

Lecture 8

Last time :

- subalgebras, homomorphic images
- limits of Ω -algebras
- congruences & kernels
- quotients by congruences
- first isomorphism theorem
- coequalisers of Ω -algebras

Proposition

The category $\Omega\text{-Alg}$ has all coproducts.

Proof

- Consider Ω -algs $(X_i)_{i \in I}$.
- We can form the coproduct $\sum_{i \in I} X_i$ as sets, which is the disjoint union, and then the free Ω -alg. on this.
- Have inclusions

$$X_i \xrightarrow{p_i} \sum_{i \in I} X_i \xrightarrow{\eta} \text{Fr} \left(\sum_{i \in I} X_i \right) \text{ but}$$

$\underbrace{\hspace{10em}}_{k_i}$

the problem is that the k_i are not Ω -alg homomorphisms.

- Have inclusions

$$X_i \xrightarrow{p_i} \sum_{i \in I} X_i \xrightarrow{\alpha} \text{Fr}(\sum_{i \in I} X_i) \text{ but}$$

$$\underbrace{\hspace{10em}}_{k_i}$$

the problem is that the k_i are not Ω -alg homomorphisms.

- To fix this, we consider the congruence E on $\text{Fr}(\sum_{i \in I} X_i)$ generated by

$$(k_i(s(x_1, \dots, x_n)), s(k_i x_1, \dots, k_i x_n)) : \left. \begin{array}{l} n \in \mathbb{N}, s \in \Omega_n, \\ i \in I, \\ x_1, \dots, x_n \in X_i \end{array} \right\}$$

& then each composite

$$X_i \xrightarrow{k_i} \text{Fr}(\sum X_i) \xrightarrow{p} \underbrace{\text{Fr}(\sum X_i)}_{\substack{E \\ C}}$$

$$\underbrace{\hspace{10em}}_{l_i}$$

is a homomorphism.

$$\text{Indeed, } p(k_i(s(x_1, \dots, x_n))) = \text{as } p \text{ quotient map}$$

$$p(s(k_i x_1, \dots, k_i x_n)) = \text{map}$$

$$s(pk_i x_1, \dots, pk_i x_n) = \text{as } p \text{ hom.}$$

- Then given $(X_i \xrightarrow{f_i} A)_{i \in I} \exists!$

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & A \\ \downarrow p_i & \searrow & \downarrow \exists! f \\ \Sigma X_i & \dashrightarrow & \exists! f \end{array} \in \text{Set}$$

- This extends uniquely along π to

$$\bar{f}: \text{Fr}(\Sigma X_i) \rightarrow A \in \Omega\text{-Alg}$$

- Then $X_i \xrightarrow{k_i} \text{Fr}(\Sigma X_i)$

$$\begin{array}{ccc} & & \downarrow \bar{f} \\ & \searrow & \\ & & A \end{array}$$

Moreover as each f_i a homomorphism, \bar{f} identifies the elements

$$(k_i(s(x_1, \dots, x_n)), s(k_i x_1, \dots, k_i x_n)) : \left. \begin{array}{l} n \in \mathbb{N}, s \in \Omega_n \\ i \in I, \\ x_1, \dots, x_n \in X_i \end{array} \right\}$$

of E ; hence \bar{f} factors uniquely through $\text{Fr}(\Sigma X_i)/E$ as required \square

Since all colimits can be constructed from coproducts & coequalisers, we have :

Corollary

$\mathcal{R}\text{-Alg}$ has all colimits.

providing the dual to the easier result we showed that $\mathcal{R}\text{-Alg}$ has all limits.

- Before turning to closure properties of (Ω, E) -algebras, it will be useful to consider projectivity.

Def.) An (Ω, E) -alg A is projective if given $B \xrightarrow{F} C \in (\Omega, E)\text{-Alg}$ surjective & $A \twoheadrightarrow C$, $\exists A \xrightarrow{\bar{f}} B$ such that

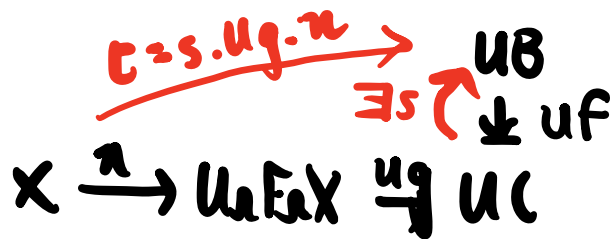
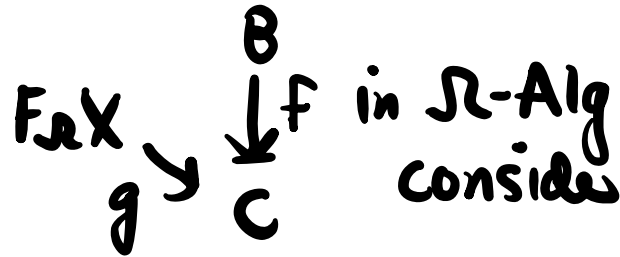
$$\begin{array}{ccc}
 A & \xrightarrow{\bar{f}} & B \\
 \searrow g & & \downarrow F \\
 & & C
 \end{array}$$

commutes.

