Lecture 13 - Projective & injective modules

- We already met projectives in universal algebra - recall that an (Ω,E)-alg A is **projective** if given $B \xrightarrow{f} C \in (\Omega,E)\text{-Alg}$ **surjective** & $A \xrightarrow{g} C$, $\exists$ $A \xrightarrow{\bar{g}} B$ such that

$$\begin{tikzcd} A \arrow[r, "\bar{g}"] \arrow[dr, "g"'] & B \arrow[d, "f"] \\ & C \end{tikzcd}$$

commutes.

- Projective modules are instance of above when $(\Omega,E)\text{-Alg} = \mathrm{Mod}_R$.

- We proved already that each free algebra is projective. For the converse, recall that in a cat $\mathcal{C}$, we say <u>A is a retract of B</u> if $\exists\, A \xrightarrow{i} B$, $B \xrightarrow{p} A$ such that

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "1"'] & B \arrow[d, "p"] \arrow[dl, phantom, "="] \\ & A \end{tikzcd}.$$

- In this case i is mono & p is epi.

Theorem

In (Ω, E)-Alg,
projectives $\equiv$ retracts of frees.

Proof

We know free $\Rightarrow$ projective. So for one dir^n, suff. to show retract of projs are proj.

Let $A \underset{f}{\overset{g}{\rightleftarrows}} B$ sat $gf = 1$ & **B projective**.

Consider $p : M \twoheadrightarrow N$ surj &

$$\begin{tikzcd} & A \arrow[d, "k"] \\ M \arrow[r, two heads, "p"'] & N \end{tikzcd}$$

Then as B proj. $\exists\ \bar{k}$ as in

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "\bar{k}"'] \arrow[dr, phantom, "="] & A \arrow[d, "k"] \\ M \arrow[r, "p"'] & N \end{tikzcd}$$

& take

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "1"] \arrow[dr, phantom, "=", shift right=3] \arrow[ddr, bend left=60, "k"] & \\ B \arrow[r, "g"] \arrow[d, "\bar{k}"'] \arrow[dr, phantom, "="] & A \arrow[d, "k"] \\ M \arrow[r, "p"'] & N \end{tikzcd}$$

as required.

Conversely, let M be projective. Then by the u.p. of $\mathbf{F}UM$, $\exists!\, \mathbf{F}UM \xrightarrow{\varepsilon_M} M$

st
$$\begin{tikzcd} UFUM \arrow[dr, "U\varepsilon_M"] & \\ UM \arrow[u, "\eta_{UM}"] \arrow[r] \arrow[ur, phantom, "="] & UM \end{tikzcd}$$

Hence ε_M is surj, so as M proj,

$$\exists \quad \begin{tikzcd} & M \arrow[dl, dashed, "i"'] \arrow[d, "1"] \\ FUM \arrow[r, two heads, "\varepsilon_M"'] \arrow[ur, phantom, "="] & M \end{tikzcd}. \quad \square$$

- In particular, an R-module is projective $\Longleftrightarrow$ it is a retract of a free module, but in fact we can say a bit more.

Here is a useful characterisation of retracts in the additive setting.

Fact) In Mod_R, A is a retract of B
$\iff$ A is a direct summand of B
(that is, $\exists C$ such that $A \oplus C \cong B$)

I will only prove one direction of this here due to time, but see Alg 4.

Lemma

Suppose $\exists C$ st $A \oplus C \cong B$. Then A is a retract of B.

~~Proof~~ Suffices to show A a retract of $A \oplus C$.

Certainly $A \xrightarrow{i_A} A \oplus C$, $A \oplus C \xrightarrow{p_A} A$
$a \longmapsto (a, 0)$ $\quad$ $(a, c) \longmapsto a$

sat $p_A i_A = 1$ so direct summands are retracts. $\square$

This lets us find examples.

Example

- Consider the ring $\mathbb{Z}_6$.
- $\mathbb{Z}_2$ is $\mathbb{Z}_6$-module $m.n = (mn) \bmod 2$
- $\mathbb{Z}_3$ is $\mathbb{Z}_6$-module $m.n = (mn) \bmod 3$
- $\mathbb{Z}_6$ is a free $\mathbb{Z}_6$-module.

$$\begin{aligned} \mathbb{Z}_6 &\longmapsto \mathbb{Z}_2 \oplus \mathbb{Z}_3 \\ n &\longmapsto (n \bmod 2, n \bmod 3) \end{aligned}$$

an iso of $\mathbb{Z}_6$-modules

Since these direct summands are retracts of $\mathbb{Z}_6$, we have

$\mathbb{Z}_2, \mathbb{Z}_3$ projective, but neither is free.

