L5 - Adjunctions ctd

Last week introduced adjunctions

$$\begin{tikzcd} B \arrow[r, phantom, "\perp"] \arrow[r, bend right=0, leftarrow, shift left=2, "F"] & A \arrow[l, shift left=2, "U"] \end{tikzcd}.$$

This time we study them further & some of their good properties.

- Let $\begin{tikzcd} \mathcal{B} \arrow[r, shift right=2, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift right=2, "F"'] \end{tikzcd}$ be an adjunction with bijection $\mathcal{B}(Fa,b) \xrightarrow{\varphi_{a,b}} \mathcal{A}(a,Ub)$.

- Taking $b = Fa$, we obtain

$$\mathcal{B}(Fa,Fa) \xrightarrow[\varphi_{a,Fa}]{} \mathcal{A}(a,UFa)$$

$$Fa \xrightarrow{1} Fa \longmapsto \underline{a \xrightarrow{\eta_a} UFa}$$

& η_a is called the <u>unit</u> of the adjunction.

We have
$$\begin{tikzcd} a \arrow[r, "\eta_a = \varphi_{a,Fa}(1)"] \arrow[d, "\alpha"'] \arrow[dr, "\varphi_{a,Fb}(F\alpha)" description] \arrow[dr, phantom, shift left=4, "="] \arrow[dr, phantom, shift right=4, "="] & UFa \arrow[d, "UF\alpha"] \\ b \arrow[r, "\eta_b = \varphi_{b,Fb}(1)"'] & UFb \end{tikzcd}$$
where the top right triangle commutes using <u>naturality</u> at
$$\begin{tikzcd} Fa \arrow[r, "1"] \arrow[dr, "F\alpha"'] \arrow[dr, phantom, shift left=3, "="] & Fa \arrow[d, "F\alpha"] \\ & Fb \end{tikzcd}$$

& the bottom left using <u>naturality</u> at
$$\begin{tikzcd} Fa \arrow[d, "F\alpha"'] \arrow[dr, "F\alpha"] & \\ Fb \arrow[r, "1"'] & Fb \end{tikzcd}$$
.

- Therefore $\eta : 1 \Rightarrow UF$ is a <u>natural transformation</u>.

- In fact, the components
$$\varphi_{a,b} : \mathcal{B}(Fa, b) \longrightarrow \mathcal{A}(a, Ub)$$
are determined by η :

For, given $Fa \xrightarrow{\alpha} b$, naturality at
$Fa \xrightarrow{1} Fa \xrightarrow{\alpha} b$ gives

(A) $\varphi_{a,b}(\alpha) = U\alpha \circ \varphi_{a,Fa}(1) = U\alpha \circ \eta_a$.

$\boxed{\text{Theorem 1}}$ There is a bijection betw. adjunctions $(F \dashv U, \varphi)$ and natural transformations $\eta : 1 \Longrightarrow UF$ with the universal property that:

given $a \xrightarrow{\alpha} Ub$ there exists a unique $Fa \xrightarrow{\bar{\alpha}} b$ such that

$$\begin{tikzcd} & UFa \arrow[dr, "U\bar{\alpha}"] & \\ a \arrow[ur, "\eta_a"] \arrow[rr, "\alpha"] & & Ub \end{tikzcd}$$
.

$\boxed{\text{Proof}}$ - By A above, if we have an adjunction then

$$\mathcal{B}(Fa,b) \xrightarrow{\varphi_{a,b}} \mathcal{A}(a,Ub)$$
$$\alpha \longmapsto U\alpha \circ \eta_A$$

so that $\blacktriangle$ just says that this map is a bijection.

- We have already showed that the assignment $(F,G,\varphi) \longmapsto (F,G,\eta)$ is injective, so it remains to show that each (F,G,η) satisfying $\blacktriangle$ arises from an adjunction.

- Given $\eta : 1 \Longrightarrow UF$ satisfying $\blacktriangle$ as above, we must show that the maps

$$\mathcal{B}(Fa,b) \xrightarrow{\varphi_{a,b}} \mathcal{A}(a,Ub)$$
$$Fa \xrightarrow{\alpha} b \longmapsto a \xrightarrow{\eta_a} UFa \xrightarrow{U\alpha} Ub$$

form an adjunction.

Certainly, by $\blacktriangle$, they are bijections, so it remains to check naturality.

This is straightforward. $\square$

- In fact, even less is required!

