Alg 3 Exercise (Due 27.10.2023)

① Let $\mathcal{C}$ be a category admitting products $A \times B$ of all objects A, B and a terminal object 1. Prove that there is an isomorphism

$$A \times 1 \cong A.$$

② Given $f : B \to C$ $\exists$ a unique map $A \times f : A \times B \longrightarrow A \times C$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "1_A"] & A \\
A \times B \arrow[u, "p"] \arrow[r, "A \times f"] \arrow[d, "q"'] & A \times C \arrow[u, "p'"'] \arrow[d, "q'"] \\
B \arrow[r, "f"'] & C
\end{tikzcd}$$

commute

where p, q & p', q' are the defining maps of the products.

- Using this, prove that we obtain a functor $A \times - : \mathcal{C} \longrightarrow \mathcal{C}$

$$B \longmapsto A \times B$$

$$B \xrightarrow{f} C \longmapsto A \times B \xrightarrow{A \times f} A \times C$$

by checking the functor axioms.

③ - Consider the cat Rng of rings, & the ring homomorphism

$$i : \mathbb{Z} \hookrightarrow \mathbb{Q}$$
$$n \longmapsto n .$$

- Prove that this is epi in the category of rings.

④ Let $f : X \longrightarrow X$ be a function, and consider its set $\mathrm{Fix}(f) = \{x \in X : fx = x\}$ of fixpoints.

Can you describe $\mathrm{Fix}(f)$ as an equaliser of two functions from X to X.

⑤ Let $\mathcal{K}$ be the following cat:

- objects are triples (X, a, s) where X is a set, $a \in X$ and $s : X \to X$ is a function.
- a morphism $f : (X, a, s) \longrightarrow (Y, b, t)$ is a function $f : X \longrightarrow Y$ such that $fa = b$ and $fs = tf$.

Can you describe the initial

object in this category?