Equalisers & coequalisers

Def) let $A \underset{g}{\overset{f}{\rightrightarrows}} B \in \mathcal{C}$.

The equaliser E of f & g comes equipped with an arrow $i: E \to A$ such that $E \xrightarrow{i} A \underset{g}{\overset{f}{\rightrightarrows}} B$ commutes (ie $fi = gi$).

- It has the universal property that given $X \xrightarrow{h} A \underset{g}{\overset{f}{\rightrightarrows}} B$ such that $fk = gk$ then $\exists! \, X \xrightarrow{\bar{h}} E$ such that

$$\begin{tikzcd} X \arrow[d, "\bar{h}"'] \arrow[dr, "h"] \arrow[dr, phantom, "="' near start] & \\ E \arrow[r, "i"'] & A \end{tikzcd}.$$

- As usual, equalisers are unique up to iso.

Ex) In Set, the equaliser is the subset

$$\begin{tikzcd} E = \{x \in A : fx = gx\} \arrow[r, hook, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ X \arrow[u, dotted, "\bar{h}"] \arrow[ur, "h"'] & & \end{tikzcd}$$

If $fhx = ghx$ then $hx \in E$, so $\bar{h}x = hx$ gives factorisation

Ex) In the other algebraic categories such as Grp, Rng etc, equalisers are constructed as in Set.

- However, in Grp a special case is quite illuminating.
- In Grp, for each pair of groups G, H we have

$$\begin{aligned} 0 : G &\longrightarrow H \\ x &\longmapsto 0_H \end{aligned}$$

sending all elements to zero (i.e. the unit element of H.)
- Then the equaliser of

$$\{x \in G : fx = 0\} \hookrightarrow G \underset{0}{\overset{f}{\rightrightarrows}} H$$

is the kernel of f.
- Thus kernels of group homomorphisms are special cases of equalisers.

Coequalisers in $\mathcal{C}$ are equalisers in $\mathcal{C}^{op}$:
in el. terms, the coequaliser of
$A \underset{g}{\overset{f}{\rightrightarrows}} B$ is an object C eq.
w/ a morph $B \xrightarrow{k} C$ st

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "k"] \arrow[dr, "h"'] & C \arrow[d, dashed, "\exists ! h"] \\ & & D \end{tikzcd}$$

commutes

- Coequalisers capture quotients, & quotients in algebraic type categories are hard to describe explicitly, but we will mention some cases.

Example

- In Set, given an equiv. rel. E on a set X, we can view it as a subset $E \subseteq X \times X$ of elts $\{(x,y) : xEy\}$ & then we have

$$E \underset{t}{\overset{s}{\rightrightarrows}} X : (x,y) \underset{t}{\overset{s}{\mapsto}} \begin{matrix} x \\ y \end{matrix}$$

The coequaliser

$$E \underset{t}{\overset{s}{\rightrightarrows}} X \xrightarrow{p} C$$

is a set C with prop that if xEy then $px = py$, and it is the <u>universal such</u>.

- Indeed, it is

$$\begin{aligned} X &\xrightarrow{p} X/E \quad \text{set of e-classes of } E \\ x &\longmapsto \underbrace{[x]_E}_{\text{equivalence class of } x} \end{aligned}$$

Exercise: check the details of this!

Ex) Given a normal subgroup $H \trianglelefteq G$

the coequaliser of $H \underset{0}{\overset{i}{\rightrightarrows}} G$ is

by def^n, the universal $G \longrightarrow ?$, sending all elements of H to 0.

Indeed, it is the quotient

$$\begin{aligned} G &\xrightarrow{p} G/H \\ g &\longmapsto gH \end{aligned}$$

by the normal subgroup:

given $G \xrightarrow{f} A$ such that $fx = 0 \ \forall x \in H$, $\exists!\, G/H \xrightarrow{\bar{f}} A$

such that

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "p"'] & \arrow[d, phantom, "="] & A \\ & G/H \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

ie. $\bar{f}(gH) = f(g)$.

- Coequalisers are also closely related to presentations

$$\langle x_1, \ldots, x_n \mid R_1, \ldots, R_n \rangle$$

of algebraic structures via generators & relations.

