Lecture 4

Monos & epis

Can we capture injectivity & surjectivity of morphisms in categories?

Def^n) A morphism $f: X \to Y$ is **mono**/monic/a monomorphism if $\forall\, Z \underset{h}{\overset{g}{\rightrightarrows}} X$ satisfying $fg = fh$ then $g = h$.

Dually, $f: X \to Y$ is **epi** if $\forall\, Y \underset{h}{\overset{g}{\rightrightarrows}} Z$ satisfying $gf = hf$ we have $g = h$.

Example

- In Set, monos ≡ injective functions
- Indeed, if f is injective, then
  $fg = fh \Rightarrow fg(z) = fh(z)\ \forall z \in Z$
  $\Rightarrow g(z) = h(z)\ \forall z \Rightarrow g = h$, so f is mono.
  (↳ as f inj.)

Conversely, suppose f is mono: we must show that $\forall x, y \in X$ $fx = fy \Rightarrow x = y$.

Consider 1 element set $1 = (\bullet)$.

let $\bar{x} : (\cdot) \longrightarrow X : \bullet \longmapsto x$

$\bar{y} : (\cdot) \longrightarrow X : \bullet \longmapsto y$

& then $f \circ \bar{x} = f \circ \bar{y}$ since at unique element $\bullet$ we have

$$\underset{fx}{\underset{\parallel}{f \circ \bar{x}(\cdot)}} = \underset{fy}{\underset{\parallel}{f \circ \bar{y}(\cdot)}}$$

so as f is mono we have $\bar{x} = \bar{y}$,

so $$\underset{x}{\underset{\parallel}{\bar{x}(\cdot)}} = \underset{y}{\underset{\parallel}{\bar{y}(\cdot)}}.$$

Idea: notion of monomorphism abstracts injections in Set by replacing 1-element set by a general object.

Whilst in Set, $1 \longrightarrow X$ is an "element" of X, in a gen. cat we think of morphisms $A \longrightarrow X$ as generalised elements.

Ex) In all algebraic categories,
mono $\equiv$ injective homomorphisms

Proof that injective $\Rightarrow$ mono is just as in Set; we will prove mono $\Rightarrow$ injective in section on universal algebra.

Epis in Set ≡ surjections

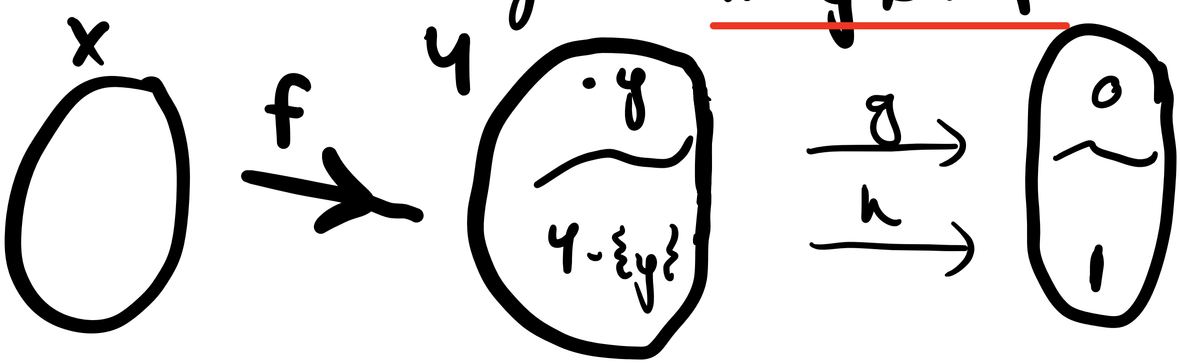
- Surjective ⇒ epi is easy (similar to injective ⇒ mono)

i.e. if $X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$ s.t. $gf = hf$

then let $y \in Y$. As f is surj., $\exists x \in X$ such that $fx = y$.

- Hence $gfx = hfx$ for all y so $g = h$.
   $gfx = gy$, $hfx = hy$

- Conversely, let $X \xrightarrow{f} Y$ be epi & consider $y \in Y$. If $y \notin \text{im} f$



where $gy = 0$ & $gz = 1$ otherwise & $hz = 1$ all $z \in Y$.

Then $gfx = 1 = hfx$ all x ⇒ $g = h$ (as f is epi) but this is false as $gy \neq hy$.

