L6 Part 1 - Last categorical titbits

① The Triangle equations

Theorem

An adjunction $\begin{tikzcd} A \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "F"'] \end{tikzcd}$

is specified by a pair of functors as above + natural transformations

$$\begin{tikzcd} B \arrow[rr, "1"] \arrow[dr, "F"'] & \eta\Downarrow & B \\ & A \arrow[ur, "U"'] & \end{tikzcd} \quad \& \quad \begin{tikzcd} & B \arrow[dr, "F"] & \\ A \arrow[ur, "U"] \arrow[rr, "1"'] & \varepsilon\Downarrow & A \end{tikzcd}$$

(unit) (counit)

Such that

$$\begin{tikzcd} B \arrow[rr, "1"] \arrow[dr, "F"'] & \eta\Downarrow & B \arrow[dr, "F"] & \\ & A \arrow[ur, "U"] \arrow[rr, "1"'] & \varepsilon\Downarrow & A \end{tikzcd} = \mathrm{id}_F \quad \& \quad \begin{tikzcd} & B \arrow[rr, "1"] \arrow[dr, "F"] & \eta\Downarrow & B \\ A \arrow[ur, "U"] \arrow[rr, "1"'] & \varepsilon\Downarrow & A \arrow[ur, "U"'] & \end{tikzcd} = \mathrm{id}_U.$$

ie.

$$\begin{tikzcd} & FUFb \arrow[d, "\varepsilon_{Fb}"] \\ Fb \arrow[ur, "F\eta_b"] \arrow[r, "1"'] & Fb \end{tikzcd} \quad \& \quad \begin{tikzcd} & UFUa \arrow[d, "U\varepsilon_a"] \\ Ua \arrow[ur, "\eta_{Ua}"] \arrow[r, "1"'] & Ua \end{tikzcd}.$$

~~Proof~~ (Sketch)

- Consider an adjunction $A(Fb, a) \xrightarrow{\varphi} B(b, Ua)$.
- We defined the unit $1 \Rightarrow UF$ last time.
- Similarly $B(Ua, Ua) \xrightarrow{\varphi^{-1}} A(FUa, a)$

$$1 \longmapsto FUa \xrightarrow{\varepsilon_a} a$$

determines the counit.

By naturality of φ^{-1}, have

@ $b \xrightarrow{\alpha} Ua$ + @ $\beta : Fa \to b$

$$\begin{tikzcd} Fb \arrow[rr, "\varphi^{-1}\alpha"] \arrow[dr, "F\alpha"'] & & a \\ & FUa \arrow[ur, "\varepsilon_a"'] & \end{tikzcd} \qquad \begin{tikzcd} a \arrow[rr, "\varphi\beta"] \arrow[dr, "\eta_a"'] & {=} & Ub \\ & UFa \arrow[ur, "U\beta"'] & \end{tikzcd}$$

(last week)

- Now consider $B(Ua, Ua) \xrightarrow{\varphi^{-1}} A(FUa, a) \xrightarrow{\varphi} B(Ua, Ua)$

sends

$$1_{Ua} \longmapsto \varepsilon_a : FUa \to a \longmapsto Ua \xrightarrow{\eta_{Ua}} UFUa \xrightarrow{U\varepsilon_a} Ua$$

so this equals the identity.

- Similarly the other triangle equation holds.

- Conversely consider ε & η satisfying triangle equations.
- We have $A(Fb,a) \xrightarrow{\varphi} B(b,Ua)$

$$Fb \xrightarrow{\alpha} a \longmapsto a \xrightarrow{\eta_a} UFa \xrightarrow{U\alpha} Ub$$

natural in a, b.
- So enough to show φ a bijection.
- Inverse given by

$$B(b,Ua) \xrightarrow{\quad\theta\quad} A(Fb,a)$$

$$b \xrightarrow{\alpha} Ua \longmapsto Fb \xrightarrow{F\alpha} FUa \xrightarrow{\varepsilon_a} a$$

& $\theta\varphi = 1$, $\varphi\theta = 1$ follow from triangle equations.

$\square$

Remark) This def^n of adjunction via unit & counit + triangle equations captures sense in which adjunction is a weaker form of iso/equivalence.

Fully & Faithful Functors

A functor $U : A \longrightarrow B$ has functions $A(a,b) \xrightarrow{U_{a,b}} B(Ua, Ub)$.

Def) U is

1. Faithful if each $U_{a,b}$ is injective
2. Full - - - - - - - - surjective
3. Fully faithful - - - - - - - - bijective.

