = (\Omega, E)\text{-Alg}$ for E a set of equations in vars ω.
② $\mathcal{C} = (\Omega, E)\text{-Alg}$ for E a set of equations.
③ $\mathcal{C} = (\Omega, E)\text{-Alg}$ for E a class of equations.

$\boxed{\text{Proof}}$ It suffices to prove $(3 \Rightarrow 1)$ & for this to show:

⊛ given $s, t \in T_\Omega X$ $\exists\ \bar{s}, \bar{t} \in T_\Omega \omega$ s.t.

$$A \models s = t \iff A \models \bar{s} = \bar{t}.$$

Indeed, then $(\Omega, E)\text{-Alg} = (\Omega, \bar{E})\text{-Alg}$ for

$$\bar{E} = \{(\bar{s}, \bar{t}) \in T_\Omega \omega^2 : (s, t) \in E\}$$

· Note that $\bar{E}$ is necessarily a set since $T_\Omega \omega$ is a set.

• Let $f : X \to Y \in \text{Set}$ be injective, & write $f^* = F_\Omega f : F_\Omega X \longrightarrow F_\Omega Y$ for the induced homomorphism.

- Firstly, we will prove:

★ given $s, t \in T_\Omega X$, $A \models s = t \iff$

Suppose $A \models s = t$. Must show $A \models f^*(s) = f^*(t)$

So let $F_\Omega Y \xrightarrow{p} A$.

Then $F_\Omega X \xrightarrow{f^*} F_\Omega Y \xrightarrow{p} A \in \Omega\text{-Alg} \Rightarrow p f^* s = p f^* t$

so $A \models f^* s = f^* t$.

Conversely, suppose $A \models f^* s = f^* t$.

- Must show $A \models s = t$.

- If A is empty, this is trivial.
- So assume $\exists a \in A$.
- Let $v : X \longrightarrow A$.

Then can find extension

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "v"'] & {} \arrow[d, phantom, "="] & Y \arrow[dl, "\exists w"] \\ & A & \end{tikzcd}$$

where $w f x = v x$ & $w(y) = a$ for $y \notin \mathrm{im}(f)$.

Then

$$\begin{tikzcd} F_\Omega X \arrow[rr, "f^*"] \arrow[dr, "\bar{v}"'] & {} \arrow[d, phantom, "="] & F_\Omega Y \arrow[dl, "\bar{w}"] \\ & A & \end{tikzcd}$$

commutes so

$\bar{v} s = \bar{w} f^* s = \bar{w} f^* t = \bar{v} t$. This proves ☆.

- Now let $s, t \in T_\Omega X$. We will define a finite subset $U \subseteq X$ s.t. $s, t \in T_\Omega U \subseteq T_\Omega X$.
- Given this, take any injective $i : U \longrightarrow \Omega$

we have

$$\begin{tikzcd} & U \arrow[dl, phantom, "\subseteq" sloped] \arrow[dr, hook, "i"] & \\ X & & \omega \end{tikzcd}$$

& by ☆

$A \models s = t$ in vars $X \iff$

$A \models s = t$ in vars $U \iff$

$\iff A \models i^* s = i^* t$ in vars ω,

completing claim.

- What is U? Well given $s \in T_\Sigma X$ we can ind. define its (finite) set of variables:

  $\mathrm{var}(x) = \{x\}$

  $\mathrm{var}(f(t_1, \dots, t_n)) = \mathrm{var}\, t_1 \cup \dots \cup \mathrm{var}\, t_n \subseteq X$.

- Then $s \in T_\Sigma(\mathrm{var}\, s)$.
- So $s, t \in T_\Sigma(\mathrm{var}\, s \cup \mathrm{var}\, t)$ so take

$$U = \mathrm{var}\, s \cup \mathrm{var}\, t. \qquad \square$$

- As a result, I will allow a class of equations.

- Given a full subcategory $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$ we define $\underline{E(\mathcal{C})} = \{ (s,t) : \text{if } A \in \mathcal{C} \text{ then } A \models s = t \}$, the <u>class of equations satisfied by objects of $\mathcal{C}$.</u>

- Then

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & E(\mathcal{C})\text{-Alg} \arrow[dl] \\ & \Omega\text{-Alg} & \end{tikzcd}$$

since if $A \in \mathcal{C}$, then A satisfies any equation satisfied by all members of $\mathcal{C}$.

- Write:
- $H\mathcal{C}$ = closure of $\mathcal{C}$ in Ω-Alg under homomorphic images;
- $S\mathcal{C}$ = - - - - - - - - - - subobjects;
- $P\mathcal{C}$ = - - - - - - - - - - products.

~~Theorem~~ For $\mathscr{C} \hookrightarrow \Omega\text{-Alg}$ as above,
$HSP(\mathscr{C}) = E(\mathscr{C})\text{-Alg}$.