Hence $y \in \text{im} f$ ⇒ f surjective.

- In algebraic categories, surjective $\Rightarrow$ epi (just as above) but the converse is not true in general

In fact, in Rng, the inclusion homomorphism
$$\mathbb{Z} \hookrightarrow \mathbb{Q}$$
is epi but not surjective.

Exercise: check the above!

However,
surjective $\equiv$ regular "epi"
which we will look at this when we study congruences & quotients in universal algebra.

Natural Transformations

Def^n) Let $F, G : A \rightrightarrows B$ be functors.

A <u>natural transformation</u> $\eta : F \Longrightarrow G$ consists of:

- For each $x \in A$ a morphism $\eta_x : Fx \longrightarrow Gx$ such that:
- For all $\alpha : x \longrightarrow y \in A$ the square

$$\begin{array}{ccc} Fx & \xrightarrow{\eta_x} & Gx \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ Fy & \xrightarrow[\eta_y]{} & Gy \end{array}$$

commutes.

Remark: We write $A \underset{G}{\overset{F}{\Downarrow \eta}} B$ for a natural transformation.

Examples

① Let $G = \boxed{1 \overset{s}{\underset{t}{\rightrightarrows}} 0}$

- A diagram $X : G \to \mathcal{C}$ consists of

$$X_1 \underset{Xt = t_X}{\overset{Xs = s_X}{\rightrightarrows}} X_0$$

& a nat. t. $\alpha : X \to Y$

consists of

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha_1} & Y_1 \\ s_X \downarrow \downarrow t_X & & s_Y \downarrow \downarrow t_Y \\ X_0 & \xrightarrow{\alpha_0} & Y_0 \end{array}$$

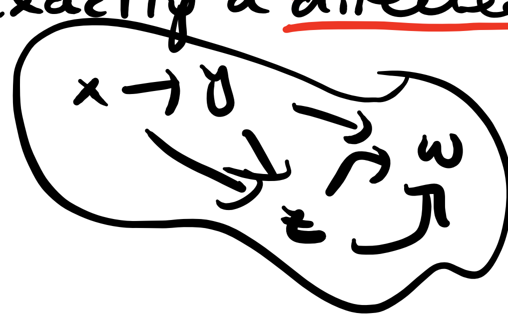
making both squares commutes.

- For $\mathcal{C} = \text{Set}$, X consists of :
  - a set X_0 of "vertices" $x, y, z \ldots$
  - a set X_1 of "directed edges" f, g, h which have "source" $s_X(f) \in X_0$ & target $t_X(f) \in X_0$

Let's write $a \xrightarrow{f} b$ if $s_X(f) = a$ & $t_X(f) = b$.

So $X : G \to \text{Set}$ is exactly a directed graph

eg.



- A natural transf.

$$\begin{array}{ccc} X_1 & \xrightarrow{\eta_1} & Y_1 \\ {\scriptstyle s_X}\downarrow\ \downarrow{\scriptstyle t_X} & & {\scriptstyle s_Y}\downarrow\ \downarrow{\scriptstyle t_Y} \\ X_0 & \xrightarrow{\eta_0} & Y_0 \end{array}$$

in a way that preserves source & target:

i.e. $x \longmapsto \eta_0 x$

$$x \xrightarrow{\alpha} y \longmapsto \eta_0 x \xrightarrow{\eta_1 \alpha} \eta_0 y$$

i.e. if $x = s_X(\alpha)$ then $\eta_0 x = \eta_0 s_X(\alpha) = s_Y \eta_1(\alpha)$, & sim. for t.

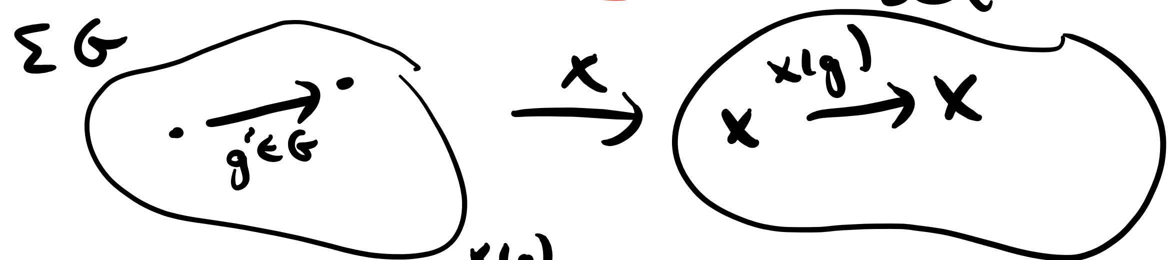
So a natural transformation is precisely a morphism of directed graphs.

