CVIČENÍ 5

5.1

1. $\int 5x^7 dx = \frac{5}{8}x^8 + c$

2. $\int -\frac{5}{x^6} dx = \int -5x^{-6} dx = -5x^{-5}\left(-\frac{1}{5}\right) + c = x^{-5} + c = \frac{1}{x^5} + c$

3. $\int \frac{50}{(5x)^3} dx = \int \frac{50}{125x^3} dx = \int \frac{2}{5} x^{-3} dx = \frac{2}{5} x^{-2}\left(-\frac{1}{2}\right) = -\frac{1}{5x^2} + c$

4. $\int e^x\left(1 + \frac{e^{-x}}{\cos^2 x}\right) dx = \int e^x dx + \int \frac{1}{\cos^2 x} dx = e^x + \tan x + c$

5. $\int (1+\sqrt{x})^2 dx = \int 1 + 2\sqrt{x} + x\, dx = \int 1 dx + 2\int x^{\frac{1}{2}} dx + \int x\, dx = x + 2x^{\frac{3}{2}}\left(\frac{2}{3}\right) + \frac{x^2}{2} + c$
$= x + \frac{4}{3} x\sqrt{x} + \frac{x^2}{2} + c$

6. $\int \left(x + \frac{1}{x} + \sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \int x\, dx + \int x^{-1} dx + \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx =$
$= \frac{x^2}{2} + \ln x + x^{\frac{3}{2}}\left(\frac{2}{3}\right) + x^{\frac{1}{2}}(\cdot 2) + c = \frac{x^2}{2} + \ln x + \frac{2x\sqrt{x}}{3} + 2\sqrt{x} + c$

7. $\int \frac{e^{2x}-1}{e^x-1} dx = \int \frac{(e^x-1)(e^x+1)}{e^x-1} dx = \int e^x + 1\, dx = e^x + x + c$

8. $\int \left(\sqrt{2x} + \sqrt{\frac{2}{x}}\right) dx = \sqrt{2}\int \sqrt{x}\, dx + \sqrt{2}\int \sqrt{\frac{1}{x}} dx = \sqrt{2}\int x^{\frac{1}{2}} dx + \sqrt{2}\int \frac{1}{\sqrt{x}} dx =$
$= \sqrt{2} \cdot x^{\frac{3}{2}}\left(\frac{2}{3}\right) + \sqrt{2}\int x^{-\frac{1}{2}} dx = \frac{2\sqrt{2}}{3}\sqrt{x^3} + \sqrt{2}\, x^{\frac{1}{2}}(2) + c =$
$= 2\sqrt{2}\left(\frac{\sqrt{x^3}}{3} + \sqrt{x}\right) + c = 2\sqrt{2}\sqrt{x}\left(\frac{x}{3} + 1\right) + c = 2\sqrt{2x}\left(\frac{x}{3} + 1\right) + c$

9. $\int \frac{(x+2)^3}{x^3} dx = \int \frac{(x^2+4x+4)(x+2)}{x^3} dx = \int \frac{x^3+4x^2+4x+2x^2+8x+8}{x^3} dx = \int \frac{x^3+6x^2+12x+8}{x^3} =$
$= \int 1 + \frac{6}{x} + \frac{12}{x^2} + \frac{8}{x^3} dx = x + 6\ln x - 12x^{-1} + 8\frac{1}{x^2}\left(-\frac{1}{2}\right) + c =$
$= x + 6\ln x - \frac{12}{x} - \frac{4}{x^2} + c$

10. $\int 8x^{\frac{3}{5}} dx = 8 \cdot x^{\frac{8}{5}}\left(\frac{5}{8}\right) + c = 5\sqrt[5]{x^8} + c = 5x\sqrt[5]{x^3} + c = 5x^{\frac{8}{5}} + c$

5.2

1. $\int (4x-3)^4 dx = \left| \begin{array}{l} t = 4x-3 \\ dt = 4dx \end{array} \right| = \int \frac{1}{4} t^4 dt = \frac{1}{4} \cdot t^5 \frac{1}{5} + k = \frac{t^5}{20} + k = \frac{(4x-3)^5}{20} + c$

