Lecture 8

Last time:

- subalgebras, homomorphic images
- limits of Ω-algebras
- congruences & kernels
- quotients by congruences
- first isomorphism theorem
- coequalisers of Ω-algebras

Proposition

The category Ω-Alg has all coproducts.

Proof

- Consider Ω-algs $(X_i)_{i\in I}$.
- We can form the coproduct $\sum_{i\in I} X_i$ as sets, which is the disjoint union, and then the free Ω-alg. on this.

· Have inclusions

$$\begin{tikzcd} X_i \arrow[r, "p_i"] \arrow[rr, bend right, "k_i"'] & \sum_{i\in I} X_i \arrow[r, "\eta"] & F_\Omega\left(\sum_{i\in I} X_i\right) \end{tikzcd}$$ but

the problem is that the K_i are not Ω-alg homomorphisms.

- Have inclusions

$$\begin{tikzcd} X_i \arrow[r, "p_i"] \arrow[rr, bend right, "k_i"'] & \sum_{i\in I} X_i \arrow[r, "u"] & Fr(\sum_{i\in I} X_i) \end{tikzcd}$$

but the problem is that the k_i are not Ω-alg homomorphisms.

- To fix this, we consider the congruence E on $Fr(\sum_{i\in I} X_i)$ generated by

$$\left\{ (k_i(s(x_1,\dots,x_n)),\ s(k_i x_1,\dots k_i x_n)) : \begin{array}{l} n\in\mathbb{N}, s\in\Omega_n, \\ i\in I, \\ x_1,\dots,x_n\in X_i \end{array} \right\}$$

& then each composite

$$\begin{tikzcd} X_i \arrow[r, "k_i"] \arrow[rr, bend right, "l_i"'] & Fr(\sum X_i) \arrow[r, "p"] & Fr(\sum X_i)/E = C \end{tikzcd}$$

is a homomorphism.

Indeed, $p(k_i(s(x_1,\dots,x_n))) =$ as p quotient map

$p(s(k_i x_1,\dots,k_i x_n)) =$

$s(pk_i x_1,\dots,pk_i x_n)$ as p hom.

- Then given $(X_i \xrightarrow{f_i} A)_{i\in I}$ $\exists!$

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "p_i"'] & UA \\ \Sigma X_i \arrow[ur, dashed, "\exists! f"'] & \end{tikzcd} \in \mathbf{Set}$$

- This extends uniquely along η to
  $\bar{f} : F_\Omega(\Sigma X_i) \longrightarrow A \in \Omega\text{-Alg}$.
- Then

$$\begin{tikzcd} X_i \arrow[r, "k_i"] \arrow[dr, "f_i"'] & F_\Omega(\Sigma X_i) \arrow[d, "\bar{f}"] \\ & A \end{tikzcd}$$

Moreover as each f_i a homomorphism, $\bar{f}$ identifies the elements

$$\left. \big(k_i(s(x_1,\dots,x_n)),\ s(k_i x_1,\dots,k_i x_n) \big) : \begin{array}{l} n\in\mathbb{N}, s\in\Omega_n, \\ i\in I, \\ x_1,\dots,x_n\in X_i \end{array} \right\}$$

of E; hence $\bar{f}$ factors uniquely through $F_\Omega(\Sigma X_i)/E$ as required $\square$

Since all colimits can be constructed from coproducts & coequalisers, we have:

Corollary

Ω-Alg has all colimits.

providing the dual to the easier result we showed that Ω-Alg has all limits.

- Before turning to closure properties of (Ω, E)-algebras, it will be useful to consider <u>projectivity</u>.

Def) An (Ω, E)-alg A is <u>projective</u> if given $B \xrightarrow{f} C \in (\Omega, E)$-Alg <u>surjective</u> & $A \xrightarrow{g} C$, $\exists$ $A \xrightarrow{\bar{g}} B$ such that

$$\begin{tikzcd} A \arrow[r, "\bar{g}"] \arrow[dr, "g"'] & B \arrow[d, "f"] \\ & C \end{tikzcd}$$

commutes.

