Cvičení 13. 5.13 (v) Řešte v $\mathbb{Z}$: $3^y = 1 + 2^x$

(MŘMV I)

Řeš: Pro $x < 0$: $1+2^x \in (1,2) \Rightarrow y \in (0,1)$ NELZE

Je tedy $x \in \mathbb{N}_0 \Rightarrow y \geq 1$.

mod 3: $L \equiv 0 \ (3)$ $\quad 1+2^x \equiv 0 \ (3)$

$2^x \equiv 2 \ (3)$

$(-1)^x \equiv (-1) \pmod 3 \Rightarrow x$ musí být liché.

mod 4: $(-1)^y \equiv 1 + 2^x \pmod 4$

když $x \geq 2$: $(-1)^y \equiv 1 \pmod 4 \Rightarrow y$ musí být sudé.

$x = 0$: $3^y = 1 + 2^0$ N.Ř.

$x = 1$: $3^y = 1 + 2^1 \Rightarrow y = 1$ $\quad [1,1]$

$x \geq 2$, liché, $y \geq 1$ je sudé $\quad y = 2k, k \in \mathbb{N}$, $x = 2l+1, l \in \mathbb{N}$

Dosadíme do zadání: $3^{2k} = 1 + 2^{2l+1}$

$2^{2l+1} = 3^{2k} - 1 = (3^k)^2 - 1^2 = (3^k+1)(3^k-1)$

mocniny 2 (liší se o 2) $\quad 2^u - 2^v = 2$

$2^{v-1}(2^{u-v}-1) = 1$

$v = 1, u - v = 1$

$u = 2$

$v = 1$

Odtud máme $k = 1, l = 1$

$[3,2]$

Pozn: $3^2 = 1 + 2^3$ je jediné řešení rovnice

$x^a - y^b = 1$, $x, y, a, b \in \mathbb{N}$

Tzv. Catalanova hypotéza (Catalan conjecture)

Vyřešil Preda Mihailescu roku 2002.

8.17 (ii) Řešte v $\mathbb{N}$ rovnici $2\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{u} + \frac{u}{x}\right) = 7$ $\quad$ (x_1, x_2, x_3, x_4)

Řeš: „složitý součet, ale jednoduchý součin" $\Rightarrow$ AG nerovnost

$x_1, \dots, x_n \in \mathbb{R}^+$: $\quad \frac{x_1 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$, rovnost pro $x_1 = \dots = x_n$

$\frac{x}{y} + \frac{y}{z} + \frac{z}{u} + \frac{u}{x} = x_1 + \dots + x_4 \geq 4 \sqrt[4]{x_1 \cdots x_4} = 4 \cdot \sqrt[4]{1} = 4$

$2 \cdot (\quad) \geq 2 \cdot 4 = 8 > 7 \Rightarrow$ rovnice nemá řešení

5.17 (iii) Řešte v $\mathbb{N}$: $(x + 2y + 3z)^2 = 14(x^2 + y^2 + z^2)$

Řeš: Cauchy-Schwarz $(x_1, \dots, x_n), (y_1, \dots, y_n)$

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \leq \left(\sum x_i^2\right)\left(\sum y_i^2\right)$$

rovnost $\Leftrightarrow \exists k \in \mathbb{R}$: $\forall i$: $x_i = k \cdot y_i$

$(1,2,3)$, (x,y,z)

C-S nerovnost $(1\cdot x+2\cdot y+3\cdot z)^2 \leq (1^2+2^2+3^2)(x^2+y^2+z^2)=14(x^2+y^2+z^2)$

Nastane tedy rovnost, tj. $\exists k\in N: x=k\cdot 1, y=k\cdot 2, z=k\cdot 3$

Množina řešení $\{[k, 2k, 3k], k\in N\}$

$(x+y)^2-4xy=(x-y)^2$

Sum Product Difference

$S^2-4P=D^2$

Symmetry-product principle

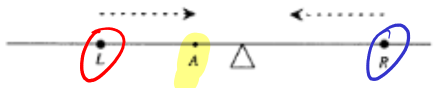
AG nerovnost

5.5.17 *AM-GM Reformulated.* Let $x_1, x_2, \ldots, x_n$ be positive real numbers with product $P = x_1x_2\cdots x_n$ and sum $S = \cancel{x}_1 + \cancel{x}_2 + \cdots + \cancel{x}_n$. Prove that the largest value of P is attained when all the x_i are equal, i.e., when

$$x_1 = x_2 = \cdots = x_n = \frac{S}{n}.$$

Solution: Imagine the n positive numbers $x_1, x_2, \ldots, x_n$ as physical points on the number line, each with unit weight. The balancing point (center of mass) of these weights is located at the arithmetic mean value $A := S/n$. Notice that if we move the points around in such a way that they continue to balance at A, that is equivalent to saying that their sum stays constant.

Our strategy, inspired by the symmetry-product principle, is to consider situations where the x_i are not all equal and show that we can make them "more equal" and increase their product *without changing their sum.* If the points are not all clustering at A, then at least one will be to the left of A (call it L) and another (call it R) will be to the right of A.[16] Of these two points, move the one that is closest to A right up to A, and move the other so that the balancing point of the two points hasn't changed. In the figure below, the arrowpoints indicate the new positions of the points.



Notice that the distance between the two points has decreased, but their balancing point is unchanged. By the symmetry-product principle, the product of the two points increased. Since the sum of the two points was unchanged, the sum of all n points has not changed. We have managed to change two of the n numbers in such a way that

- one number that originally was not equal to A became equal to A;
- the sum of all n numbers did not change;
- the product of the n numbers increased.