~~Proof~~ • Since $\mathscr{C} \subseteq E(\mathscr{C})\text{-Alg}$ & $E(\mathscr{C})\text{-Alg}$ closed under H, S, P (last week), $HSP(\mathscr{C}) \subseteq E(\mathscr{C})\text{-Alg}$.

- Must prove each $E(\mathscr{C})$-algebra belongs to $HSP(\mathscr{C})$.
- From last week, free $E(\mathscr{C})$-algebras FX exist on any set, with unit $X \xrightarrow{\eta_X} UFX$.
- For each $E(\mathscr{C})$-alg A, have $FUA \xrightarrow{\varepsilon_A} A \in E(\mathscr{C})\text{-Alg}$, unique such that

$$\begin{tikzcd} UFUA \arrow[dr, "U\varepsilon_A"] & \\ UA \arrow[u, "\eta_{UA}"] \arrow[r, "1"'] & UA \end{tikzcd}$$

commutes.

- Hence ε_A is surjective $\Rightarrow$ A a homomorphic image of free $E(\mathscr{C})$-algebra.
- So suffices to prove each free $E(\mathscr{C})$-alg belongs to $SP(\mathscr{C})$.

To this end, let $F_\Omega X$ = free Ω-alg on X &

$N = \{(s,t) \in (F_\Omega X)^2 : \mathcal{C} \not\models s = t\}$.

- Then given $(s,t) \in N$,

$\exists \quad F_\Omega X \xrightarrow{k_{s,t}} A_{s,t} \in \mathcal{C}$ such that

$k_{s,t}(s) \neq k_{s,t}(t)$.

induced map to prod.

- Consider

$$\begin{tikzcd} F_\Omega X \arrow[rr, "k"] \arrow[dr, "\eta_X"'] & & \prod_{(s,t)\in N} A_{s,t} \\ & RX \arrow[ur, hook, "i"'] & \end{tikzcd}$$

& factor through image $\mathrm{im}(k) = RX$ in Ω-Alg.

Then $RX \in SP(\mathcal{C}) \in E(\mathcal{C})$-Alg.

It remains to prove that

RX is free $E(\mathcal{C})$-algebra on X:

For this, it is enough to show that if A is an $E(\mathcal{C})$-alg, then each $F_\Omega X \xrightarrow{f} A \in \Omega$-Alg factors uniquely through η_X:

$$\begin{tikzcd}
F_\Omega X \arrow[rr, "f"] \arrow[dr, "\eta_X"'] & & A \\
& RX \arrow[ur, dashed, "\exists ! \bar{f}"'] &
\end{tikzcd}$$

since then

$$E(\mathcal{C})\text{-Alg}(RX, A) \cong \Omega\text{-Alg}(F_\Omega X, A) \cong \mathbf{Set}(X, U_\Omega A).$$

- Consider again the diagram

$$\begin{tikzcd}
F_\Omega X \arrow[rr, "k"] \arrow[dr, "\eta_X"'] & & \prod_{(s,t)\in N} A_{s,t} \\
& RX \arrow[ur, "i"'] &
\end{tikzcd}$$

- Let $(s,t) \in F_\Omega(X)^2$.

Now if $(s,t) \in N$, then $ks \neq kt$ since they are unequal in (s,t)-component $k_{s,t}$.

On other hand, if $(s,t) \in E(\mathcal{C})$, then $ks = kt$ as $\prod A_{s,t} \in HSP(\mathcal{C}) \subseteq E(\mathcal{C})$-Alg.

Therefore $ks = kt \iff (s,t) \in E(\mathcal{C})$.

• Since $k = i \circ n_X$ & i inj therefore
$n_X s = n_X t \iff (s,t) \in E(\mathcal{C})$,
where $\mathrm{Fr}X \xrightarrow{n_X} RX$.

Therefore $\mathrm{Ker}(n_X) = \{(s,t) \in \mathrm{Fr}X : (s,t) \in E(\mathcal{C})\}$
Since n_X is surjective, by the first isomorphism theorem, it is the quotient of its kernel:
therefore given $\mathrm{Fr}X \xrightarrow{f} A$ s.t.
$f(s) = f(t)$ for each $(s,t) \in E(\mathcal{C})$ $\exists!$
$\bar{f} : RX \longrightarrow A$ making

$$\begin{tikzcd}
\mathrm{Fr}X \arrow[rr, "f"] \arrow[dr, "n_X"'] & & A \\
& RX \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

commute.

In particular, if A is an $E(\mathcal{C})$-algebra then f has this prop. so we obtain the unique factorisation. $\square$

Birkhoff's Theorem

$\mathcal{C} \hookrightarrow \Omega\text{-Alg}$ is of the form $(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ $\Longleftrightarrow$ $\mathcal{C}$ closed under hom. images, products & subalgebras.