2. $\int \sin^3 x \cos x \, dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array} \right| = \int t^3 dt = \frac{1}{4} t^4 + k = \frac{t^4}{4} + k = \frac{\sin^4 x}{4} + c$

3. $\int \frac{1}{\sqrt{4x+9}} dx = \left| \begin{array}{l} t = 4x+9 \\ dt = 4dx \end{array} \right| = \int \frac{1}{4} \frac{1}{\sqrt{t}} dt = \frac{1}{4} \int t^{-\frac{1}{2}} dt = \frac{1}{4} t^{\frac{1}{2}} \cdot 2 + k = \frac{1}{2} \sqrt{t} + k = \frac{1}{2} \sqrt{4x+9} + c$

4. $\int 14 e^{7x-8} dx = \left| \begin{array}{l} t = 7x-8 \\ dt = 7dx \end{array} \right| = \int 2e^t dt = 2e^t + k = 2e^{7x-8} + c$

5. $\int 33(8-3x)^{6/5} dx = \left| \begin{array}{l} t = 8-3x \\ dt = -3dx \end{array} \right| = \int -11 t^{\frac{6}{5}} dt = -11 t^{\frac{11}{5}} \left(\frac{5}{11}\right) + k = -5 t^{\frac{11}{5}} + k = -5(8-3x)^{\frac{11}{5}} + c$

6. $\int \frac{(1+\ln x)^4}{x} dx = \left| \begin{array}{l} t = 1+\ln x \\ dt = \frac{1}{x} dx \end{array} \right| = \int t^4 dt = \frac{1}{5} t^5 + k = \frac{(1+\ln x)^5}{5} + c$

7. $\int \frac{1}{x^2-6x+9} dx = \int \frac{1}{(x-3)^2} dx = \left| \begin{array}{l} t = x-3 \\ dt = dx \end{array} \right| = \int \frac{1}{t^2} dt = \int t^{-2} dt = -t^{-1} + k = -\frac{1}{(x-3)} + c$

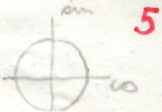
8. $\int \frac{3\sqrt{\ln x}}{x} dx = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x} dx \end{array} \right| = \int 3 t^{\frac{1}{2}} dx = 3 \cdot t^{\frac{3}{2}} \cdot \frac{2}{3} + k = 2\sqrt{t^3} + k = 2\sqrt{\ln^3 x} + c$

$= 2 \ln^{3/2} x + c$

9. $\int \frac{3\cos x}{\sin^4 x} dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array} \right| = \int \frac{3}{t^4} dt = 3 \int t^{-4} dt = 3 \cdot \frac{t^{-3}}{-3} + k = -t^{-3} + k = -\frac{1}{\sin^3 x} + c$

10. $\int \frac{-2}{\tan x \sin^2 x} dx = \int \frac{-2}{\frac{\sin x}{\cos x} \sin^2 x} dx = \int \frac{-2\cos x}{\sin^3 x} dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array} \right| =$

$= \int \frac{-2}{t^3} dt = -2 \int t^{-3} dt = -2 \cdot \frac{t^{-2}}{-2} + k = t^{-2} + k = \frac{1}{\sin^2 x} + c$



5.3

1. $\int_0^{\pi} \sin x \, dx = \left[-\cos x\right]_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2$

2. $\int_0^4 12\sqrt{x+\frac{1}{4}}\,dx = \left| \begin{array}{l} t = x + \frac{1}{4} \\ dt = dx \end{array} \right| = \int_{\frac{1}{4}}^{4\frac{1}{4}} 12\sqrt{t}\,dt = \int_{\frac{1}{4}}^{4\frac{1}{4}} 12\,t^{\frac{1}{2}}\,dt = \left[12\,t^{\frac{3}{2}}\frac{2}{3}\right]_{\frac{1}{4}}^{4\frac{1}{4}} = \left[8\sqrt{t^3}\right]_{\frac{1}{4}}^{4\frac{1}{4}} = \left[8\sqrt{(x+\frac{1}{4})^3}\right]_0^4 =$