Proposition

Free algebras in Ω-Alg are projective.

Proof

Given
$$\begin{tikzcd} & B \arrow[d, "f"] \\ F_\Omega X \arrow[r, "g"'] & C \end{tikzcd}$$
in Ω-Alg, consider

$$\begin{tikzcd} & & UB \arrow[d, "Uf"] \\ X \arrow[r, "\eta"] \arrow[urr, bend left, "t = s.Ug.\eta"] & U_\Omega F_\Omega X \arrow[r, "Ug"] & UC \arrow[u, bend left, "\exists s"] \end{tikzcd}$$

where $sc \in f^{-1}(c)$ for each $c \in C$ exists since f is surjective.

- Then $\exists!\ F_\Omega X \xrightarrow{\bar{t}} B$ such that

$$\begin{tikzcd} & U_\Omega F_\Omega X \arrow[dr, "U\bar{t}"] \arrow[d, phantom, "="] & \\ X \arrow[ur, "\eta_X"] \arrow[rr, "t"'] & & UB \end{tikzcd}$$

- Then
$$\begin{tikzcd} & U_\Omega F_\Omega X \arrow[d, "U\bar{t}"] \\ X \arrow[ur, "\eta_X"] \arrow[r, "t"] \arrow[d, "\eta_X"'] & U_\Omega B \arrow[d, "Uf"] \\ U_\Omega F_\Omega X \arrow[r, "Ug"] & U_\Omega C \end{tikzcd}$$
so $f.\bar{t} = g$ by freeness.

$\square$

(Ω, E)-Algebras - Closure properties

Proposition

The full subcategory $(\Omega,E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ is closed under products, subalgebras and quotients (aka homomorphic images)

Proof

- Let $(s,t) \in E$. Then $s,t \in F_\Omega X$ some X.
- Then $A \models s = t \iff$ each $f: F_\Omega(X) \to A \in \Omega\text{-Alg}$ satisfies $f(s) = f(t)$
- Consider product $\prod_{i\in I} A_i \xrightarrow{p_i} A_i \in \Omega\text{-Alg}$ where each A_i an (Ω,E)-algebra, & consider $F_\Omega(X) \xrightarrow{f} \prod_{i \in I} A_i \in \Omega\text{-Alg}$.
  - Then $\underset{f(s)_i}{\underset{\shortparallel}{p_i f(s)}} = \underset{f(t)_i}{\underset{\shortparallel}{p_i f(t)}}$ as $A_i \in (\Omega,E)\text{-Alg}$.
  - But then $f(s) = f(t)$ as agree in each comp.
  - So $\prod A_i$ an (Ω,E)-alg.

- Let $A \in (\Omega, E)$-Alg & $B \overset{j}{\hookrightarrow} A$ a subalgebra.
- Consider $Fr(X) \xrightarrow{f} B$.
  As $jf \in \Omega$-Alg, $jf(s) = jf(t)$.
  As j injective, $f(s) = f(t)$.
  Hence $B \in (\Omega, E)$-Alg.

- Let $A \overset{p}{\twoheadrightarrow} B \in \Omega$-Alg be surjective
  & A an (Ω, E)-algebra.
- Consider $Fr X \xrightarrow{f} B \in \Omega$-Alg.
- By projectivity of $Fr X$,

$$\begin{tikzcd} & A \arrow[d, "p"] \\ Fr X \arrow[ur, dashed, "\exists \bar{f}"] \arrow[r, "f"'] & B \end{tikzcd} \in \Omega\text{-Alg}.$$

  Since A an (Ω, E)-Alg, $\bar{f}(s) = \bar{f}(t)$
  Hence $f(s) = p\bar{f}(s) = p\bar{f}(t) = f(t)$
  as required. $\square$

Corollary

(Ω, E)-Alg $\hookrightarrow$ Ω-Alg closed under limits & coequalisers.