Ex 2

- If A, B are posets viewed as categories where $\exists!\, a \to b \iff a \leq b$
- Given $F, G : A \rightrightarrows B$ are order-preserving functions viewed as functors then a natural transformation $\alpha : F \Rightarrow G$ means $Fx \leq Gx \;\forall x$.
- The commutativity condition is redundant since all diagrams commute in B.

Ex 3

If G a group, we have 1-object category ΣG. A functor $\Sigma G \xrightarrow{X} \text{Set}$ is a G-sets:



where $X \xrightarrow{X(g)} X : x \mapsto g \cdot x$.

A natural transformation is a function $f : X \to Y$ s.t. $f(g \cdot x) = g \cdot f(x)$, a G-equivariant map.

Ex 4

Consider

$$\mathbf{Set} \xrightarrow{F} \mathbf{Mon} \xrightarrow{U} \mathbf{Set}, \qquad W = UF$$

(F: word/list monoid; U: forgetful)

We can define a natural transformation $\eta : 1_{\mathbf{Set}} \Longrightarrow W$ whose component at X is

$$X \xrightarrow{\eta_X} WX$$
$$x \longmapsto [x]$$

($[x]$: word of length 1)

At $f : X \to Y$ we need

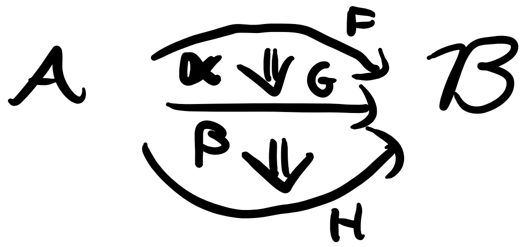
$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & WX \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle Wf} \\ Y & \xrightarrow[\eta_Y]{} & WY \end{array}$$

By def^n, $Wf[x_1, \dots, x_n] = [fx_1, \dots, fx_n]$.

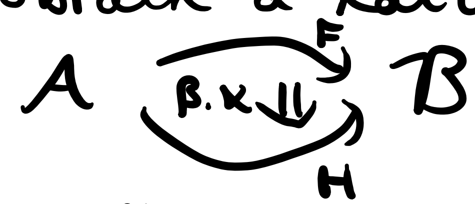
so $Wf\,\eta_x(x) = Wf[x] = [fx]$
$= \eta_Y(fx) = \eta_Y \circ f(x)$.

Functor categories

- Consider categories $\mathcal{A}$ & $\mathcal{B}$.
- Given natural transformations



we can compose them (vertically) to obtain a natural tr.

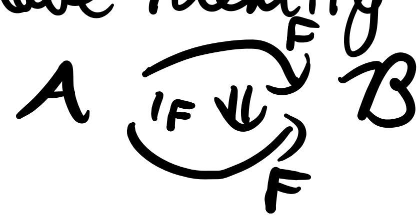


with components

$$FX \xrightarrow{\alpha_X} GX \xrightarrow{\beta_X} HX \text{ for each } X.$$

The "naturality cond." is easy to check, & this composition of natural transformations is associative (as composition in $\mathcal{B}$ is associative)

- Also have identity nat. transf.



with components

$$1_{FX} : FX \longrightarrow FX.$$

- Altogether, we obtain a category $[A, B]$ called the <u>functor category</u>:

  objects - functors $A \to B$

  arrows - natural transformations.

<u>Example</u>

- $[1 \rightrightarrows 0, Set]$ is the category of <u>directed graphs</u>.

  - For G a group, what is $[\Sigma G, Set]$?

    Or $[\Sigma G, Vect]$?

Horizontal composition

- Given $A \underset{G}{\overset{F}{\Downarrow \eta}} B \xrightarrow{H} \mathcal{C}$

define $A \underset{HG}{\overset{HF}{\Downarrow H\eta}} \mathcal{C}$ to be the nat. transf. with components: at $x \in A$, $HFX \xrightarrow{H\eta_x} HGX$.