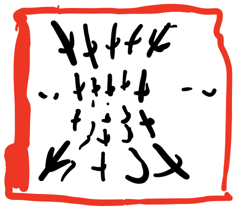
Example from geometry

- $\mathbf{Man}$ = cat of smooth manifolds & smooth functions.
- Vector bundle $p : E \to M \in \mathbf{Man}$ (each $p^{-1}x$ a vect. space..)
- Section s of p

$$\begin{tikzcd} & E \arrow[d, "p"] \\ M \arrow[ur, "s"] \arrow[r, "1"'] & M \end{tikzcd}$$

- E.g. $TM \xrightarrow{p} M$ - Tangent bundle of M, a section is a smooth vector field

e.g.



- $C^\infty(M) := \mathbf{Man}(M, \mathbb{R})$ a ring

w/ pointwise operations $f \cdot g\,(x) = f(x) \cdot g(x)$ etc.

- $\mathrm{Sect}(p)$ the set of sections of p a $C^\infty(M)$-module -

given $s, s' : M \to E$, $sx, s'x \in p^{-1}(x) \in \mathrm{Vect}$

$\Rightarrow$ $sx + s'x \in p^{-1}(x)$, $rs(x) \in p^{-1}(x)$

$\Rightarrow$ sections $s + s'$, $r.s \in \mathrm{Sect}(p)$.

Thm (Serre-Swan)

$\mathrm{Sect}(p)$ is a finitely gen. projective module &, if M is connected, then every fin. gen. projective module arises in this way.

Injective modules

- Dual to projective modules, we have injective ones:

Def^n) A module M is injective if for each mono $A \overset{i}{\hookrightarrow} B$ &
$A \xrightarrow{f} M$ $\exists$ $B \xrightarrow{\bar{f}} M$ such that

$$\begin{tikzcd} A \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & B \arrow[dl, "\bar{f}"] \\ & M & \end{tikzcd}.$$

- To understand, injective modules in practice the following is a useful tool:

Theorem (Baer criterion)

M is injective if and only if it has the extension prop. wrt submodules $I \hookrightarrow R$ (i.e. ideals of R)

Proof

One direction is trivial. Conversely, it suffices to show that M has extension prop wrt submodule inclusions

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[dl] \\ M & \end{tikzcd}$$

- Consider pairs (B_i, f_i) where $A \subseteq B_i \subseteq B$ a submod & $f_i : B_i \to M$ sat

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & B_i \arrow[d, "f_i"] \\ & M \end{tikzcd}$$

- Say $(B_i, f_i) \leq (B_j, f_j)$ if $B_i \subseteq B_j$ & $f_j|B_i = f_i$.
- This is a poset & we claim it has a max^l element, which we will obtain by Zorn's Lemma.

- So suppose $(B_i, f_i)_{i \in I}$ is a chain in this poset.
- Form $\bigcup B_i = \{ x \in B : \exists i \in I, x \in B_i \}$

This is clearly a submodule, & we can define $f^* : \bigcup_{i \in I} B_i \longrightarrow M$ by
$f^*(x) = f_i(x)$ where $x \in B_i$.

This is well .

Therefore the chain has an upper bound.

- Hence, by Zorn, it has a maximal elt.

(N, g) as pictured:
$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] & N \arrow[r, hook] \arrow[d, "g"] & B \\ & M & \end{tikzcd}$$

- Suppose $N \neq B$. We will show this leads to a contradiction, completing proof.

- Indeed, let $b \in B - N$, & consider the pullback

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "-\cdot b"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "-\cdot b"] \\ N \arrow[r, hook] & B \end{tikzcd}$$

which has $I = \{r \in R : rb \in N\}$

- Then have
$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "-\cdot b"'] & R \arrow[d, "-\cdot b"] \\ N \arrow[r, hook] & N+Rb \subseteq B \end{tikzcd}$$

submodule $N+Rb = \{n + rb : n \in N, r \in R\} \subseteq B$.

- If we can show $g : N \to M$ extends to $N+Rb$, we will contrad max. of N, & be done.

Consider
$$\begin{tikzcd}
I \arrow[r, hook] \arrow[d, "-\cdot b"'] & R \arrow[d, "-\cdot b"] \arrow[ddl, bend left=60, "\exists h \text{ by assumption}"] \\
N \arrow[r, hook] \arrow[d, "g"'] & N+Rb \\
M
\end{tikzcd}$$

- Define $N+Rb \xrightarrow{\ell} M$ by
$$\ell(n+rb) = gn + hr.$$
  - It clearly extends g.
  - Let's check well defined:

Suppose $n+rb = n'+r'b$
$\Rightarrow (r-r')b = n'-n \in N$ so
$(r-r') \in I$.

Then $\ell(n+rb) - \ell(n'+r'b) =$
$(gn+hr) - (gn'+hr') =$
$g(n-n') + h(r-r') =$
$g(n-n') + g((r-r')b) =$
$= 0.$ $\square$

Corollary

In Ab, A is injective $\Leftrightarrow$ it is divisible: that is, $\forall a \in A$, $n \neq 0 \in \mathbb{N}$ $\exists b \in A$ st $nb = a$.