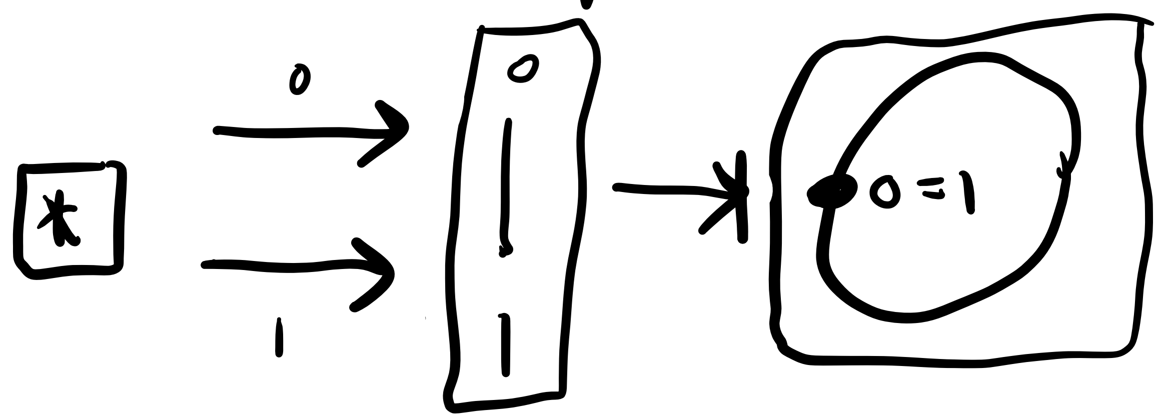
We will discuss this in the section on universal algebra.

Ex) In topology, coequalisers capture quotient spaces:

For instance, the circle is obtained by glueing together the two endpoints of the interval

$0 \text{ ——— } 1$;

it is the coequaliser



Limits & colimits in general

Let J be a small cat & $\mathcal{C}$ a cat. A functor $J \to \mathcal{C}$ is called a diagram of shape J in $\mathcal{C}$.

Example) $J = \boxed{0 \rightrightarrows 1}$

A diagram $J \longrightarrow \mathcal{C}$ is specif. by morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$

Limits & colimits in general

Def^n) Given a diagram $D : J \longrightarrow \mathcal{C}$ a <u>cone</u> over D is an object $A \in \mathcal{C}$ together with morphisms $A \xrightarrow{f_j} D_j$ for each $j \in J$ such that for all $\alpha : j \longrightarrow k \in J$ the triangle

$$\begin{tikzcd} & D_i \arrow[d, "D\alpha"] \\ A \arrow[ur, "f_i"] \arrow[r, "f_j"'] & D_j \end{tikzcd}$$

commutes.

- A <u>limit</u> of D is a cone $(L \xrightarrow{p_j} D_j ; j \in J)$ with the universal property that given any other cone $(A \xrightarrow{f_j} D_j, j \in J)$ there exists a unique $A \xrightarrow{k} L$ such that

$$\begin{tikzcd} A \arrow[d, "k"'] \arrow[dr, "f_j"] & \\ L \arrow[r, "p_j"'] & D_j \end{tikzcd}$$

commutes for all $j \in J$.

Examples 1

$J = \boxed{0 \overset{u}{\underset{v}{\rightrightarrows}} 1}$ A diagram $J \xrightarrow{D} \mathcal{C}$ is a parallel pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$

(ie. $D0 = A$, $D1 = B$, $Du = f$, $Dv = g$)

& a cone consists of maps $X \xrightarrow{p_0} A$, $X \xrightarrow{p_1} B$ such that

$$\begin{tikzcd} & A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ X \arrow[ur, "p_0"] \arrow[r, "p_1"'] & B \end{tikzcd}$$

commutes (ie. $fp_0 = p_1 = gp_0$)

or, equivalently, a single map $X \xrightarrow{p_0} A$ such that $fp_0 = gp_0$

(since p_1 is forced to be this composite).

In this way we see that the limit of D is precisely the equaliser of f & g.

Example 2

- $J = \boxed{0 \quad 1}$

J-shaped limits are products.

Example 3

$J = \boxed{}$ the empty cat.