- Given $A \xrightarrow{F} B \underset{H}{\overset{G}{\Downarrow \eta}} \mathcal{C}$ we define $A \underset{HF}{\overset{GF}{\Downarrow \eta_F}} \mathcal{C}$ as the natural transf. with component $\eta_{Fx} : GFX \longrightarrow HFX$ at $x \in A$.

Given $\mathcal{A} \underset{G}{\overset{F}{\underset{\alpha \Downarrow}{\rightrightarrows}}} \mathcal{B} \underset{I}{\overset{H}{\underset{\beta \Downarrow}{\rightrightarrows}}} \mathcal{C}$

we have two ways of defining a composite, as either path in

$$\begin{array}{ccc} HF & \xrightarrow{\beta F} & IF \\ {\scriptstyle H\alpha}\Downarrow & & \Downarrow{\scriptstyle I\alpha} \\ HG & \xrightarrow[\beta G]{} & IG \end{array} \quad ;$$

(i.e. $I\alpha . \beta F$) pointing to IF

these agree by naturality of β

$$\begin{array}{ccc} HFX & \xrightarrow{\beta FX} & IFX \\ {\scriptstyle H\alpha_X}\downarrow & & \downarrow{\scriptstyle I\alpha_X} \\ HGX & \xrightarrow[\beta GX]{} & IGX \end{array}$$

at the morphism $\alpha_X : FX \to GX \in \mathcal{B}$

The resulting nat t. $\mathcal{A} \underset{IG}{\overset{HF}{\underset{\beta * \alpha \Downarrow}{\rightrightarrows}}} \mathcal{C}$

is called horizontal composite.

Remark Categories, Functors & natural transformations form a 2-category!

Equivalence of categories

- When are two categories *the same*?
- CAT is a category, so we can speak of isos of cats., but this is too strong a notion.
- Better notion: equivalence of cats.

Def^n) A natural transformation $\mathcal{A} \underset{G}{\overset{F}{\underset{\eta \Downarrow}{\rightrightarrows}}} \mathcal{B}$ is a natural isomorphism if it is an isomorphism in $[\mathcal{A}, \mathcal{B}]$.

We write $\eta : F \cong G$ for a nat. isomorphism.

Lemma $\eta : F \Rightarrow G$ is a nat. iso. $\iff$ $\eta_X : FX \to GX$ is an isomorphism in $\mathcal{B}$.

Proof Exercise.

Defn A functor $F : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if $\exists$ functor

$$G : \mathcal{B} \to \mathcal{A}$$

and natural isos

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} & & \\ & \overset{\alpha}{\cong} \searrow^{1_{\mathcal{A}}} & \downarrow G \; \beta & \cong \searrow^{1_{\mathcal{B}}} & \\ & & \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

.

Example

$$\mathbf{Mon} \xrightarrow{\Sigma} \mathbf{Cat}_{\bullet}$$ — 1-object categories & functors

not an iso as not surjective on obs.

$$\mathbf{Cat}_{\bullet} \xrightarrow{H} \mathbf{Mon}$$

$$C \longmapsto C(x,x)$$ where x is unique object of C.

Check $H\Sigma \cong 1$ &

$1 \cong \Sigma H$.

Adjoint Functors

- Key concepts in cat. theory: adjunctions

Def^n) An adjunction consists of functors

$$A \underset{U}{\overset{F}{\rightleftarrows}} B$$ together

with bijections

$$B(Fa, b) \xrightarrow{\varphi_{a,b}} A(a, Ub)$$

for each $a \in A$, $b \in B$ and these are natural in each variable.

- naturality in b means: given $Fa \xrightarrow{\alpha} b$ & $b \xrightarrow{\beta} b'$ we have $\varphi_{a,b'}(\beta \circ \alpha) = U\beta \circ \varphi_{a,b}(\alpha)$
- naturality in a means: given $Fa \xrightarrow{\alpha} b$ & $a' \xrightarrow{\beta} a$ we have $\varphi_{a',b}(\alpha \circ F\beta) = \varphi_{a,b}(\alpha) \circ \beta$.

Remark: Key point is that if we an adjunction, maps $Fa \longrightarrow b$ bijectively corresp. to maps $a \longrightarrow Ub$ in a natural way.

- In the examples below, we won't check naturality conds - you should do it!

Notation: We say that F is left adjoint to U, & write $F \dashv U$.