Theorem 2 An adjunction is specified by a functor
$U : \mathcal{B} \longrightarrow \mathcal{A}$ and
for each object $a \in \mathcal{A}$ an object $Fa \in \mathcal{B}$
and morphism $a \xrightarrow{\eta_a} UFa$
with the universal property

given $a \xrightarrow{\alpha} Ub$ there exists
a unique $Fa \xrightarrow{\bar{\alpha}} b$ such that

$$\begin{tikzcd} & UFa \arrow[dr, "U\bar{\alpha}"] & \\ a \arrow[ur, "\eta_a"] \arrow[rr, "\alpha"'] & & Ub \end{tikzcd}$$

Proof Certainly $\eta : 1 \Rightarrow UF$ as in Th. 1
gives rise to this.

Conversely, given the above, we define $F\alpha : Fa \to Fb$ as the unique map such that the square

$$\bullet \qquad \begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & UFa \arrow[d, "UF\alpha"] \\ b \arrow[r, "\eta_b"'] & UFb \end{tikzcd}$$

commutes. (Such a unique map exists by $\blacktriangle$.)

- Indeed we are forced to define $F\alpha$ this way in order for η to be natural.

- Therefore, we will be finished if we can show F is a functor.

- To see that F preserves identities, observe that

$$\begin{tikzcd} a \arrow[r, "\eta_a"] \arrow[d, "1_a"'] & UFa \arrow[d, "U1 = 1_{Fa}"] \\ a \arrow[r, "\eta_a"'] & UFa \end{tikzcd}$$

commutes, so $F1_a = 1_{Fa}$ by $\bullet$

- To see that $F\beta \circ F\alpha = F(\beta \circ \alpha)$ consider

$$
\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "//"] & UFa \arrow[d, "UF\alpha"] \arrow[dd, bend left=60, "U(F\beta \circ F\alpha)"] \\
b \arrow[r, "\eta_b"] \arrow[d, "\beta"'] \arrow[dr, phantom, "//"] & UFb \arrow[d, "UF\beta"] \\
c \arrow[r, "\eta_c"] & UFc
\end{tikzcd}
$$

Since the outside commutes, by we have $F\beta \circ F\alpha = F(\beta \circ \alpha)$.

Thus F is a functor & $\eta : 1 \Rightarrow UF$ a nat. t.

$\square$

Corollary

$U : B \longrightarrow A$ has a left adjoint

if:

$\forall a \in A \ \exists\, Fa \in B$ & $\eta_a : a \longrightarrow UFa$

with the universal property:

given $a \xrightarrow{\alpha} Ub$ there exists

a unique $Fa \xrightarrow{\bar{\alpha}} b$ such that

$$\begin{tikzcd} & UFa \arrow[dr, "U\bar{\alpha}"] & \\ a \arrow[ur, "\eta_a"] \arrow[rr, "\alpha"'] & & Ub \end{tikzcd}$$

.

Corollary

$U : \mathcal{B} \longrightarrow \mathcal{A}$ has a left adjoint

if:

$\forall a \in \mathcal{A}$ $\exists$ $Fa \in \mathcal{B}$ & $\eta_a : a \to UFa$

with the universal property:

given $a \xrightarrow{\alpha} Ub$ there exists a unique $Fa \xrightarrow{\bar{\alpha}} b$ such that

$$\begin{tikzcd}
 & UFa \arrow[dr, "U\bar{\alpha}"] & \\
a \arrow[ur, "\eta_a"] \arrow[rr, "\alpha"'] & & Ub
\end{tikzcd}$$
.

Remark) This is often easiest way to check a functor has a left adjoint in practise.

E.g. consider $U : \mathrm{Mon} \to \mathrm{Set}$ & a set X.

Have
$$\begin{aligned} X &\xrightarrow{\eta_X} UFX \\ x &\longmapsto [x] \end{aligned}$$
where FX is list monoid.

$$\begin{tikzcd} & UFX \arrow[dr, "U\bar{f}"] & \\ X \arrow[ur, "\eta_X"] \arrow[rr, "f"'] & & UM \end{tikzcd}$$

where
$$\bar{f}[x_1, \ldots, x_n] = fx_1 \cdot \ldots \cdot fx_n.$$

- Hence U has left adjoint F

Another important corollary of the above is that

Thm) Let $U : \mathcal{B} \to \mathcal{A}$. Then its left adjoint, if it exists, is unique up to natural isomorphism.

$\boxed{\text{Proof}}$ Suppose F_1, F_2 are left adj. to U, with units

$$a \xrightarrow{\eta_1 a} UF_1 a \quad \& \quad a \xrightarrow{\eta_2 a} UF_2 a$$

satisfying $\blacktriangle$.