- In elementary terms, faithful means given $a \underset{\beta}{\overset{\alpha}{\rightrightarrows}} b$ s.t. $U\alpha = U\beta$ then $\alpha = \beta$.
- Full means given $Ua \xrightarrow{\alpha} Ub$ $\exists\, a \xrightarrow{\beta} b$ st $U\beta = \alpha$.

$\boxed{\text{Def}^n}$ - A subcategory A of B is a subcollection of objects and arrows in B closed under identities and composition.

- Then $i : A \hookrightarrow B$ a functor. If i is fully faithful it is a full subcategory.

Examples

- U : Grp, Mon, Ring $\longrightarrow$ Set are all faithful.
- i : Grp $\hookrightarrow$ Mon is a full subcategory.

Lecture 6 Part 2 - Universal Algebra

- Universal algebra : study of "sets with operations satisf. equations"

E.g. groups, monoids, rings . .
operations : $x+y, 0 \ldots$
equations : $x+0=x \ldots$

But not things like Fields :
$0 \neq 1$ not an equation

- "Universal algebra" also called "general algebra" . . .

- Two excellent books I like :
- Algebra Chapter 0 by Aluffi
- An invitation to General Alg. by Bergman
(I am using this book a little)

Def^n) A signature $\Omega = (\Omega_n)_{n \in \mathbb{N}}$ consists of a set Ω_n for each $n \in \mathbb{N}$.

Remark) The elements of Ω_n are thought of as "n-ary operation symbol".
0-ary operations = "constants".

Examples

① Signature Ω for monoids, groups... has
$\Omega = \{e, m\}$.

e nullary
$e \in \Omega(0)$

m binary
$m \in \Omega(2)$

ie. $\Omega(n) = \phi$ for $n \neq 0, 2$.

② Signature for rings
$\Omega = \{\underbrace{0, 1}_{\text{0-ary}}, \underbrace{+, \times}_{\text{2-ary}}\}$

ie. $\Omega(0) = \{0, 1\}$
$\Omega(2) = \{+, \times\}$
$\Omega(n) = \phi$ otherwise.

$\boxed{\text{Def}^n}$ Let Ω be a signature.
An Ω-algebra is a set X together with, for each $n \in \mathbb{N}$ & $m \in \Omega(n)$, a function

$$m_X : X^n \longrightarrow X \quad : \quad (a_1, \ldots, a_n) \longmapsto m_X(a_1, \ldots, a_n).$$

$\boxed{\text{Remark}}$ What is X^0? Well $X^n \cong \mathrm{Set}(n, X)$, the set of functions from the n elt set $\to X$.

- In partic., $X^0 = \mathrm{Set}(\underset{\substack{\text{empty set}\\ \text{which is initial}}}{0}, X) \cong 1$.
- So given $e \in \Omega(0)$ & X an Ω-algebra, we have $e_X : 1 \longrightarrow X$, so this is just an element e_X of X.
- This is why elts of $\Omega(0)$ are thought of as nullary op. symbols = constants.

Notation) For an Ω-alg.

$$(X, (m_X)_{m \in \Omega_n, n \in \mathbb{N}})$$

I will denote it by $\overline{X}$, or perhaps just X.

Example

For $\Omega = \{e, m\}$, where e is nullary and m is binary,

an Ω-algebra is a magma:

- we have:
  - a binary function $m_X : X^2 \to X : (a,b) \mapsto m_X(a,b)$

    $e_X : 1 \to X : \cdot \mapsto e_X \in X$

Magmas do not need to satisfy any equations.

Exercise) Find a signature Ω s.t. $\mathbb{R}$-vector spaces are certain Ω-algebras.

Def^n) A homomorphism $f : \bar{X} \to \bar{Y}$ of Ω-algebras is a function $f : X \to Y$ such that $\forall n \in \mathbb{N}$, $m \in \Omega_n$

we have
$$\begin{tikzcd} X^n \arrow[r, "f^n"] \arrow[d, "m_X"'] \arrow[dr, phantom, "="] & Y^n \arrow[d, "m_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

This says $\forall a_1, \ldots, a_n \in X$, we have
$$f(m_X(a_1, \ldots, a_n)) = m_Y(fa_1, \ldots, fa_n).$$

Remark This captures the general notion of homomorphism in monoids, groups, rings etc when we choose the appropriate signature.