Examples

Free $\dashv$ Forgetful!

a) Consider

$F \dashv U$ $\quad$ $\mathrm{Vect} \underset{U \text{ forgetful}}{\overset{F \text{ free vector sp.}}{\rightleftarrows}} \mathrm{Set}$ $\quad$ ($\perp$)

- $FX = \{ \lambda_1 x_1 + \dots + \lambda_n x_n : x_1, \dots, x_n \in X, \lambda_1, \dots, \lambda_n \in k \}$

  the vector space with basis X

- At a function $f : X \to Y$,

  $Ff : FX \longrightarrow FY : \sum_i \lambda_i x_i \longmapsto \sum_i \lambda_i f(x_i)$ is obtained by linear extension.

- We have a bijection

  $$\mathrm{Vect}(FX, Y) \longrightarrow \mathrm{Set}(X, UY)$$
  $$\left(FX \xrightarrow{f} Y\right) \longmapsto \left(X \longrightarrow UY, \; x \longmapsto f(x)\right)$$

- This is a bij^n, whose inverse sends $g : X \to UY$ to the linear map $FX \longrightarrow Y$:

  $$\sum \lambda_i x_i \longmapsto \sum \lambda_i g(x_i).$$

- why inverse parts of a bij^n? since $f(\sum_i \lambda_i x_i) = \sum_i \lambda_i f(x_i)$ is linear.

b) Similarly, we have an adjunction

$$\mathbf{Mon} \underset{U = \text{forgetful}}{\overset{F = \text{free}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Set}$$

- Recall FX = list monoid .. $[x_1, \ldots, x_n] \in FX$
- Given $FX \xrightarrow{f} Y$, corresponding map

  $X \longrightarrow UY : x \longmapsto f[x]$ (word of length 1.)

- Given $X \xrightarrow{g} UY$, the corresponding map $\bar{g} : FX \longrightarrow Y$:

  $[x_1, \ldots, x_n] \longmapsto g(x_1) \cdot \ldots \ldots \cdot g(x_n)$

  where $\cdot$ denotes multiplication in monoid Y.

- Note that this def^n is forced on us since $[x_1, \ldots, x_n] = [x_1] \cdot \ldots \cdot [x_n]$ & $\bar{g}$ must preserve multiplication & sat. $\bar{g}[x] = g(x)$ for $x \in X$ in order to have a bij^n.

c) Similarly, the forgetful functors from Grp or Rng to Set have left adjoints, sending a set to free group / free ring ...
More generally, forgetful functors in universal algebra have left adjoints – we will see this soon.

Other examples

- The forgetful functor $U: \mathrm{Grp} \longrightarrow \mathrm{Mon}$ has a left adjoint, & a right adjoint R: this sends M to subgroup RM of invertible elements in M

If G a group, M a monoid, then a monoid map $U(G) \longrightarrow M$ takes elements of G to invertible elements of M, & so factors as $G \longrightarrow R(M)$ through $R(M) \hookrightarrow M$.

We obtain
$$\frac{U(G) \longrightarrow M}{G \longrightarrow R(M)}$$
a bij^n.

- We have adjoint functors

$$\mathrm{Top} \;\underset{I \,=\, \text{indiscrete}}{\overset{D \,=\, \text{discrete}}{\leftleftarrows \!\!\! \rightarrow}}\; \mathrm{Set} \qquad D \dashv U \dashv I$$

DX = set X with all subsets open.
IX = set X with $X, \emptyset$ open.
Then any function $X \to UY$ is cts
wrt discrete topology so

$$\frac{X \to UY}{DX \to Y} \quad \text{bij}^n.$$

Sim. any function $UX \to Y$ is cts
wrt indiscrete top. on Y

so
$$\frac{UX \to Y}{X \to IY}.$$

- Let $A \in \mathbf{Set}$. We have a functor

$$A\times - : \mathbf{Set} \longrightarrow \mathbf{Set}$$
$$X \longmapsto A \times X.$$

Now there is a bij^n between functions $A \times X \xrightarrow{f} Y$

&

Functions $X \xrightarrow{\bar{f}} \mathbf{Set}(A, Y) = Y^A$ (functions $A \to Y$)

where $\bar{f}_x : A \longrightarrow Y$
$$a \longmapsto f(x, a).$$

This process is called currying.

Therefore we have adjunction

$A\times - \dashv (-)^A$ where

$$(-)^A : \mathbf{Set} \longrightarrow \mathbf{Set}$$
$$X \longmapsto X^A.$$