$= 8\sqrt{(4+\frac{1}{4})^3} - 8\sqrt{\frac{1}{4^3}} = 8\sqrt{(\frac{17}{4})^3} - 8\sqrt{\frac{1}{4^3}} = 8\,\frac{17}{4}\sqrt{\frac{17}{4}} - 8\,\frac{1}{4}\sqrt{\frac{1}{4}} = 8\cdot\frac{17}{4}\cdot\frac{1}{2}\sqrt{17} - 8\cdot\frac{1}{4}\cdot\frac{1}{2}$

$= 17\sqrt{17} - 1$

3. $\int_1^2 \frac{6}{6x-1}\,dx = \left| \begin{array}{l} t = 6x-1 \\ dt = 6dx \end{array} \right| = \int_5^{11} \frac{1}{t}\,dt = \left[\ln t\right]_5^{11} = \ln 11 - \ln 5 = \ln\frac{11}{5}$

4. $\int_0^{\frac{\pi}{2}} 4\sin x\cos^3 x\,dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \end{array} \right| = \int_1^0 -4\,t^3\,dt = \int_0^1 4t^3\,dt = \left[4\frac{t^4}{4}\right]_0^1 = \left[t^4\right]_0^1 = 1 - 0 = 1$

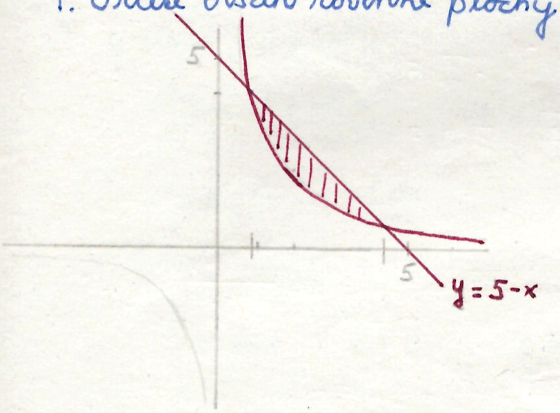
5. $\int_{-1}^3 (x^3 - 3x^2 + 1)\,dx = \left[\frac{x^4}{4} - \frac{3}{3}x^3 + x\right]_{-1}^3 = \left[\frac{x^4}{4} - x^3 + x\right]_{-1}^3 = \frac{3^4}{4} - 3^3 + 3 - \left(\frac{1}{4} - 1 + 1\right) = \frac{81}{4} - 27 + 3 - \frac{1}{4} =$

$= 20 - 24 = -4$

5.4

1. Určete obsah rovinné plochy ohraničené křivkami $yx = 4$ a $x + y = 5$.

$y = \frac{4}{x}$ $\quad y = 5 - x$



1. průsečíky obou křivek: $y = \frac{4}{x}$ $\quad x = 5 - y$

$y = \frac{4}{5-y}$

$5y - y^2 = 4$

$y^2 - 5y + 4 = 0$

$D = b^2 - 4ac$
$= 25 - 4\cdot 4 = 25 - 16$
$= 9$

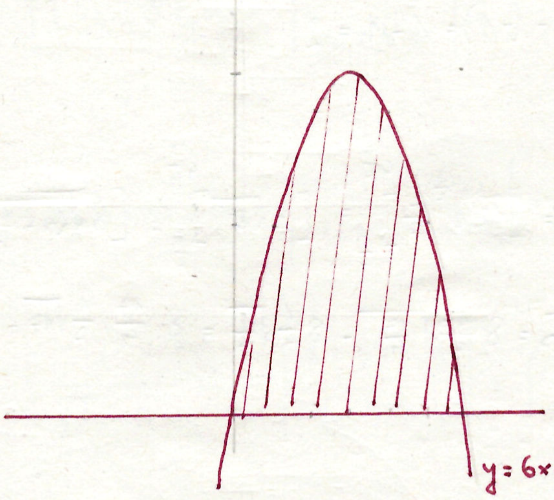
$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{5 \pm 3}{2}$