~~Proof~~ • Since limits can be constructed from products & equalisers, enough to establish these cases.

• We know (Ω,E)-Alg closed under prods.

• Equalisers in Ω-Alg are subalgebras. Since (Ω,E)-Alg closed under these two, closed under equalisers.

• Given $A \underset{g}{\overset{f}{\rightrightarrows}} B \in (\Omega,E)$-Alg, the coequaliser in Ω-Alg is

$$B \xrightarrow{p} B/E_{f,g}$$

but since this is surjective, & (Ω,E)-Alg closed under homomorphic images, $B/E_{f,g} \in (\Omega,E)$-Alg too. $\square$

$\boxed{\text{Remark}}$ A congruence on an (Ω, E)-alg A is, by definition, a congruence on the Ω-algebra A.

This makes sense since such an $E \subseteq A \times A$ is automatically an (Ω, E)-alg too, as it is a subobject of a product of (Ω, E)-algebras.

Also have

First isomorphism theorem

Given $f : A \longrightarrow B \in (\Omega, E)\text{-Alg}$ the induced map

$t : A/K_f \longrightarrow \operatorname{im} f : [a] \longmapsto fa$

is an isomorphism in $(\Omega, E)\text{-Alg}$.

Proof) As (Ω, E)-Alg closed under images (subalgebras), congruences & quotients, this follows from First iso thm for Ω-algebras.

Free (Ω, E)-algebras

We would like to prove

Prop^n The inclusion $(\Omega,E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ has a left adjoint.

Since $(\Omega,E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ is closed under products & <u>subobjects</u>, a more general statement is:

Prop^n Let $\mathcal{C} \xhookrightarrow{j} \Omega\text{-Alg}$ be a full subcategory closed under products & subalgebras. Then j has a left adjoint.

- In order to prove it, we need to consider, for an Ω-alg A, the collection $Q(A)$ of surjections $A \twoheadrightarrow B$ where B is any Ω-algebra.

- Problem is that $Q(A)$ is a proper class – just consider maps to each 1 element Ω-algebra.

  - let us say that $(f,B) \sim (g,C)$ if $\exists$ iso $B \xrightarrow{h} C$ such that

$$
\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[dr, two heads, "g"'] & B \arrow[d, "h"] \\
& C
\end{tikzcd}
$$

  - By the first iso. theorem, $(f,B) \cong (A/\ker f, p_{\ker f})$ so we have a surj. function

$$
\begin{aligned}
\mathrm{Cong}(A) &\longrightarrow Q(A)/\sim \\
E &\longmapsto A \to A/E
\end{aligned}
$$

(which is in fact a bij^n).

- Since $\mathrm{Cong}(A) \subseteq \mathrm{Powerset}(A \times A)$ is a set, therefore

  $Q(A)/\sim$ is a set.

Prop^n Let $\mathcal{C} \overset{j}{\hookrightarrow} \Omega\text{-Alg}$ be a full subcategory closed under products & subalgebras.
Then j has a left adjoint.

Proof • Given $X \in \Omega\text{-Alg}$, need $RX \in \mathcal{C}$ & $X \xrightarrow{\eta_X} RX$ such that:
given $X \xrightarrow{f} Y$ with $Y \in \mathcal{C}$ $\exists!$ $RX \xrightarrow{\bar{f}} Y$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\eta_X"'] & & Y \\ & RX \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

- Since $\mathbb{Q}(X)/\sim$ is a set, consider a rep $\{X \overset{p_i}{\twoheadrightarrow} X_i : i \in I\}$ of each e-class; thus each surjection $X \overset{k}{\twoheadrightarrow} Z$ is equiv. to one of the p_i's.

