Exercises 8

- A complete lattice is a poset such that every subset has a least upper & greater lower bound.

① Let A be an (Ω, E)-algebra.
Show that $\mathrm{Cong}(A)$ is a complete lattice.

② Use this to conclude that $Q(A)/\sim$ is a complete lattice too.
Describe its poset structure & greatest lower bounds without referring to $\mathrm{Cong}(A)$.

(3) Prove that each (Ω, E)-alg is a quotient of a free (Ω, E)-alg; ie. $\exists\, FX \twoheadrightarrow A$ surjective in (Ω, E)-Alg.

(4) Set $= (\Omega, E)$-Alg for $\Omega = \emptyset$. Which sets are projective?

(4) Show that an (Ω, E)-algebra is projective $\Longleftrightarrow$ it is a retract of a free algebra: ie. $\exists\, p, q$ st

$$\begin{tikzcd} & FX \arrow[dr, "p"] \arrow[d, phantom, "="] & \\ A \arrow[ur, "q"] \arrow[rr, "1"'] & {} & A \end{tikzcd}$$