$x_1 = 1 \quad x_2 = 4$

$\int_1^4 5 - x\,dx - \int_1^4 \frac{4}{x}\,dx = \left[5x - \frac{x^2}{2}\right]_1^4 - \left[4\ln x\right]_1^4 = 20 - \frac{16}{2} - 5 + \frac{1}{2} - 4\ln 4 + 4\ln 1 = 20 - 8 - 5 + \frac{1}{2} - 4\ln 4 =$

$= 7 + \frac{1}{2} - 4\ln 2^2 = \frac{15}{2} - 8\ln 2$

2. Určete obsah rovinné plochy ohraničené křivkami $y = 6x - x^2$, $y = 0$.



Průsečíky obou křivek: $0 = 6x - x^2$

$0 = x(6-x) \ldots x=0$

$x = 6$

$$\int_0^6 6x - x^2\,dx - \int_0^6 0\,dx = \left[6\frac{x^2}{2} - \frac{x^3}{3}\right]_0^6 - 0 = \left[3x^2 - \frac{x^3}{3}\right]_0^6 = 3\cdot 6^2 - \frac{6^3}{3} - 0 + 0 = 3\cdot 36 - \frac{6\cdot 36}{3} =$$

$$= 108 - 72 = 36$$

CVIČENÍ 6

6.1

1. $y' = x^3 + 3$

$\frac{dy}{dx} = x^3 + 3$

$dy = x^3 + 3\,dx$

$\int dy = \int x^3 + 3\,dx$

$y = \frac{x^4}{4} + 3x + C \qquad x \in \mathbb{R},\ c \in \mathbb{R}$

2. $\frac{1}{y-3} y' = 6x^2$ $\quad K = 0 \Rightarrow y = 3 \Rightarrow \frac{1}{0} y' = 6x^2 \ldots$ nejde $\Rightarrow K \neq 0$

$\frac{1}{y-3} dy = 6x^2 dx$

$\int \frac{1}{y-3} dy = \int 6x^2 dx$ $\qquad \int \frac{1}{y-3} dy = \left| \begin{matrix} t = y-3 \\ dt = dy \end{matrix} \right| = \int \frac{1}{t} dy = \ln t + C = \ln|y-3| + C$

$\ln|y-3| = 6 \cdot \frac{x^3}{3} + c$

$\ln|y-3| = 2x^3 + c$

$|y-3| = e^{2x^3 + c}$

$y = e^{2x^3 + c} + 3$

$y = e^{2x^3} \cdot e^c + 3$

$y = Ke^{2x^3} + 3 \qquad K \neq 0 \quad x \in \mathbb{R}$

3. $y' \cos^2 x = 1 + \cos^2 x$ $\quad x = \frac{\pi}{2} \Rightarrow y' \cdot 0 = 1 + 0 \Rightarrow 0 = 1 \to$ spor $\to x \neq \frac{\pi}{2} + k$, $k \in \mathbb{Z}$

$\frac{dy}{dx} = \frac{1 + \cos^2 x}{\cos^2 x} \qquad \cos^2 x \neq 0$

$dy = \frac{1}{\cos^2 x} + 1\,dx$

$\int dy = \int \frac{1}{\cos^2 x} + 1\,dx$

$y = \tan x + x + C \qquad c \in \mathbb{R} \quad x \neq \frac{\pi}{2} + k,\ k \in \mathbb{Z}$

4. $(1-x^2)y' = 2x$ $x = \pm 1 \rightarrow 0 \cdot y' = \pm 2 \rightarrow$ spor $\rightarrow x \neq \pm 1$

$\frac{dy}{dx} = \frac{2x}{1-x^2}$ $x \neq \pm 1$

$dy = \frac{2x}{1-x^2}\,dx$

$\int dy = \int \frac{2x}{1-x^2}\,dx$

$\int \frac{2x}{1-x^2}\,dx = \left| \begin{matrix} t = 1-x^2 \\ dt = -2x\,dx \end{matrix} \right| = -\int \frac{1}{t}\,dt = -\ln|t| + k = -\ln|1-x^2| + c$

$y = -\ln|1-x^2| + c$ $x \neq \pm 1,\ c \in \mathbb{R}$

5. $1 + y^2 + xyy' = 0$ $K < x^2 \rightarrow y$ nedává smysl. $K = x = 0 \rightarrow y = \frac{0}{0}$ nejde $\rightarrow K \leq x^2$
$x \neq 0$