Proof

By last week $(\Omega, E)\text{-Alg}$ has these closure ~~props~~ props. Conversely,
$\mathcal{C} = \underline{HSP(\mathcal{C})} = \underline{(\Omega, E(C))\text{-Alg}}$ using previous theorem. $\square$

Algebraic Functors have left adjoints

By an <u>algebraic functor</u> $W : (\Omega', E')\text{-Alg} \to (\Omega, E)\text{-Alg}$ I mean one which commutes with the forgetful functors to Set

$$\begin{tikzcd}
(\Omega', E')\text{-Alg} \arrow[rr, "W"] \arrow[dr, "U"'] & \arrow[d, phantom, "="] & (\Omega, E)\text{-Alg} \arrow[dl, "U'"] \\
& \mathrm{Set} &
\end{tikzcd}$$

Examples) $U : \mathrm{Rng} \longrightarrow \mathrm{Ab}$,
$U : \mathrm{Grp} \longrightarrow \mathrm{Mon}$, ...
any such (partially) forgetful functor,

Theorem
Each algebraic functor has a left adjoint.

Proof
- A <u>W-reflection</u> of $X \in (\Omega', E')\text{-Alg}$ is a map $\exists\ X \xrightarrow{\eta_X} WX'$ <u>universal</u>, in the sense that given $X \xrightarrow{f} WY$ $\exists!\ X' \xrightarrow{\bar{f}} Y$ st $W\bar{f} \circ \eta_X = f$.

(Equivalently, that we have a bijection
$(\Omega', E')\text{-Alg}(X', Y) \cong (\Omega, E)\text{-Alg}(X, WY)$ natural in Y.)

- Must show that each X has a W-reflection. Then W has left adj.

3 steps

1. Each free algebra has a W-reflection
2. If $D: J \to (\Omega, E)$-Alg & each Di has W-reflection, then colD has W-reflection.
3. Each algebra is a coequaliser of frees.

+ Follows that each alg has W-reflection $\Rightarrow$ W has a left adjoint.

(1) FX has reflection $F'X$:
indeed, the natural bijection
$(\Omega', E')\text{-Alg}(F'X, A) \cong$
$\mathrm{Set}(X, U'A) = \mathrm{Set}(X, UWA)$
$\cong (\Omega, E)\text{-Alg}(FX, WA)$ proves it.

② If D_i has reflection $D_i \xrightarrow{\eta_i} \omega(D_i')$, then @ $i \xrightarrow{\alpha} j$ $\exists!$ $D_i' \xrightarrow{D\alpha'} D_j'$ st

$$\begin{tikzcd}
D_i \arrow[r, "\eta_i"] \arrow[d, "D\alpha"'] \arrow[dr, phantom, "="] & \omega(D_i') \arrow[d, "\omega(D\alpha')"] \\
D_j \arrow[r, "\eta_j"] & \omega(D_j')
\end{tikzcd}$$

Then $D' : J \longrightarrow (\Omega', E')\text{-Alg}$
$i \longmapsto Dx'$,
$\alpha \longmapsto D\alpha'$ is a functor,

so can form colimit $\mathrm{col}D'$,
which we claim is the ω-reflection of $\mathrm{col}D$.

- Have bijection between

① Maps $\mathrm{col}D' \longrightarrow Y$ (def of colim)

② cocones $(D'i \xrightarrow{p_i} Y)_{i \in J}$ (ω-reflection θ_i)

③ Cocones $(Di \xrightarrow{\bar{p}_i} \omega Y)_{i \in J}$ (def of colim)

④ Maps $\mathrm{col}D \longrightarrow \omega Y$

natural in Y.

So $(\Omega', E')\text{-Alg}(\mathrm{colD}', \Psi) \cong (\Omega, E)\text{-Alg}(\mathrm{colD}, W\Psi)$ proving the claim.

③ $\exists!\ FUA \xrightarrow{\varepsilon_A} A \in (\Omega, E)\text{-Alg}$

st

$$\begin{tikzcd} & UFUA \arrow[dr, dotted, "U\varepsilon_A"] & \\ UA \arrow[ur, "\eta_{UA}"] \arrow[rr, "1"'] \arrow[rr, phantom, shift left=3, "="] & & UA \end{tikzcd}$$

- Therefore $\varepsilon_A : FUA \longrightarrow A$ is surjective as $\varepsilon_A(\eta_A(x)) = x$.

- So by first iso. theorem, it is the coequaliser

$$K = \ker(\varepsilon_A) \overset{d}{\underset{c}{\rightrightarrows}} FUA \xrightarrow{\varepsilon_A} A$$

of its kernel.

- Now consider

$$FUK \xrightarrow{\varepsilon_K} K \overset{d}{\underset{c}{\rightrightarrows}} FUA \xrightarrow{\varepsilon_A} A$$

Since ε_K is surj., it is epi, hence the coequaliser of $d\varepsilon_K$ & $c\varepsilon_K$ is the coequaliser of d & c, that is, A!