Example If $\Omega = \{ \bullet, e \}$

$$\underbrace{\bullet}_{2\text{-ary}} \qquad \underbrace{e}_{0\text{-ary}}$$

A homomorphism of Ω-algs is a fn

$f : X \longrightarrow Y$ satisfying

$$f(a \cdot_X b) = f(a) \cdot_Y f(b) \;\&\; f(e_X) = e_Y.$$

(Here I write $a \cdot_X b = \cdot_X(a,b)$ for convenience.)

- There is a category Ω-Alg of Ω-algebras & Ω-algebra homomorphisms, & it has a forgetful functor

$$U : \Omega\text{-Alg} \longrightarrow \mathbf{Set}.$$

- Our goal now is to describe its left adjoint.

Terms & term algebras

- Let Ω be a signature & X a set.
  We define a set
  $T_\Omega(X)$ of "Ω-terms" in X
  as follows:

• $T_\Omega^0(X) = X$

• $T_\Omega^{i+1}(X) =$

formal expression

$$T_\Omega^i(X) \cup \{m(t_1,\dots,t_n) : n \in \mathbb{N},\ m \in \Omega(n),\ t_1,\dots,t_n \in T_\Omega^i(X)\}$$

• $T_\Omega(X) = \bigcup_{i=0}^{\infty} T_\Omega^i(X)$.

The depth of a term $t \in T_\Omega(X)$,
$d(t) = \min\{n \in \mathbb{N} : t \in T_\Omega^n(X)\}$.

Example) $\Omega = \{ \underset{\text{2-ary}}{m}, \underset{\text{0-ary}}{e} \}$, $X = \{a, b, c\}$

$T_\Omega(X)$

$= \{ a, b, c, \ldots$ depth 0

$e, m(a,b), m(a,a), m(b,b), m(b,a) \ldots$ depth 1

$m(a,e), m(a, m(b,c)), \ldots$ depth 2 ..

$m(m(a, m(b,e)), c)$ etc ... $\}$

- Remark: $T_\Omega(X)$ consists of all expressions one can build from "variables" X and operation symbols in Ω.

- In fact, $T_\Omega(X)$ is an Ω-algebra:

  given $m \in \Omega_n$, we define

$$T_\Omega(X)^n \xrightarrow{m} T_\Omega(X):$$
$$(t_1, \ldots, t_n) \longmapsto m(t_1, \ldots, t_n)$$

- We also have a function

$$\eta_X : X \longrightarrow T_\Omega(X) : x \longmapsto x.$$

$\boxed{\text{Theorem}}$ $U : \Omega\text{-Alg} \longrightarrow \text{Set}$ has a left adjoint, whose value at X is the Ω-algebra $T_\Omega(X)$ of Ω-terms.

Proof

Given $X \xrightarrow{f} UY$ we must show that $\exists!$ Ω-algebra map $T_\Omega(X) \xrightarrow{\bar{f}} Y$ such that

$$\begin{tikzcd}
 & UT_\Omega(X) \arrow[dr, "U\bar{f}"] \arrow[d, phantom, "="] & \\
X \arrow[ur, "\eta_X"] \arrow[rr, "f"'] & {} & UY
\end{tikzcd}$$

- Since the triangle must commute, we must define $\bar{f}x = fx$ for $x \in X$.
- **We define $\bar{f}$ inductively on depth:**
- depth 0 terms are elements of X, where $\bar{f}x = fx$ as above.
- Suppose we have defined it on terms of depth $\leq i$. Then consider a term $m(t_1, \ldots, t_n)$ of depth $i+1$.

- Then $t_1,\dots,t_n$ are of depth $\leq i$.
  In order that $\bar{F}$ is a homomorphism, we must set

(*) $$\bar{F}(m(t_1,\dots,t_n)) = m^Y(\underbrace{\bar{F}t_1,\dots,\bar{F}t_n}_{\text{already defined}})$$

- This completes the definition of $\bar{f}$ & (*) shows that it is a homomorphism.
- Since we were forced to define $\bar{f}$ in this way, it is the unique extension of f to a homomorphism as required. $\square$

$\boxed{\text{Example}}$ If $\Omega = \{m, e\}$, $X = \{a, b, c\}$ &

$X \xrightarrow{f} UY$, we obtain

$$\begin{aligned}
T_\Omega X &\xrightarrow{\bar{f}} Y \\
a, b, c &\longmapsto fa, fb, fc \\
e &\longmapsto e_Y \\
m(a,b) &\longmapsto m_Y(fa, fb) \\
m(a, m(b,c)) &\longmapsto m_Y(fa, m_Y(fb, fc))
\end{aligned}$$

- - - - .