$xy\frac{dy}{dx} = -y^2 - 1$

$\frac{y}{y^2+1}\,dy = -\frac{1}{x}\,dx$

$\int \frac{y}{y^2+1}\,dy = \left| \begin{matrix} t = y^2+1 \\ dt = 2y\,dy \end{matrix} \right| = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\ln|t| = \frac{1}{2}\ln|y^2+1|$

$\int \frac{y}{y^2+1}\,dy = -\int \frac{1}{x}\,dx$ $x \neq 0$

$\frac{1}{2}\ln|y^2+1| = -\ln|x| + c$

$\ln|y^2+1| = -2\ln|x| + c$

$y^2 + 1 = e^{-2\ln|x| + c}$

$y^2 = e^{\ln x^{-2}} \cdot e^c - 1$

$y^2 = \frac{1}{x^2} \cdot K - 1$ $K \neq 0$ $x \neq 0$

$y^2 = \frac{K - x^2}{x^2}$

$y = \frac{\sqrt{K - x^2}}{x}$ $K \geq x^2$

6. $y' = y$ $K = 0 \rightarrow y = 0 \rightarrow 0' = 0$ ... OK $\rightarrow K \in \mathbb{R},\ x \in \mathbb{R}$

$\frac{dy}{dx} = y$

$\frac{1}{y}\,dy = dx$

$\int \frac{1}{y}\,dy = \int dx$ $y \neq 0$

$\ln|y| = x + c$

$y = e^{x+c}$

$y = k \cdot e^x$ $k \overset{?}{\neq} 0$ ... pokud $k = 0$, potom $0' = 0$ a to je OK $\rightarrow k \in \mathbb{R},\ x \in \mathbb{R}$

7. $x^5 y' = -3 \qquad x \neq 0$

$\frac{dy}{dx} = -3 \frac{1}{x^5}$

$dy = -3 x^{-5} dx$

$\int dy = -3 \int x^{-5} dx$

$y = -3\, x^{-4} \left(-\frac{1}{4}\right) + c$

$y = \frac{3}{4} x^{-4} + c$

$y = \frac{3}{4x^4} + c \qquad x \neq 0,\ c \in \mathbb{R}$

6.2

1. $y \ln y + x y' = 0 \quad y(1) = 1$

$K = 0 \Rightarrow y = 0 \Rightarrow 0 \cdot \ln 0 + x \cdot 0 =$ nejde $\Rightarrow K \neq 0$

$x = 0 \Rightarrow y \ln y = 0$ ... nejde $\Rightarrow x \neq 0$

$y \ln y + x \frac{dy}{dx} = 0$

$y \ln y = -\frac{x}{dx} dy$

$-\frac{1}{x} dx = \frac{1}{y \ln y} dy$

$\int \frac{1}{y \ln y} dy = \int -\frac{1}{x} dx$

$\int \frac{1}{y \ln y} dy = \left| \begin{matrix} t = \ln y \\ dt = \frac{1}{y} dy \end{matrix} \right| = \frac{1}{t} dt = t^{-1} dt \quad \ln t + k$

$\ln \ln y + c$

$\ln|\ln|y|| = -\ln|x| + c$

$\ln|y| = e^{-\ln x + c}$

$\ln|y| = e^{\ln x^{-1}} \cdot e^c$

$\ln|y| = \frac{1}{x} \cdot k \qquad \boxed{k \neq 0 \quad x \neq 0}$

$y = e^{\frac{1}{x} k}$

$y = e^{\frac{1}{x}} \cdot e^k$

$y = K e^{\frac{1}{x}} \qquad K \neq 0$

P.ř. pro $y(1) = 1$: $\quad 1 = K \cdot e^1$

$1 = K \cdot e \rightarrow K = \frac{1}{e}$

Partikulární řešení pro $y(1) = 1$ je $y = \frac{1}{e} \cdot e^{\frac{1}{x}}$, $x \neq 0$.