Proposition

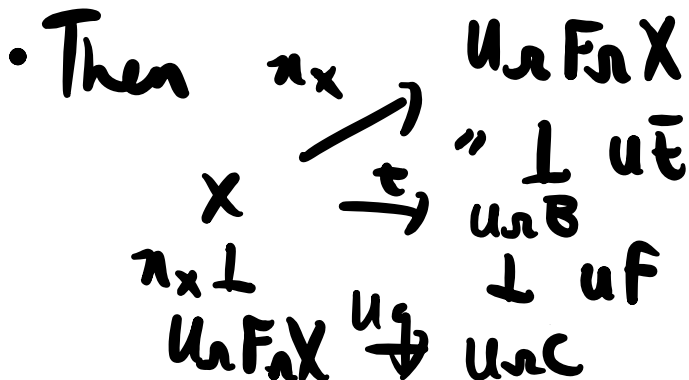
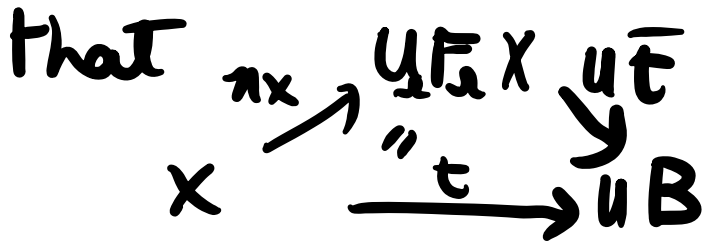
Free algebras in Ω -Alg are projective.

Proof
Given



where $sc \in f^{-1}(c)$ for each $c \in C$ exists since f is surjective.

• Then $\exists!$ $F_n X \xrightarrow{\bar{t}} B$ such



so $f \cdot \bar{t} = g$
by freeness.

□

(Ω, E) -Algebras - Closure properties

Proposition

The full subcategory

$(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ is closed under products, subalgebras and quotients (aka homomorphic images)

Proof

- let $(s, t) \in E$. Then $s, t \in \text{Fr} X$ some X .
- Then $A \models s = t \iff$
each $f: \text{Fr}(X) \rightarrow A \in \Omega\text{-Alg}$ satisfies $f(s) = f(t)$
- Consider product $\prod_{i \in I} A_i$; $p_i \rightarrow A_i \in \Omega\text{-Alg}$
where each A_i an (Ω, E) -algebra, &
consider $\text{Fr}(X) \xrightarrow{f} \prod_{i \in I} A_i \in \Omega\text{-Alg}$.
- Then $p_i f(s) = p_i f(t)$ as $A_i \in (\Omega, E)\text{-Alg}$.
 $f''(s); f''(t);$
- But then $f(s) = f(t)$ as agree in each comp.
- So $\prod A_i$ an (Ω, E) -alg.

• Let $A \in (\Omega, E)\text{-Alg}$ & $B \hookrightarrow A$ a subalgebra.

- Consider $F_{\Omega}X \xrightarrow{f} B$.

As $j f \in \Omega\text{-Alg}$, $j f(s) = j f(t)$.

As j injective, $f(s) = f(t)$.

Hence $B \in (\Omega, E)\text{-Alg}$.

• Let $A \xrightarrow{p} B \in \Omega\text{-Alg}$ be surjective
& A an $(\Omega, E)\text{-algebra}$.

• Consider $F_{\Omega}X \xrightarrow{f} B \in \Omega\text{-Alg}$.

• By projectivity of $F_{\Omega}X$,

$$\begin{array}{ccc}
 F_{\Omega}X & \xrightarrow{\exists \bar{f}} & A \\
 & \searrow f & \downarrow p \\
 & & B
 \end{array}
 \quad p \in \Omega\text{-Alg}.$$

Since A an $(\Omega, E)\text{-Alg}$, $\bar{f}(s) = \bar{f}(t)$

Hence $f(s) = p\bar{f}(s) = p\bar{f}(t) = f(t)$

as required. \square

Corollary

(Ω, E) -Alg \longleftrightarrow Ω -Alg closed under limits & coequalisers.