Equations

- Let Ω be a signature.
- Then given a set X, we have the set $T_\Omega(X)$ of Ω-terms in X.

Def^n) An Ω-equation in variables X is a pair $(s,t) \in T_\Omega(X)^2$.

Remark) We often informally write an Ω-equation as "$s = t$."

Example

For $\Omega = (\underset{\text{nullary}}{e}, \underset{\text{bin}}{\cdot})$ the signature for magmas, the following are equations

$$\left.\begin{array}{l} \bullet\ x \cdot e = x \\ \bullet\ e \cdot x = x \\ \bullet\ (x \cdot y) \cdot z = x \cdot (y \cdot z) \end{array}\right\} \text{in variables } \{x, y, z\}.$$

Satisfying equations

- Consider an X-tuple $a = (a_x)_{x \in X}$ of an Ω-algebra A.
- In other words a function $a : X \longrightarrow UA$
  $x \longmapsto a_x$.

- We have

$$\begin{tikzcd} U T_\Omega(X) \arrow[r, "U\bar{a}"] & UA \\ X \arrow[u, "\eta_X"] \arrow[ur, phantom, "="] \arrow[r, "a"'] & UA \end{tikzcd}$$

$\boxed{\text{Def}^n}$ Let $(s,t) \in T_\Omega(X)$ be an equation & A an Ω-algebra. Then the X-tuple a satisfies the equation $s = t$ if $\bar{a}(s) = \bar{a}(t)$.

- The Ω-algebra A satisfies $s = t$ if each X-tuple a of A does so.

One writes $A \models s = t$.

Example

Given $\{x, y, z\} \xrightarrow{a} A$ a magma,

$x, y, z \longmapsto a_1, a_2, a_3$

the extension

$$T_\Omega\{x, y, z\} \longrightarrow A$$
$$(x.y).z \longmapsto (a_1.a_2).a_3$$
$$x.(y.z) \longmapsto a_1.(a_2.a_3)$$

so A satisfies the equation $(x.y).z = x.(y.z)$ iff A is associative.

Def) A **universal algebraic theory** (Ω, E) is a signature Ω and a set E of Ω-equations.

- By an (Ω, E)-algebra we mean an Ω-algebra A such that $A \models s = t$ for each equation $(s, t) \in E$.

Examples

- For $\Omega = \{\cdot, e\}$ as before, and

$$E = \left\{ \begin{array}{l} (x.y).z = x.(y.z) \\ x.e = x \\ e.x = x \end{array} \right\}$$

an (Ω, E)-algebra is a monoid.

- If we take $\Omega = \{\cdot, e, (-)^{-1}\}$ unary and add the equations

$$x.x^{-1} = e \qquad x^{-1}.x = e$$

to E, then an (Ω, E)-algebra is a group.

Def^n) For (Ω, E) as above, we write

$$(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$$

for the full subcategory of (Ω, E)-algebras.

This means that the objects are (Ω, E)-algebras and the morphisms simply the Ω-algebra homomorphisms between them.

- The inclusion $(\Omega, \mathbf{E})\text{-Alg} \xhookrightarrow{i} \Omega\text{-Alg}$ is a (fully faithful) functor.
- We obtain a composite forgetful functor to Set, as in the diagram below:

$$\begin{tikzcd}
(\Omega,E)\text{-Alg} \arrow[rr, hook, "i"] \arrow[dr, "U"'] & & \Omega\text{-Alg} \arrow[dl, "U_\Omega"] \\
& \mathrm{Set} &
\end{tikzcd}$$

- Later we will show that i and U have left adjoints - in particular, there exist free (Ω,E)-algebras.

<u>Examples</u>

- All of the algebraic categories earlier considered - eg. Vect, Grp, Rng, Mon etc - are of the form (Ω,E)-Alg for suitable Ω and E.

- This is the framework and scope of what people typically call <u>universal algebra</u>.

- Our goal now is to study some of the good properties of categories of the form Ω-**Alg** and, more generally, (Ω, E)-**Alg**.
- In fact, (Ω,E)-Alg is closed in Ω-Alg under all of the constructions we will care about - limits, coequalisers, so we can concentrate on Ω-**Alg** firstly.

First, a definition:

Def^n) Let $A \in \Omega$-Alg. A subalgebra $B \hookrightarrow A$ is a subset B of A such that if $s \in \Omega_n$ & $b_1, \dots, b_n \in B$ then $s(b_1, \dots, b_n) \in B$.

- In particular a subalgebra B is itself an Ω-algebra & the inclusion $B \hookrightarrow A$ a homomorphism.