Since there are finitely many numbers, this process will end when all of them are equal to A; then the product will be maximal. ■

The proof is called "algorithmic" because the argument used describes a concrete procedure that optimizes the product in a step-by-step way, ending after a finite number of steps. Another distinctive feature of this proof was that we altered our point of view and recast an inequality as an *optimization* problem. This is a powerful strategy, well worth remembering.

[15] This simple idea of reformulating AM-GM is not very well known. Our treatment here is inspired by Kazarinoff's wonderful monograph [25]; this short book is highly recommended.

[16] Observe that this is a neat "physical" proof of the average principle (5.5.12).

5.20(N) Řešte v $\mathbb{Z}$, s parametrem p – prvočíslo

$$2x^2 - x - 36 = p^2$$
$$(2x-9)(x+4) = p^2$$

$p^2 \cdot 1$
$p \cdot p$
$1 \cdot p^2$
$(-p^2)\cdot(-1)$
$(-p)(-p)$
$(-1)\cdot(-p^2)$

Řešme 6 soustav 2 rovnic ohledně x:

	$x+4$	$2x-9$	x	
$p^2\cdot 1$	1	p^2	-3	$p^2 = -15$ ✗
$p\cdot p$	p	p	$p-4$	$2(p-4)-9 = p \iff p = 17,\ x = 13$
$1\cdot p^2$	p^2	1	5	$x+4 = 9 = p^2 \iff p = 3,\ x = 5$
$(-p^2)(-1)$	-1	$-p^2$	-5	$2(-5)-9 = -p^2 \iff p^2 = 19$ ✗
$(-p)(-p)$	$-p$	$-p$	$-p-4$	$2(-p-4)-9 = -p \iff p = -17$ ✗
$(-1)(-p^2)$	$-p^2$	-1	4	$x+4 = 8 = -p^2$ ✗

6.3(Vi) Řešte v $\mathbb{Z}$: $2^x + 5^y = 19^z$

Řeš: analogicky jako dříve: $x,y,z \geq 0$

mod 2: $x = 0$: $1+1 \equiv 1\ (2)$ neplatí!

$x \geq 1$

mod 5: $2^x + 5^y \equiv (-1)^z \pmod 5$

x	0	1	2	3
$2^x(5)$	1	2	4	3

$y=0$: $2^x + 1 \equiv (-1)^z$

$2 \mid z$: nemá ř., $2 \nmid z$: $2^x \equiv 3(5) \iff x \equiv 3(4)$

$2^x + 1 = 19^z \iff 2^x = 19^z - 1 = \underbrace{(19-1)}_{18}(19^{z-1} - 19^{z-2} \ldots$

NELZE

$y \geq 1$: $2^x \equiv (-1)^z \pmod 5$

může platit pouze pro $x \equiv 0(4) \wedge z \equiv 0(2)$
nebo $x \equiv 2(4) \wedge z \equiv 1(2)$

$x \equiv 0(2)$

mod 3: $2^x + 5^y = 19^z$

$(-1)^x + (-1)^y \equiv 1^z \pmod 3$

x sudé: $1 + (-1)^y \equiv 1 \pmod 3$ NELZE

Zadaná rovnice nemá řešení.

6.5 (iii) Řešte v $\mathbb{Z}$: $x^4+y^4+z^4=9u^4$

Řeš:

evidentně $x=y=z=u=0$.

Předp., že máme řešení (x,y,z,u), kde $u\neq 0$, položme $d:=u^4>0$

Čtvrté mocniny nabývají podivně se na levé straně rovnice modulo 5.

$5\nmid u \Rightarrow$ podle MFV je $u^4\equiv 1\,(5)$

x	0	±1	±2
$x^4\,(5)$	0	1	1

Tj. $P\equiv 4 \pmod 5$

Ale $L\equiv \{{}^0_1+{}^0_1+{}^0_1\}=\{0,1,2,3\} \pmod 5$

Tedy $L\not\equiv P \pmod 5$

Je tedy $u\equiv 0\,(5) \Rightarrow P=9u^4\equiv 0 \pmod 5$

$\Rightarrow x\equiv y\equiv z\equiv 0 \pmod 5$

Subst. $x=5x_1$, $y=5y_1$, $z=5z_1$, $u=5u_1$, $x_1,y_1,z_1,u_1\in\mathbb{Z}$

$(5x_1)^4+(5y_1)^4+(5z_1)^4=9\cdot(5u_1)^4 \quad /:5^4$

$x_1^4+y_1^4+z_1^4=9u_1^4$, kde $d'=u_1^4=\left(\frac{u}{5}\right)^4<u^4=:d$

Tím jsme ukázali, že rovnice nemá řešení kromě $x=y=z=u=0$.

DÚ: Dk, že rovnice $x^3+x=3y^4+1$ nemá řešení v $\mathbb{Z}$

Řeš:

modulo 5: $y\equiv 0\,(5) \Rightarrow P\equiv 1\,(5)$

$y\not\equiv 0\,(5) \Rightarrow P\equiv 3\cdot 1+1\equiv 4\,(5)$

x	0	1	-1	2	-2
x^3	0	1	-1	-2	2
$x+x^3$ (mod 5)	0	2	-2	0	0

Tj. $L\equiv 0,\pm2 \pmod 5$ $\quad P\equiv \pm1\,(5)$

Tedy $L\not\equiv P \pmod 5$.