J-shaped limits are terminal objects.

Def) The colimit of a diagram $D : J \longrightarrow \mathcal{C}$ is a cocone

$$\begin{tikzcd} Dj \arrow[d, "D\alpha"'] \arrow[dr, "p_j"] \arrow[d, phantom, shift left=4ex, "="] & \\ Dk \arrow[r, "p_k"'] & X \end{tikzcd}$$

with the dual universal property;

equiv., the limit of $D^{op} : J^{op} \to \mathcal{C}^{op}$.

- Details are left to the reader

Ex) Pullbacks & pushouts

- **Pullbacks** are J-limits for

$$J = \boxed{\begin{tikzcd} & 0 \arrow[d] \\ 2 \arrow[r] & 1 \end{tikzcd}}$$

• In elementary terms, given
$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$
its pullback P comes with a commuting square

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "="] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

such that given

$$\begin{tikzcd} D \arrow[r, "r"] \arrow[d, "s"'] \arrow[dr, phantom, "="] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

$\exists ! \, D \xrightarrow{t} P$ st $pt = r$, $qt = s$.

- Pushouts are colimits over J^{op} -

$$J^{op} = \boxed{\begin{tikzcd} 1 \arrow[r] \arrow[d] & 0 \\ 2 & \end{tikzcd}}$$

$\boxed{\text{Theorem}}$ Limits & colimits are unique up to unique isomorphism.

$\boxed{\text{Pf}}$ The two cases are dual - we will do limits.

- Consider $D : J \longrightarrow \mathcal{C}$ & let $(A, p_i : A \to D_i)_{i \in I}$ & $(B, q_i : B \to D_i)_{i \in I}$ be limits of D.

- By their universal properties,

$\exists! \, k : A \to B$ such that
$$\begin{tikzcd} A \arrow[rr, "k"] \arrow[dr, "p_i"'] & \arrow[d, phantom, "="] & B \arrow[dl, "q_i"] \\ & D_i & \end{tikzcd}$$
for all i

& $\exists! \, l : B \to A$ such that
$$\begin{tikzcd} B \arrow[rr, "l"] \arrow[dr, "q_i"'] & \arrow[d, phantom, "="] & A \arrow[dl, "p_i"] \\ & D_i & \end{tikzcd}$$
for all i.

- Then
$$\begin{tikzcd} A \arrow[r, "k"] \arrow[dr, "p_i"'] & B \arrow[r, "l"] \arrow[d, "q_i"] & A \arrow[dl, "p_i"] \\ & D_i & \end{tikzcd}$$
&
$$\begin{tikzcd} A \arrow[rr, "1"] \arrow[dr, "p_i"'] & \arrow[d, phantom, "="] & A \arrow[dl, "p_i"] \\ & D_i & \end{tikzcd}$$

so by universal property of A, $l \circ k = 1_A$

- Similarly $k \circ l = 1_B \Rightarrow k$ an iso. $\square$

Infinite products

- Before moving away from limits & colimits, we mention infinite products.
- Given a set X, we can view it as a discrete category: all arrows are identities.

Then a diagram
$$A : X \longrightarrow \mathcal{C}$$
consists of a family of objects $(A_x : x \in X)$ & its limit $\prod_{x \in X} A_x$ is the (X-indexed) product of the obs $(A_x : x \in X)$

This comes equipped with maps $\prod_{x \in X} A_x \xrightarrow{p_x} A_x$ for all $x \in X$, & it is universal amongst such objects.

Example

- In Set, $\prod_{x \in X} A_x = \{ (a_x)_{x \in X} : a_x \in A_x \}$

As special cases,

- when $X = \boxed{0 \quad 1}$, we obtain ordinary (binary) products
- when $X = \boxed{}$, we obtain the terminal object.

Def^n) A category $\mathcal{C}$ is <u>complete</u> if it has <u>limits</u> of all diagrams, and <u>cocomplete</u> if it has <u>colimits</u> of all diagrams.

Remark) • All algebraic categories are both complete & cocomplete.

• It is not hard to see that they are complete.

• It is harder to show that they are cocomplete, since colimits in algebraic cats are more complex.

• Will return to this later.