Def^n) A homomorphic image is a homomorphism $f: A \to B$ which is surjective.

- Let $f : A \longrightarrow B$ be a homomorphism.
  Let $\text{im} f = \{ b \in B : \exists a \text{ with } fa = b \}$.
  Then $\text{im} f \hookrightarrow B$ is a subalgebra of B,
  since if $b_1 = fa_1, \ldots, b_n = fa_n$ & $s \in \Omega_n$,
  then
  $$\begin{aligned} s(b_1, \ldots, b_n) &= s(fa_1, \ldots, fa_n) \\ &= f\, s(a_1, \ldots, a_n) \in \text{im} f. \end{aligned}$$
- In particular, we obtain a factorisation

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\ & \text{im} f \arrow[ur, hook, "i"'] & \end{tikzcd} \qquad a \mapsto fa \qquad \text{in } \Omega\text{-Alg}.$$

homomorphic image

subalgebra inclusion

Proposition

Ω-Alg has all limits & $U : \Omega\text{-Alg} \longrightarrow \text{Set}$ preserves them.

Proof – I will just check that Ω-Alg has (possibly infinite) products.

and equalisers, since these generate all limits (not proven in course) and are important for us.

- Consider a family $(A_i)_{i\in I}$ of Ω-algebras.
- Their product as sets is the direct product

$$\prod_{i\in I} A_i = \{\bar{a} = (a_i)_{i\in I} : a_i \in A_i\} \xrightarrow{p_i} A_i$$
$$(a_i)_{i\in I} \longmapsto a_i .$$

- We want to show $\prod_{i\in I} A_i$ has the structure of an Ω-algebra such that each p_i is a homomorphism: this says: given $s \in \Omega_n$ and $\bar{a}^1, \dots, \bar{a}^n$ we have $s(\bar{a}^1, \dots, \bar{a}^n)_i = s(\bar{a}^1_i, \dots, \bar{a}^n_i)$ ~

ie. the componentwise Ω-algebra structure.

- Given an Ω-algebra B and $(f_i : B \to A_i)_{i\in I}$ we have a unique function $B \xrightarrow{f} \prod A_i$ such that $p_i \circ f = f_i$ for each $i \in I$: namely $(fb)_i = f_i(b)$, and we must check that f is a homomorphism if each f_i is one: ie. $f\,s(b^1, \dots, b^n) = s(fb^1, \dots, fb^n)$
but the components at $i \in I$ are
$$f_i s(b^1, \dots, b^n) = s(f_i b^1, \dots, f_i b^n)$$
which are equal since f_i a homomorphism.

- Given $A \underset{g}{\overset{f}{\rightrightarrows}} B$ the equaliser
$$E = \{x \in A : fx = gx\} \xrightarrow{i} A \text{ in Set}$$
is a subalgebra of A:
ie. given $x_1, \dots, x_n$ st $fx_i = gx_i$ then

$f(s(x_1,\ldots,x_n)) = s(fx_1,\ldots,fx_n) = s(gx_1,\ldots,gx_n) = g(s(x_1,\ldots,x_n))$
so $s(x_1,\ldots,x_n) \in E$
In particular, $i : E \to A$ is a homomorphism and it is easy to check it has the u.p. of the equaliser.

$\square$

What about colimits? The key kind are the simple quotients by congruences.

- These generalise:
  - quotients by equiv. rels (for sets)
  - quotients by normal subgroups (for groups)
    ideals (for rings)

these these last two are less obvious.

Def^n) - Let A be an Ω-algebra. An equivalence relation $E \subseteq A$ is called a congruence if E is a subalgebra of A.

- In elementary terms, a cong. is an e-rel E:
(so xEx, $xEy \Rightarrow yEx$, $xEy \wedge yEz \Rightarrow xEz$)
such that if
$x_1 E y_1, \ldots, x_n E y_n$ & $s \in \Omega_n$ then
$s(x_1,\ldots,x_n) \, E \, s(y_1,\ldots,y_n)$.

Exercise: • For groups, show that there

is a bij^n bet. cong. on A & normal subgroups of A :

$$\begin{array}{c|c}
\begin{tikzcd} E \arrow[d, hook] & \mapsto & \bar{E} \arrow[d, hook] \\ A\times A & & A \end{tikzcd} \quad \bar{E} = x\in A \text{ s.t. } xE0 &
\begin{tikzcd} N \arrow[d, hook] & \mapsto & E_N \arrow[d, hook] \\ A & & A\times A \end{tikzcd} \quad E_N = \{(x,y) : xy^{-1}\in N\}
\end{array}$$

- What about rings?

