Lect 7

Last time:

- Ω a signature
- Cat $\Omega\text{-Alg} \xrightarrow{U_\Omega} \text{Set}$
- $T_\Omega(X)$ Ω-alg of terms in variables X
- Free Ω-alg on X : $F_\Omega \dashv U_\Omega$
  & $T_\Omega(X) = U_\Omega F_\Omega(X)$
- $$\begin{tikzcd} & T_\Omega(X) \arrow[dr, "U\bar{a}"] & \\ X \arrow[ur, "\eta"] \arrow[rr, "a"'] & & UA \end{tikzcd}$$
- Ω-Equation in variables is a pair $(s,t) \in T_\Omega(X)^2$.
- $A \models s=t$ if $\forall a : X \to UA$, $\bar{a}(s) = \bar{a}(t)$. (A satisfies the eq^n)

  or $\forall F_\Omega(X) \xrightarrow{f} A \in \Omega\text{-Alg}$, $f(s) = f(t)$.

- $(\underset{\text{sig}}{\Omega}, \underbrace{E})$ set of equations
- (Ω, E)-alg is an Ω-alg satisfying the equations in E.

<u>Examples</u>

- For $\Omega = (\cdot, e)$ &

$$E = \left\{ \begin{array}{r} (x.y).z = x.(y.z) \\ x.e = x \\ e.x = e \end{array} \right\}$$

an (Ω, E)-alg is a <u>monoid</u>.

For $\Omega' = (\cdot, e, \underbrace{(-)^{-1}})$ unary op.

& $E' = E \cup \{ x.x^{-1} = e,\ x^{-1}.x = e \}$

then (Ω, E)-algebra is a <u>group</u>.

Def^n) For (Ω, E) as above, we define
$(\Omega,E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$
as the Full subcategory of (Ω,E)-algebras.

- We obtain a composite Forgetful Functor to Set, as depicted below

$$\begin{tikzcd}
(\Omega,E)\text{-Alg} \arrow[rr, hook, "i"] \arrow[dr, "U"'] & {} \arrow[d, phantom, "="] & \Omega\text{-Alg} \arrow[dl, "U_\Omega"] \\
& \text{Set} &
\end{tikzcd}$$

.

Examples

- When I spoke of "algebraic categories" earlier in course, the precise meaning is cat. of the form (Ω, E)-Alg.
This framework captures all of the examples we have been talking about –
Vect, Grp, Ring, Mon, G-Set.

Today: study good props of cats of form Ω-Alg & (Ω,E)-Alg.

- Firstly (today) we look at Ω-Alg – the case of (Ω,E)-Alg follows easily from Ω-Alg.
- In particular, will study limits, kernels and quotients.

Firstly, subalgebras, homomorphic images & image factorisation.

Subalgebras

Def) - Let $A \in \Omega$-Alg. A subalgebra $B \hookrightarrow A$ of A is a subset B of A such that - if $s \in \Omega_n$ & $b_1, \dots, b_n \in B$ then $s^A(b_1, \dots, b_n) \in B$.

- In particular, B is then an Ω-alg & the incl. $B \hookrightarrow A$ a injective homomorphism.

Homomorphic images

Def^n) Let $A \in \Omega$-Alg. A homomorphic image of A is a surjective homomorphism
$$f : A \twoheadrightarrow B .$$
means surjective $\twoheadrightarrow$

- Let $f : A \longrightarrow B \in \Omega$-Alg.

Then let $\operatorname{im} f = \{ b \in B : \exists a \in A \text{ with } fa = b \}$

Then $\operatorname{im} f \hookrightarrow B$ is a subalgebra of B:

indeed, if $s \in \Omega_n$, $b_1 = fa_1, \ldots, b_n = fa_n$,
then $s(b_1, \ldots, b_n) = s(fa_1, \ldots, fa_n)$
$= fs(a_1, \ldots, a_n) \in \operatorname{im} f$.