2. $y \ln y + x' y = 0 \qquad y(0) = 3$

$\vdots$

$y = K \cdot e^{\frac{1}{x}}$

Part. ř. pro $y(0) = 3$: $\quad 3 = K \cdot e^{\frac{1}{0}} \Rightarrow$ Partikulární řešení pro $y(0) = 3$ neexistuje, protože $x \neq 0$.

$K=0 \to y=0 \to (1+e^x)\frac{0'}{0}+e^x=0$

$\to$ nejde $\to K\neq 0$

3. $(1+e^x)\frac{y'}{y}+e^x=0 \qquad y(0)=1$

$$\frac{(1+e^x)}{y}\frac{dy}{dx}=-e^x$$

$$\frac{1}{y}dy=\frac{-e^x}{1+e^x}dx$$

$$\int\frac{1}{y}dy=-\int\frac{e^x}{1+e^x}dx$$

$$-\int\frac{e^x}{1+e^x}dx=\left|\begin{matrix} t=1+e^x \\ dt=e^x dx\end{matrix}\right|=-\int\frac{1}{t}dt=-\int t^{-1}dt=-\ln|t|+k=-\ln|1+e^x|+c$$

$$\ln|y|=-\ln|1+e^x|+c$$

$$y=e^{-\ln|1+e^x|+c}$$

$$y=\frac{1}{1+e^x}\cdot K \qquad K\neq 0 \qquad x\in\mathbb{R}$$

Part. řešení pro $y(0)=1$:

$$1=\frac{1}{1+e^0}\cdot k$$

$$1=\frac{1}{1+1}\cdot K$$

$$1=\frac{1}{2}K$$

$$2=K$$

Partikulární řešení pro $y(0)=1$ je $y=\frac{2}{1+e^x}$, $x\in\mathbb{R}$.

4. $y'=x^3+3, \quad y(-2)=5$

$\vdots$

$$y=\frac{x^4}{4}+3x+C, \quad x\in\mathbb{R},\ C\in\mathbb{R}$$

Part. řešení pro $y(-2)=5$:

$$5=\frac{(-2)^4}{4}+3(-2)+C$$

$$5=\frac{16}{4}-6+C$$

$$C=5-4+6$$

$$C=7$$

Partikulární řešení pro $y(-2)=5$ je $y=\frac{x^4}{4}+3x+7$, $x\in\mathbb{R}$.

5. $1 + y^2 + xyy' = 0, \; y(-3) = 1$

$\vdots$

$y = \dfrac{\sqrt{K - x^2}}{x}$

Part. řešení pro $y(-3) = 1$: $1 = \dfrac{\sqrt{K - (-3)^2}}{-3}$

$-3 = \sqrt{K - 9}$ ... nemá řešení, protože $\sqrt{K-9} \geq 0$

Partikulární řešení pro $y(-3) = 1$ neexistuje.

6. $y' = y \qquad y(-1) = \frac{1}{4}$

$\vdots$

$y = K \cdot e^x \quad K \in \mathbb{R}, \; x \in \mathbb{R}$

Part. řešení pro $y(-1) = \frac{1}{4}$: $\frac{1}{4} = K \cdot e^{-1}$

$\frac{e}{4} = K$

Partikulární řešení pro $y(-1) = \frac{1}{4}$ je $y = \frac{e}{4} \cdot e^x$, $x \in \mathbb{R}$.

7. $x^5 y' = -3, \; y\left(\frac{1}{2}\right) = 11$:

$\vdots$

$y = \dfrac{3}{4x^4} + c, \quad x \neq 0, \; c \in \mathbb{R}$

Part. řešení pro $y\left(\frac{1}{2}\right) = 11$: $11 = \dfrac{3}{4\left(\frac{1}{2}\right)^4} + c$

$11 = \dfrac{3}{4 \cdot \frac{1}{16}} + c$

$11 = \dfrac{3}{\frac{1}{4}} + c$

$11 = 3 \cdot 4 + c$

$-1 = c$

Partikulární řešení pro $y\left(\frac{1}{2}\right) = 11$ je $y = \dfrac{3}{4x^4} - 1$, $x \neq 0$.