Then by the u.p. $\blacktriangle$ of $F_1 a$, $\exists! \, k_a : F_1 a \to F_2 a$ such that

$$\begin{tikzcd} & UF_1 a \arrow[dr, "Uk_a"] & \\ a \arrow[ur, "\eta_1 a"] \arrow[rr, "\eta_2 a"'] & & UF_2 a \end{tikzcd} \quad \& \text{ likewise } \quad \begin{tikzcd} & UF_2 a \arrow[dr, "Ul_a"] & \\ a \arrow[ur, "\eta_2 a"] \arrow[rr, "\eta_1 a"'] & & UF_1 a \end{tikzcd}$$

& then $\blacktriangle$ again, twice applied, shows that k_a is an isomorphism, with inverse l_a.

For naturality, must show

$$\begin{tikzcd} F_1 a \arrow[r, "K_a"] \arrow[d, "F_1\alpha"'] \arrow[dr, phantom, "="] & F_2 a \arrow[d, "F_2\alpha"] \\ F_1 b \arrow[r, "K_b"'] & F_2 b \end{tikzcd}$$

which by u.p. of $F_1 a$

is to show that the two outer paths of

$$\begin{tikzcd} a \arrow[dr, "\eta_{1a}"] & & \\ & UF_1 a \arrow[r, "UK_a"] \arrow[d, "UF_1\alpha"] & UF_2 a \arrow[d, "UF_2\alpha"] \\ & UF_1 b \arrow[r, "UK_b"'] & UF_2 b \end{tikzcd}$$

agree.

The upper path is

$$\begin{aligned} UF_2\alpha \circ UK_a \circ \eta_{1a} &= UF_2\alpha \circ \eta_{2a} \quad \text{by nat.} \\ &= \eta_{2b} \circ \alpha \end{aligned}$$

The lower path is

$$\begin{aligned} UK_b \circ UF_1\alpha \circ \eta_{1a} &= UK_b \circ \eta_{1b} \circ \alpha \\ &= \eta_{2b} \circ \alpha . \quad \square \end{aligned}$$

Remark:

This says something incredible:
something very simple like
forgetful functor,
uniquely determines (up to iso)
something more complex:
free structures!

What do adjoint functors preserve?

- let $D: J \longrightarrow \mathcal{A}$ be a diagram.
- A cone over D consists of maps $L \xrightarrow{p_i} D_i$ for $i \in J$ sat.

$$\begin{tikzcd}
 & D_i \arrow[d, "D\alpha"] \\
L \arrow[ur, "p_i"] \arrow[dr, "p_j"'] \arrow[r, phantom, "="] & D_j
\end{tikzcd} \quad \text{for} \quad \begin{tikzcd} i \arrow[d, "\alpha"] \\ j \end{tikzcd}$$

- It is a limit cone (or just that L is the limit) if given a cone $(k_i : A \longrightarrow D_i)_{i \in J}$ $\exists!\ k : A \longrightarrow L$ such that $p_i \circ k = k_i$ for all i.

- If $U: \mathcal{A} \longrightarrow \mathcal{B}$ is a functor it takes the cone $(p_i : L \to D_i)_{i \in J}$ to a cone $(Up_i : UL \longrightarrow UD_i)_{i \in J}$ for UD.

Def) We say that U preserves the limit L of D if the cone $(UL \xrightarrow{Up_i} UD_i)_{i \in J}$ is a limit cone.

- Similarly, we can speak of a functor preserving colimits.

Theorem) Right adjoints preserve limits.

Proof) Consider

$$J \xrightarrow{D} A \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} B \quad \& \text{ a}$$

limit cone $(L \xrightarrow{p_i} D_i)_{i \in J}$.

We must show that

$(UL \xrightarrow{Up_i} UD_i)_{i \in J}$ is a limit cone.

- Consider cone $(x \xrightarrow{k_i} UD_i)_{i \in J}$.
- Using bijections $A(Fx, Di) \xrightarrow{\varphi} B(x, UDi)$ we obtain maps $Fx \xrightarrow{\varphi^{-1}k_i} D_i$ & claim these form a cone to D:

we must show

$$\begin{tikzcd} & D_i \arrow[d, "D\alpha"] \\ FX \arrow[ur, "\varphi^{-1}k_i"] \arrow[dr, "\varphi^{-1}k_j"'] \arrow[r, phantom, "="] & D_j \end{tikzcd}$$

but this is equivalent to showing images of these maps $FX \rightrightarrows D_j$ under $A(FX, D_j) \xrightarrow{\varphi} B(X, UD_j)$ are equal.