Image Factorisation

- In particular, we obtain a factorisation

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \text{ in } \Omega\text{-Alg} \\
& \operatorname{im} f \arrow[ur, "i"'] &
\end{tikzcd}$$

$$a \mapsto fa$$

homomorphic image

subalgebra inclusion

- Later, we will look at the first isomorphism theorem which explains how to view $\operatorname{im} f$ as a quotient.

Limits of Ω-algebras

Proposition

Ω-Alg has (infinite) products and equalisers & $U : \Omega\text{-Alg} \longrightarrow \text{Set}$ preserves them.

(Remark: these generate all limits - not proved in course.)

Proof

- Consider a set I and family $(A_i)_{i \in I}$ of Ω-algebras. (i.e. $A : I \longrightarrow \Omega\text{-Alg}$)
- Their product as sets is the direct product

$$\prod_{i \in I} A_i = \{ \bar{a} = (a_i)_{i \in I} : a_i \in A_i \} \xrightarrow{p_i} A_i$$
$$(a_i)_{i \in I} \longmapsto a_i$$

- We want to show that $\prod_{i \in I} A_i$ has the structure of Ω-algebra such that each p_i is a homomorphism:

this says given $s \in \Omega_n$ and $\bar{a}^1, \dots, \bar{a}^n$ we have $s(\bar{a}^1, \dots, \bar{a}^n)_i = s^{A_i}(\bar{a}^1_i, \dots, \bar{a}^n_i) \in A_i$.

In other words, we are forced to equip $\prod A_i$ with component-wise Ω-algebra structure.

- Given Ω-alg. B & homs $(f_i : B \longrightarrow A_i)_{i \in I}$ we have a unique function
$$f : B \longrightarrow \prod_{i \in I} A_i \text{ such that } p_i \circ f = f_i \, ;$$
namely $(fb)_i = f_i(b)$.

- Must check f is a homomorphism :
$$f s(b^1, \ldots, b^n) = s(fb^1, \ldots, fb^n) \in \prod_{i \in I} A_i$$

$$f s(b^1, \ldots, b^n)_i \stackrel{\text{def}}{=} f_i s(b^1, \ldots, b^n) \stackrel{\text{hom}}{=} s(f_i b^1, \ldots, f_i b^n)$$

$$s(fb^1, \ldots, fb^n)_i \stackrel{\text{def}}{=} s((fb^1)_i, \ldots, (fb^n)_i) \stackrel{\text{def}}{=} s(f_i b^1, \ldots, f_i b^n)$$

• Given $A \overset{f}{\underset{g}{\rightrightarrows}} B$ their equaliser is $E = \{x \in A : fx = gx\} \xhookrightarrow{i} A$ <u>in Set</u>.

In fact, E is a <u>subalgebra</u> of A:

if $s \in \Omega_n$ & $x_1, \dots, x_n$ st $fx_i = gx_i$ then

$$fs(x_1,\dots,x_n) \underset{\text{f a homom.}}{=} s(fx_1,\dots,fx_n) \underset{\text{assumption}}{=} s(gx_1,\dots,gx_n) \underset{\text{g a homom.}}{=} gs(x_1,\dots,x_n)$$

In partic., $i : E \hookrightarrow A$ is a homomorphism and easy to check <u>univ. prop.</u> of the equaliser. $\square$

Quotients of congruences

- What about colimits?
- Key sort - quotients by congruences.

Congruences generalise
- equiv. rels for sets
- normal subgroups of groups
- (2-sided) ideals for rings

Def) Let A be an Ω-algebra. An equivalence relation $E \subseteq A^2$ is called a congruence if E is a subalgebra of A^2.

- In elementary terms, a cong. is an equivalence relation E such that $s \in \Omega_n$, $(x_1,y_1) \in E, \ldots, (x_n,y_n) \in E$, $(s(x_1,\ldots,x_n), s(y_1,\ldots,y_n)) \in E$.
- I will write xEy to mean $(x,y) \in E$.