~~Proof~~. Since limits can be constructed from products & equalisers, enough to establish these cases.

- We know (Ω, E) -Alg closed under prods.
- Equalisers in Ω -Alg are subalgebras.

Since (Ω, E) -Alg closed

under these two, closed under equalisers.

- Given $A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \in (\Omega, E)\text{-Alg}$,

the coequaliser in Ω -Alg is

$$B \xrightarrow{p} B/E_{f,g} \text{ but}$$

since this is surjective,

& (Ω, E) -Alg closed under homomorphic

images, $B/E_{f,g} \in (\Omega, E)\text{-Alg}$ too. \square

Remark A congruence on an (Ω, E) -alg A is, by definition, a congruence on the Ω -algebra A .

This makes sense since such an $E \subseteq A \times A$ is automatically an (Ω, E) -alg too, as it is a subobject of a product of (Ω, E) -algebras.

Also have

First isomorphism theorem

Given $f: A \longrightarrow B \in (\Omega, E)\text{-Alg}$
the induced map
 $t: A/K_f \longrightarrow \text{im } f : [a] \mapsto fa$
is an isomorphism in $(\Omega, E)\text{-Alg}$.

Proof) As $(\Omega, E)\text{-Alg}$ closed under images (subalgebras), congruences & quotients, this follows from first iso thm for Ω -algebras.

Free (Ω, E) -algebras

We would like to prove

Propⁿ The inclusion $(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ has a left adjoint.

Since $(\Omega, E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$ is closed under products & subobjects, a more general statement is:

Propⁿ Let $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$ be a Full subcategory closed under products & subalgebras.

Then j has a left adjoint.

• In order to prove it, we need to consider, for an Ω -alg A , the collection $\mathcal{Q}(A)$ of surjections $A \twoheadrightarrow B$ where B is any Ω -algebra.

• Problem is that $\mathcal{Q}(A)$ is a proper class - just consider maps to each 1 element \mathcal{R} -algebra.

• let us say that $(f, B) \sim (g, C)$ if \exists iso $B \xrightarrow{h} C$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \downarrow h \\ & & C \\ & \swarrow & \uparrow g \end{array}$$

• By the first iso. Theorem, $(f, B) \cong (A/\ker f, \text{pr}_{\ker f})$ so we have a surj. function

$$\begin{array}{ccc} \text{Cong}(A) & \longrightarrow & \mathcal{Q}(A)/\sim \\ \downarrow E & \searrow & \downarrow \\ & & A \longrightarrow A/E \end{array}$$

(which is in fact a bijⁿ).

• Since $\text{Cong}(A) \subseteq \text{Powerset}(A \times A)$ is a set, therefore $\mathcal{Q}(A)/\sim$ is a set.

Propⁿ Let $\mathcal{C} \hookrightarrow \Omega\text{-Alg}$ be a full subcategory closed under products & subalgebras.

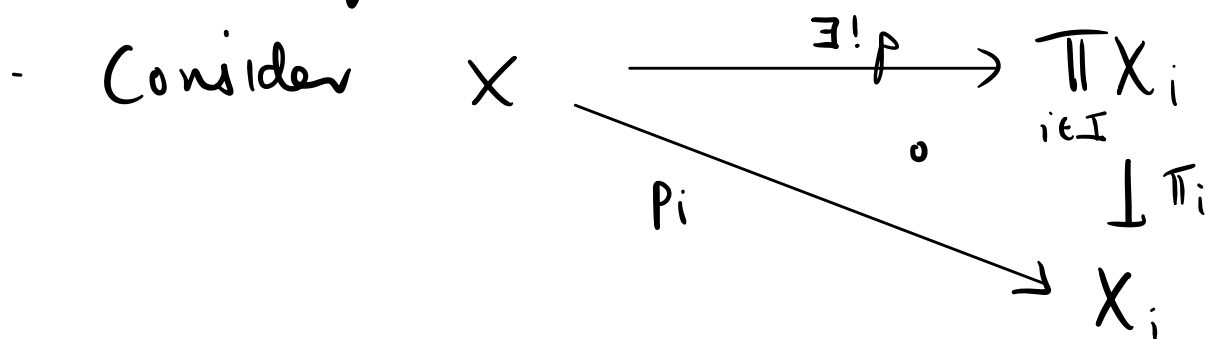
Then j has a left adjoint.