- If $E \overset{i}{\hookrightarrow} A^2$ is a congruence on A, then we can form the diagram
$$E \overset{i}{\hookrightarrow} A^2 \underset{p_2}{\overset{p_1}{\rightrightarrows}} A \text{ in } \Omega\text{-Alg},$$
and so obtain a pair of Ω-alg. homomorphisms
$$E \underset{c}{\overset{d}{\rightrightarrows}} A : (x,y)\in E \;\begin{array}{l}\longmapsto x \\ \longmapsto y\end{array}.$$
- We are interested in their coequaliser
$$A \xrightarrow{p} A/E$$
defined as follows:
  - elements of A/E are equiv. classes $[a]$, with $p(a) = [a] = \{x : xEa\}$
  - Observe that p is surjective. Therefore if p is to be a homomorphism, we are forced to define
$$s^{A/E}([a_1], \ldots, [a_n]) = [s^A(a_1, \ldots, a_n)].$$
- Is this well defined?
Suppose $[b_1] = [a_1], \ldots, [b_n] = [a_n]$, so that $b_1 E a_1, \ldots, b_n E a_n$.
Then $s(b_1, \ldots, b_n) E s(a_1, \ldots, a_n)$ as we have a congruence, so that

$[s(b_1,\dots,b_n)] = [s(a_1,\dots,a_n)]$ as required.

In particular, A/E is an Ω-algebra and $p : A \to A/E$ a surjective homomorphism.

Proposition

$$E \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{p} A/E$$

is a coequaliser in Ω-Alg.

$\boxed{\text{Proof}}$

- Firstly, if $(x,y) \in E$, then $pd(x,y) = [x] = [y] = pc(x,y)$ so $pd = pc$.
- Given $A \xrightarrow{f} B$ with $fd = fc$, this means if $(x,y) \in E$ then $fx = fy$, so that $[x] = [y] \Rightarrow fx = fy$.
- Define $A/E \xrightarrow{\bar{f}} B$, $[a] \longmapsto f(a)$ for some choice of representatives.
- The def.^{n} of $\bar{f}$ is indep. of the choice as f is.
- The triangle

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "f"'] & A/E \arrow[d, "\bar{f}"] \\ & B \end{tikzcd}$$

commutes for this reason too.

- Clearly $\bar{f}$ is a homomorphism, since f is.

Def^{n}) Let $f : A \longrightarrow B$ be an Ω-alg homomorphism. The Kernel of f is the congruence

$$K_f = \{(x,y) : fx = fy\} \hookrightarrow A^2.$$

- It is easy to see that this is a congruence. Check it!

Categorically, K_f is the pullback

$$
\begin{tikzcd}
(x,y) \arrow[rr, mapsto] \arrow[dd, mapsto] & & x \\
& K_f \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
y & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

so, in particular, we have $fd = fc$.

Therefore we get a unique factorisation of f through the coequaliser

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\
& A/K_f \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[dr, mapsto] & & fa \\
& {[a]} \arrow[ur, mapsto] &
\end{tikzcd}
$$

First isomorphism theorem

- Given a homomorphism $f : A \longrightarrow B$, we have an isomorphism
  $t : A/K_f \longrightarrow \mathrm{im} f$
  making the diagram

$$\begin{tikzcd}
& A/K_f \arrow[dr, "\bar{f}"] \arrow[dd, "s"', "t"] & \\
A \arrow[ur, "p"] \arrow[dr, "f_1"'] \arrow[r, phantom, "="] & & B \arrow[l, phantom, "="] \\
& \mathrm{im} f \arrow[ur, hook, "f_2"'] &
\end{tikzcd}$$

$\boxed{\text{Proof}}$ - The formula is

$$[a] \overset{t}{\longmapsto} fa$$

It is clearly a homomorph as $\bar{F}$ is.

- Let us show t is injective. Suppose $t[a] = t[b]$.
- That is, $fa = fb$.
- Then $(a,b) \in K_f$ so $[a] = [b]$.
- For surjectivity, if $b \in \mathrm{im} f$, we have $b = fa$ so

$$t[a] = fa = b. \qquad \square$$

Corollary

If $f : A \longrightarrow B$ is a surjective homom., then $A/K_f \cong B$. In particular

$$K_f \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{f} B \text{ is a coequaliser}$$

diagram.

<u>Proof</u> / In this case, $B = \inf$.