- If $E \xhookrightarrow{i} A^2$ is a congruence, can form diagram $E \xhookrightarrow{i} A^2 \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$ in Ω-Alg

& so obtain $E \overset{d}{\underset{c}{\rightrightarrows}} A \in \Omega\text{-Alg}$

where $d(x,y) = x$, $c(x,y) = y$.

- We will form coequaliser

$$E \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{p} A/E \text{ in } \Omega\text{-Alg}$$

- elements of A/E are equiv. classes $[a]$ with $p(a) = [a] = \{x : xEa\}$.

- Observe p is surjective. Therefore if p is to be a homomorphism we are forced to define

$$s^{A/E}([a_1], \ldots, [a_n]) = [s^A(a_1, \ldots, a_n)].$$

- Is this well defined?

Suppose $[b_1] = [a_1], \ldots, [b_n] = [a_n]$. Then $b_1 E a_1, \ldots, b_n E a_n$ so as E a congruence we have

$$s(b_1, \ldots, b_n) \; E \; s(a_1, \ldots, a_n)$$

so $[s(b_1, \ldots, b_n)] = [s(a_1, \ldots, a_n)]$ as required.

- In particular, A/E is a Ω-algebra & $p : A \longrightarrow A/E$ a surjective homomorphism.

Proposition

$$E \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{p} A/E$$ is a coequaliser in Ω-Alg.

Proof

- Firstly if $(x,y) \in E$ then
  $pd(x,y) = [x] = [y] = pc(x,y)$
  so $pd = pc$.
- Given $A \xrightarrow{f} B$ with $fd = fc$.
  This means precisely that if $(x,y) \in E$ then $fx = fy$.
  Therefore $[x] = [y] \implies fx = fy$.
- Therefore we can extend f along p by

$$\begin{tikzcd} A \arrow[r, two heads, "p"] \arrow[rr, bend right, "f"'] & A/E \arrow[r, "\bar{f}"] & B \end{tikzcd}$$

  where $\bar{f}[a] = fa$
- Clearly $\bar{f}$ is a homomorphism, since f is.
- Since p is surjective, $\bar{f}$ is only map extending f along p $\square$

Kernels

Def) Let $f : A \to B \in \Omega\text{-Alg}$.
The kernel of f is the congruence
$$K_f = \{(x,y) : fx = fy\} \hookrightarrow A^2.$$

- Easy to see this is a congruence: check it!

Categorically, K_f is the pullback

$$\begin{tikzcd}
(x,y) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & x \\
& K_f \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] & \\
& A \arrow[r, "f"'] & B & \\
y & & &
\end{tikzcd}$$

so, in particular, we have $fd = fc$.

Therefore, we get a unique factorisation of f through the coequaliser

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\
& A/K_f \arrow[ur, "\bar{f}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, mapsto] & {[a]} \arrow[r, mapsto] & fa
\end{tikzcd}$$

First isomorphism theorem

Given $f : A \longrightarrow B \in \Omega\text{-Alg}$

the induced map

$t : A/K_f \longrightarrow \mathrm{im}f : [a] \longmapsto fa$

is an isomorphism in Ω-Alg.

$\boxed{\text{Proof}}$ Since $K_f \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{e} \mathrm{im}f$

commutes as $(x,y) \in K_f \iff fx = fy$ we get ! map t from the coequaliser

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "e"'] & A/K_f \arrow[d, "t"] \\ & \mathrm{im}f \end{tikzcd}$$

commutes.

This says $t[a] = fa$.

- For surj., if $b \in \mathrm{im}f$, then $b = fa$ so $t[a] = fa = b$.
- For inj, suppose $t[a] = t[b]$.
  That is, $fa = fb$.
  Then $(a,b) \in K_f$ so $[a] = [b]$. $\square$

Corollary

- If $f : A \twoheadrightarrow B \in \Omega\text{-Alg}$ is surjective, then $A/K_f \cong B$.

In particular,

$$K_f \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{f} B$$ is a coequaliser.

~~Proof~~ In this case $\operatorname{im} f = B$.

Hence $A/K_f \xrightarrow{t} B$ an iso by prev. result.