- Well $\varphi\varphi^{-1}k_j = k_j$.
- $\varphi(D\alpha \circ \varphi^{-1}k_i) = UD\alpha \circ \varphi\varphi^{-1}k_i = UD\alpha \circ k_i$

by naturality of φ

so their images are the two paths of

$$\begin{tikzcd} & UD_i \arrow[d, "UD\alpha"] \\ X \arrow[ur, "k_i"] \arrow[dr, "k_j"'] \arrow[r, phantom, "="] & UD_j \end{tikzcd}$$

which agree, since the k_i are a cone

- Since the maps $\varphi^{-1}k_i : FX \longrightarrow D_i$ form a cone we obtain a unique $l : FX \longrightarrow L$

such that

$$\begin{tikzcd} FX \arrow[r, "l"] \arrow[dr, "\varphi^{-1}k_i"'] \arrow[r, phantom, "=", shift right=4] & L \arrow[d, "p_i"] \\ & D_i \end{tikzcd}$$

for all i.

- Using the bijection $A(FX, L) \xrightarrow{\varphi} B(X, UL)$ this corresponds to a map

$X \xrightarrow{\varphi_\ell} UL$ & the equations (*) corresp. to the equations

(**)
$$\begin{tikzcd} X \arrow[r, "\varphi_\ell"] & UL \arrow[d, "Up_i"] \\ K_i \arrow[r] & UD_i \end{tikzcd}$$
using naturality of φ.

- In partic, $\varphi_\ell : X \to UL$ is the unique map s.t. (**) commutes; therefore $(UL \xrightarrow{Up_i} UD_i)_{i \in J}$ is a limit cone. $\square$

- For example, $U : \mathbf{Grp} \to \mathbf{Set}$ preserves products, equalisers, terminal object etc. More generally, forgetful functors from algebraic cats to $\mathbf{Set}$ preserve all limits.

Dually

Theorem

Left adjoints preserve colimits.

Exercise

Prove that forgetful functor $U : \mathrm{Grp} \longrightarrow \mathrm{Set}$ does not have a right adjoint (i.e. is not a left adjoint.)

Example

- Let Field = cat. of fields & homomorphisms : preserve addition, mult., 0 & 1.
- Fields are commutative rings s.t. $0 \neq 1$ &
  $x \neq 0 \Rightarrow \exists y : xy = 1$

These equations involve negation
$\Rightarrow$ not universal algebra

- The cat. of Fields is bad.

I will show the forgetful functor $U : \text{Field} \longrightarrow \text{Set}$ does not have a left adjoint.

- If it did have left adj F, then F would send the init. ob $\phi \in \text{Set}$ to an init object in Field (as left adjoints preserves colimits). So it suffices to show Field does not have an initial object.

- Firstly, let $f: R \to S$ be a field homomorphism. We claim f is injective.

Indeed, suppose $fx = 0$ for $x \neq 0$. Then $\ker f \hookrightarrow R$ is an ideal of R, non-zero, so as R is a field, $\ker f = R$.

Therefore $f1 = 0$ so $1 = f1 = 0$ which is a contradiction; hence f is injective.

- Let $\mathbb{Z}_p$ Field of integers modulo p, so $p.1 = 0$, for p a prime.
- If F is initial, $\exists$

$$\begin{tikzcd} & \mathbb{Z}_p \\ F \arrow[ur, "\mathrm{inj}"] \arrow[dr, "\mathrm{inj}"'] & \\ & \mathbb{Z}_q \end{tikzcd}$$

for p, q coprime.

- Since $F \hookrightarrow \mathbb{Z}_p, \mathbb{Z}_q$ are injective they reflect equations $p.1 = 0$ & $q.1 = 0$, so these equations hold in F. But as p, q coprime $\underline{1 = np + mq}$, by

Bezout's identity, so in F,

$$1 = 1.1 = (np+mq).1 = n(p1) + m(q1) = n0 + m0 = 0.$$

Hence $1=0$ in F, so F not a field. $\square$

Note: In universal algebra all forgetful functors have left adjoints.

- We have seen: right adjoints preserve limits

Proposition

Right adjoints preserve monos.

Proof Let $U : A \to B$ have left adj. F, and consider mono $a \xrightarrow{f} b \in A$.

Consider $x \underset{v}{\overset{u}{\rightrightarrows}} Ua \xrightarrow{Uf} Ub$

satisfying $Uf.u = Uf.v$. We must show $u = v$.

Then we obtain maps

$$Fx \underset{\varphi^{-1}v}{\overset{\varphi^{-1}u}{\rightrightarrows}} a \xrightarrow{f} b$$ &

the diagram commutes by naturality of φ.

Since f is mono, therefore $\varphi^{-1}u = \varphi^{-1}v$.

Therefore $u = v$ so that uf is mono, as claimed. $\square$

Dually

Proposition

Left adjoints preserve epis.