- Consider

$$\begin{tikzcd} X \arrow[rr, "\exists ! p"] \arrow[drr, "p_i"'] & & \prod_{i \in I} X_i \arrow[d, "\pi_i"] \\ & & X_i \end{tikzcd}$$

- Then $\prod_{i \in I} X_i \in \mathcal{C}$ as closed under prods.

- Now factor

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, two heads, "\eta"'] & & \prod_{i \in I} X_i \\ & RX = \mathrm{im}(p) \arrow[ur, hook] & \end{tikzcd}$$

Then $RX \in \mathcal{C}$ as closed under subalgs.

- Then consider $X \xrightarrow{f} Y$ with $Y \in \mathcal{C}$.

- Factor as $X \overset{u}{\twoheadrightarrow} \mathrm{im} f \overset{v}{\hookrightarrow} Y$.

Then $\mathrm{im} f \in \mathcal{C}$ as closed under subalgs.

- Then

$$\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[rrrd, two heads, "u"] \arrow[dd, "\eta_X"'] \arrow[ddr, "p"] \arrow[ddrr, "p_i"] & & & Y \\
& & & \mathrm{im} f \arrow[u, hook, "v"'] \\
RX \arrow[r] \arrow[rrruu, bend right=60, "\bar{f}"'] & \prod X_i \arrow[r, "\pi_i"'] & X_i \arrow[ur, "s"'] &
\end{tikzcd}$$

by above

as $\mathrm{im} f \in \mathcal{C}$

- Moreover $\bar{f}$ is unique factorisation since η_X is surjective $\Rightarrow$ epi. $\square$

<u>Corollary</u>

The inclusion $(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ has a left adjoint.

Remark

- Here is another construction of the adjoint:
- let A be an Ω-alg & $E(X)$ the set of equations in E in variables X.
- Problem is: given $(s,t) \in E(X)$, there may exist $f : F_\Omega X \longrightarrow A$ st. $fs \neq ft$.
- So consider the set

$$\{(f(s), f(t)) \in A^2 : (s,t) \in E(X),\ f : F_\Omega X \to A\}$$

& let $\bar{E} \subseteq A^2$ be the congruence it generates

- Now form $p : A \twoheadrightarrow A/\bar{E}$.
- Now consider $FX \xrightarrow{k} A/E$. Must show $k(s) = k(t)$. But as p surj can

Find

$$\begin{tikzcd} FX \arrow[d, "\bar{k}"'] \arrow[dr, "k"] \arrow[dr, phantom, shift right=2, "="] & \\ A \arrow[r, "p"'] & A/E \end{tikzcd}$$

(FX projective)

But then $(\bar{k}(s), \bar{k}(t)) \in \bar{E}$ so

$k(s) = p\bar{k}(s) = p\bar{k}(t) = k(t) \Rightarrow$ A/E an (Ω, E)-alg.

- Easy to check universal property.

Corollary

$U : (\Omega, E)\text{-Alg} \to \mathrm{Set}$ has a left adjoint.

Proof

We have

$$\begin{tikzcd}
(\Omega,E)\text{-Alg} \arrow[rr, hook, "i"] \arrow[dr, "U"'] & & \Omega\text{-Alg} \arrow[ll, bend right, "R"'] \arrow[ll, phantom, bend right=20, "\perp"] \arrow[dl, "U_\Omega"] \\
& \mathrm{Set} \arrow[ur, bend right, "F_\Omega"'] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

& adjoints compose:

$$\begin{aligned}
(\Omega,E)\text{-Alg}(RF_\Omega X, Y) &\cong \\
\Omega\text{-Alg}(F_\Omega X, iY) &\cong \\
\mathrm{Set}(X, U_\Omega i Y) &= \\
\mathrm{Set}(X, UY). &
\end{aligned}$$

So $RF_\Omega \dashv U$. $\square$

Proposition

Free (Ω, E)-algebras are projective.

~~Proof~~ The proof is just as for free Ω-algebras - it just uses the universal property of freeness.