Proof

- Here $R = \mathbb{Z}$ & the ideals of $\mathbb{Z}$ are all principal:

$$n\mathbb{Z} \hookrightarrow \mathbb{Z}.$$

- Consider

$$\begin{tikzcd} n\mathbb{Z} \arrow[r, hook] \arrow[d, "f"'] & \mathbb{Z} \arrow[dl, dashed, "g"] \\ A & \end{tikzcd}$$

- A map f is spec. by $a \in A$ st. $f(n) = a$.
- To give an extension is to give $g : \mathbb{Z} \longrightarrow A$ with $1 \longmapsto b$ with

$$a = g(n) = \underbrace{g(1) + \dots + g(1)}_{n \text{ times}} = n.b$$

Then A is inj $\iff$ all such extensions exist - that is, A is <u>divisible</u>.

Projective & injective resolutions

- In homological algebra, an important point is that each module M has a proj. & inj. resolution.

- The key here is:
  - given M we can find a proj M' & surj $M' \twoheadrightarrow M$.
  - : - - - - - - - - - - - - an inj. M' & mono $M \rightarrowtail M'$.

- The first we know - take $\underset{\text{proj}}{FUM} \xrightarrow{\varepsilon_M} \underset{\text{surj}}{M}$

- The second is more complex & I will outline it now - it is a special case of a construction called the small object argument.

- The small object argument

- Suppose we are given a class J of morphs $j: A \to B$ in a cat $\mathcal{C}$.
- We say an object is J-injective if given

$$\begin{tikzcd} A \arrow[r, "j \in J"] \arrow[dr, "f"'] & B \arrow[d, dashed, "\exists"] \\ & X \end{tikzcd}$$

Example

In Mod_R Take $J = \{\text{all monos}\}$

Then X is J-injective $\Leftrightarrow$ injective.

Or by Baer criterion, we can take $J = \{\text{all submodules of } R\}$ & J-injective $\Leftrightarrow$ injective.

- Goal: given X, construct $X \to X^{\#}$ with $X^{\#}$ J-injective.

- Assume J a set.

- How do we construct this $X^{\#}$?

- The idea is that we will construct it as a colimit of a chain:

$$\begin{tikzcd}
X = X_0 \arrow[r] \arrow[drrr, bend right] & X_1 \arrow[r] \arrow[drr, bend right] & \cdots \arrow[r] & X_n \arrow[r, "p_{n,n+1}"] \arrow[d, "p_n"'] & X_{n+1} \arrow[r] \arrow[dl, "p_{n+1}"] & \cdots \\
 & & & X^{\#} & &
\end{tikzcd}$$

where $X_0 = X$. & where X_{n+1} contains solutions to extension problems in X_n: in other words,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X_n \arrow[d, "p_{n,n+1}"] \\
B \arrow[r, "\exists"'] & X_{n+1}
\end{tikzcd}$$

given $A \xrightarrow{f} X_n$. let $S_n = \{\text{pairs } (j,F) \text{ as above}\}$

- How to construct such an object X_{n+1}? As a **pushout**

$$
\begin{tikzcd}
A \arrow[r, "i_{(j,f)}"] \arrow[d, "j"'] \arrow[rr, dashed, bend left, "f"] & \sum\limits_{(j,f)\in S_n} A \arrow[r, "\varepsilon_n"] \arrow[d, "\sum\limits_{S_n} f"'] & X_n \arrow[d, "p_{n,n+1}"] \\
B \arrow[r] & \sum\limits_{(j,f)\in S_n} B \arrow[r, "\varphi_n"] \arrow[ur, phantom, "\ulcorner" very near end] & X_{n+1}
\end{tikzcd}
$$

where $\varepsilon_n \circ i_{(j,f)} = f$.

- Now consider
$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & X^\# \\
B &
\end{tikzcd}
$$

- If f factors through earlier stage as
$$
\begin{tikzcd}
& X_n \arrow[d, "p_n"] \\
A \arrow[ur, dashed, "f_n"] \arrow[r, "f"'] & X^\#
\end{tikzcd}
$$

then
$$
\begin{tikzcd}
A \arrow[r, "f_n"] \arrow[d, "j"'] \arrow[rr, bend left, "f"] & X_n \arrow[d, "p_{n,n+1}"] \arrow[r, "p_n"] & X^\# \\
B \arrow[r, "\varphi_n(j,f_n)"] & X_{n+1} \arrow[ur, "p_{n+1}"'] &
\end{tikzcd}
$$

& solved problem!

- Indeed, if each $j : A \to B$ has A finitely presented, this will work, since if A has generators $\langle a_1, \dots, a_n \rangle$, then each $fa_1, \dots, fa_n \in X^\# = \bigcup X_i$ will belong to some X_i & we can take max of these m.

Then
$$\begin{tikzcd} A \arrow[r, dashed, "\exists"] \arrow[dr, "f"'] & X_m \arrow[d] \\ & X^\# \end{tikzcd}$$

- If the $j : A \to B$ don't have finitely pres. domain, we just have to take colimit of a longer chain.