Proof • Given $X \in \Omega\text{-Alg}$, need $RX \in \mathcal{C}$ & $X \xrightarrow{\pi_X} RX$ such that:

given $X \xrightarrow{f} Y$ with $Y \in \mathcal{C} \exists!$
 $RX \xrightarrow{\bar{f}} Y$ such that

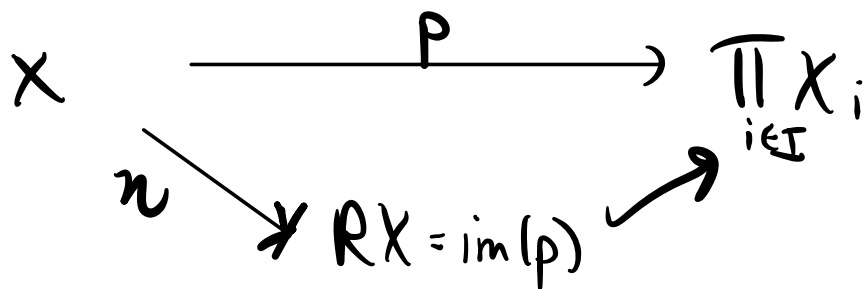
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_X \searrow & & \nearrow \bar{f} \\ & RX & \end{array}$$

- Since $\mathcal{O}(X)/\sim$ is a set, consider a rep $\{X \xrightarrow{p_i} X_i : i \in I\}$ of each \sim -class; thus each surjection $X \xrightarrow{k} Z$ is equiv. to one of the p_i 's.



- Then $\prod_{i \in I} X_i \in \mathcal{C}$ as closed under prods.