Now $K_f \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{p} A/K_f$ a coeq.

As t is an iso & coequalisers invariant up to iso

$$\begin{tikzcd} K_f \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & A \arrow[r, "p"] \arrow[rr, bend right, "= \; f"'] & A/K_f \arrow[r, "t"] & B \end{tikzcd}$$

a coequaliser. $\square$

Generating congruences & colimits

Def^n) Let A be an Ω-algebra. Let $\mathrm{Cong}(A)$ denote the set of congruences on A.

Rk) $\mathrm{Cong}(A)$ is a poset, ordered by inclusion.

Lemma) If $(E_i)_{i \in I}$ is a set of congruences, then $\bigcap_{i \in I} E_i$ is a congruence.

Proof) Routine check.

Lemma) Let $X \subseteq A \times A$. Then $\exists$ a smallest cong E_X containing X.

$\boxed{\text{Proof}}$ Consider the set

$I = \{ E \text{ a congruence on } A : X \subseteq E \}$
which is non-empty as it contains $A \times A$.

Form the congr. $E_X = \bigcap_{E \in I} E$.

- Then as $X \subseteq E$ all $E \in I$, $X \subseteq E_X$.
- Also $E_X \subseteq E$ for any E cont. X, by construction.

Proposition

Ω-Alg has coequalisers.

~~Proof~~ Consider $X \overset{f}{\underset{g}{\rightrightarrows}} Y \in \Omega\text{-Alg}$.

Let $E_{f,g} \subseteq Y \times Y$ be the congruence generated by $\{(fx, gx) : x \in X\}$.

- Then we have a comm. diagram:

$$\begin{tikzcd}
(fx,gx) & E_{f,g} \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & Y \arrow[r, "p"] & Y/E_{f,g} \\
x \arrow[u, mapsto] & X \arrow[u] \arrow[ur, shift left, "f"] \arrow[ur, shift right, "g"'] & &
\end{tikzcd}$$

where $d(x,y) = x$ & $c(x,y) = y$.

- We proved the top row is a coequaliser, and we must show that the bottom row $pf = pg$ is too.

- So let $Y \xrightarrow{h} Z \in \Omega\text{-Alg}$ sat $hf = hg$.

Recall the congruence $\ker h = \{(a,b) : ha = hb\}$ on Z.

Since $hfx = hgx$, $\{(fx, gx) : x \in X\} \subseteq \ker h$

so that $E_{f,g} \subseteq \ker h$:

That is, given $(x,y) \in E_{f,g}$ we have $hx = hy$, which is to say that

$$E_{f,g} \overset{d}{\underset{c}{\rightrightarrows}} X \xrightarrow[h]{} Z \text{ commutes.}$$

- Therefore we obtain unique $Y/E_{f,g} \xrightarrow{\bar{h}} Z$ from the coeq. of d & c such that

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[dr, "f"'] & Y/E_{f,g} \arrow[d, "\bar{f}"] \\ & Z \end{tikzcd}.$$

In particular $Y/E_{f,g}$ is also the coequaliser of f & g. $\square$

Example

- We can "present" algebraic structures using coequalisers.
- Often one speaks of a an algebra

$$\langle x_1, \dots, x_n \mid s_1 = t_1, \dots, s_m = t_m \rangle$$

generated by elements $x_1, \dots, x_n$ subject to equations $s_i = t_i$ where

$$s_i, t_i \in T_\Omega\{x_1, \dots, x_n\} = U_\Omega F_\Omega\{x_1, \dots, x_n\}$$

- These correspond to functions

$$m \underset{t}{\overset{s}{\rightrightarrows}} U_\Omega F_\Omega n \qquad \text{for } n = \{x_1, \dots, x_n\}$$

and so

$$F_\Omega m \underset{\bar{t}}{\overset{\bar{s}}{\rightrightarrows}} F_\Omega n \in \Omega\text{-Alg},$$

whose coequaliser is

$$\langle x_1, \dots, x_n \mid s_1 = t_1, \dots, s_n = t_n \rangle.$$