- Now factor

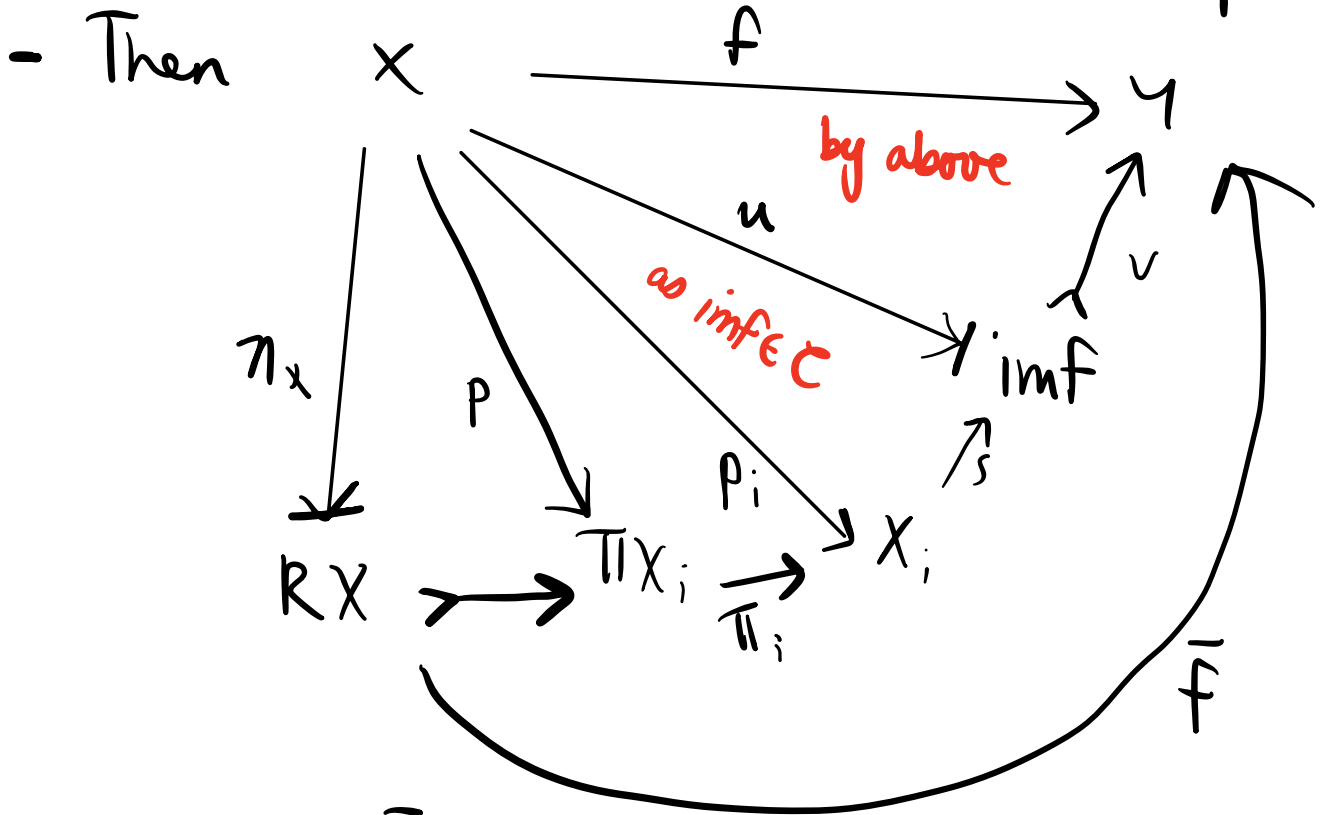


Then $RX \in \mathcal{C}$ as closed under subalgs.

- Then consider $X \xrightarrow{f} Y$ with $Y \in \mathcal{C}$.

- Factor as $X \xrightarrow{u} \text{im} f \xleftarrow{v} Y$.

Then $\text{im} f \in \mathcal{C}$ as closed under subalgs.



- Moreover \bar{f} is unique factorisation since π_X is surjective \Rightarrow epi. \square

Corollary

The inclusion $(\mathcal{R}, \mathcal{E})\text{-Alg} \hookrightarrow \mathcal{R}\text{-Alg}$ has a left adjoint.

Remark

- Here is another construction of the adjoint:
- let A be an Ω -alg & $E(X)$ the set of equations in E in variables X .
- Problem is: given $(s, t) \in E(X)$, there may exist $f: F_{\Omega}X \rightarrow A$ st. $fs \neq ft$.
- So consider the set
 $\{(f(s), f(t)) \in A^2 : (s, t) \in E(X), f: F_{\Omega}X \rightarrow A\}$
& let $\bar{E} \subseteq A^2$ be the congruence it generates.
- Now form $p: A \rightarrow A/\bar{E}$.

- Now consider $FX \xrightarrow{k} A/\bar{E}$. Must show $k(s) = k(t)$. But as p surj can

Find

$$\begin{array}{ccc} FX & \xrightarrow{k} & A/\bar{E} \\ \bar{k} \downarrow & \parallel & \uparrow p \\ A & \xrightarrow{\quad} & \end{array} \quad \left(\begin{array}{l} FX \\ \text{projective} \end{array} \right)$$

But then $(\bar{k}(s), \bar{k}(t)) \in \bar{E}$ so A/\bar{E} on
 $k(s) = p\bar{k}(s) = p\bar{k}(t) = k(t) \Rightarrow \underline{(\Omega, E)\text{-alg}}$.

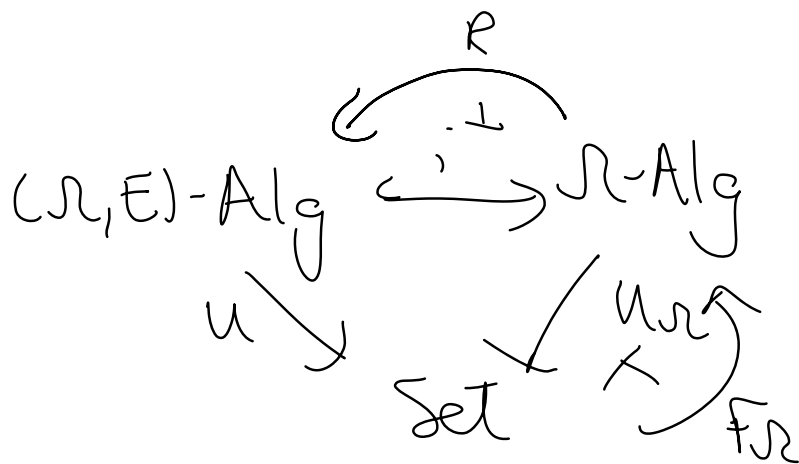
- Easy to check universal property.

Corollary

$U: (\Omega, E)\text{-Alg} \rightarrow \text{Set}$ has a left adjoint.

Proof

We have



& adjoints compose:

$$\begin{aligned}
 (\Omega, E)\text{-Alg} (R F \Omega X, \eta) &\cong \\
 \Omega\text{-Alg} (F \Omega X, i \eta) &\cong \\
 \text{Set} (X, U i \eta) &= \\
 \text{Set} (X, U \eta). &
 \end{aligned}$$

So $R F \Omega \dashv U$. \square

Proposition

Free (R, E) -algebras are projective.

~~Proof~~ The proof is just as for free R -algebras - it just uses the universal property of freeness.