CVIČENÍ 3

3.1

$f(x) = -x^2 + \frac{9}{4}$

1. $D(f) = \mathbb{R}$

2. $H(f)$: $-x^2 + \frac{9}{4} = y$

$x^2 = \frac{9}{4} - y$

$x = \pm\sqrt{\frac{9}{4} - y}$

$\frac{9}{4} - y \geq 0$

$\frac{9}{4} \geq y$

$y \leq \frac{9}{4}$

$H(f) = (-\infty; \frac{9}{4}\rangle$

3. $f(-x) = -x^2 + \frac{9}{4}$ $\quad f(-x) = f(x)$ ... sudá

$-f(x) = x^2 - \frac{9}{4}$ $\quad f(-x) \neq -f(x)$ ... není lichá

4. neperiodická, neobsahuje periodickou funkci

5. BN nemá

6. $-x^2 + \frac{9}{4} = 0$

$x^2 = \frac{9}{4}$

$x = \pm\frac{3}{2}$

$\ominus \quad -\frac{3}{2} \quad \oplus \quad \frac{3}{2} \quad \ominus$

7. $(-x^2 + \frac{9}{4})' = 0$

$-2x = 0$

$x = 0$

$\nearrow \oplus \quad 0 \quad \ominus \searrow$

8. $(-x^2 + \frac{9}{4})'' = 0$

$(-2x)' = 0$

$-2 = 0$

$-\infty \quad \ominus \; \cap \quad \infty$

9. ABS: nemá BN → nemá ABS

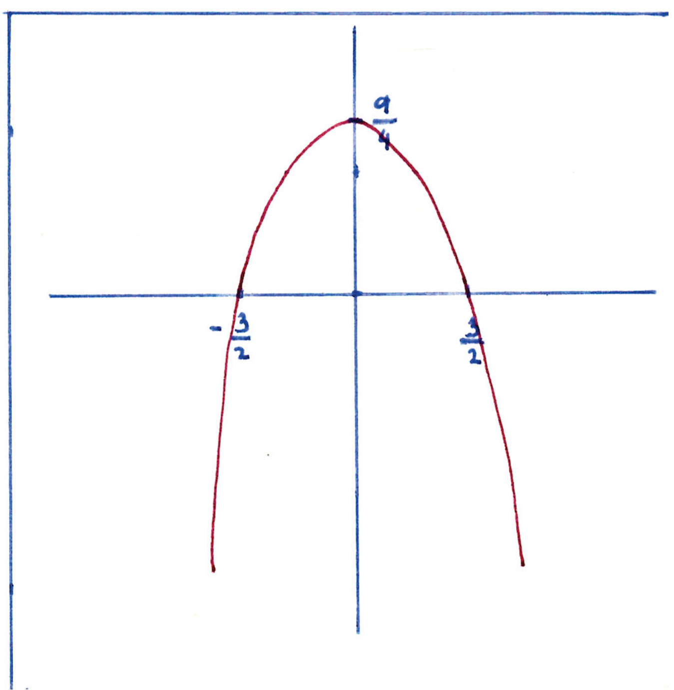
ASS: $\lim\limits_{x \to \infty} (-x^2 + \frac{9}{4}) = -\infty$

$\lim\limits_{x \to -\infty} (-x^2 + \frac{9}{4}) = -\infty$ → ASS nemá

10. $f(0) = \frac{9}{4}$

$f(-\frac{3}{2}) = 0$

$f(\frac{3}{2}) = 0$



3.2

$f(x) = x^3 + 27$

1. $D(f) = \mathbb{R}$
2. $H(f)$: $x^3 + 27 = y$

$$x^3 = y - 27$$
$$x = \sqrt[3]{y-27}$$

$H(f) = \mathbb{R}$

3. $f(-x) = -x^3 + 27$ $\quad$ $f(-x) \neq f(x) \rightarrow$ není sudá

$-f(x) = -x^3 - 27$ $\quad$ $f(-x) \neq -f(x) \rightarrow$ není lichá

4. neperiodická, neobsahuje periodickou fci
5. BN nemá
6. $x^3 + 27 = 0$

$$x^3 = -27$$
$$x = \sqrt[3]{-27}$$
$$x = -3$$

⊖ ⊕
-3

7. $(x^3 + 27)' = 0$

$3x^2 = 0 \ldots x = 0$

↗ ↗
⊕ ⊕
0

8. $(x^3 + 27)'' = 0$

$(3x^2)' = 0$

$6x = 0 \ldots x = 0$

∩ ∪
⊖ ⊕
0

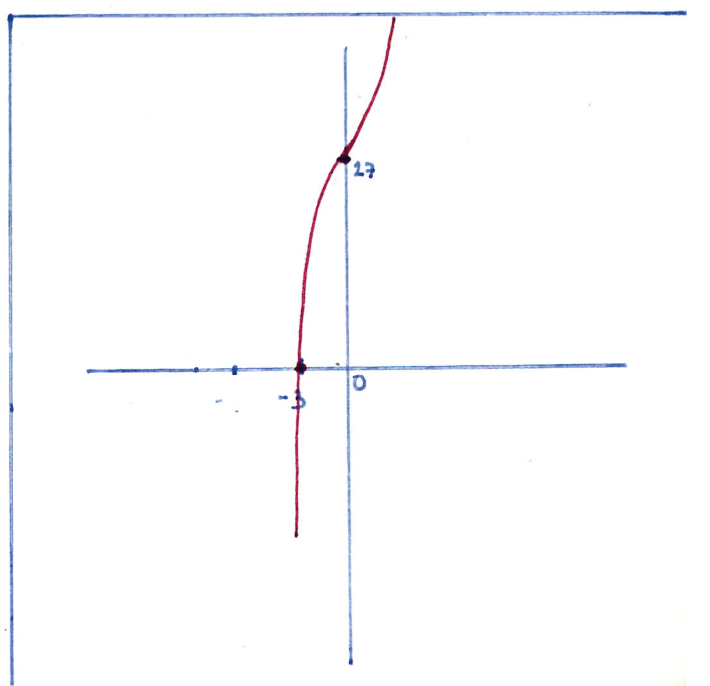
9. ABS: nemá BN $\rightarrow$ nemá ABS

ASS: $\lim\limits_{x \to \infty} x^3 + 27 = \infty$

$\lim\limits_{x \to -\infty} x^3 + 27 = -\infty \longrightarrow$ ASS nemá

10. $f(0) = 27$

$f(-3) = 0$



3.3

$f(x) = -1 + \frac{4}{x^2}$

1. $D(f) = \mathbb{R} \setminus \{0\}$

2. $H(f)$: $-1 + \frac{4}{x^2} = y$ $\qquad \frac{4}{y+1} = x^2 \qquad y \neq -1 \qquad \frac{4}{y+1} \geq 0$

$\frac{4}{x^2} = y+1 \qquad x = \pm\sqrt{\frac{4}{y+1}} \qquad y+1 \geq 0$

$y \geq -1 \quad \wedge \quad y \neq -1$

$H(f) = (-1; \infty)$

3. $f(-x) = -1 + \frac{4}{(-x)^2} = -1 + \frac{4}{x^2}$ $\qquad f(-x) = f(x) \ldots$ sudá

$-f(x) = 1 - \frac{4}{x^2}$ $\qquad f(-x) \neq -f(x) \ldots$ není lichá

4. neperiodická, neskládá se z periodické funkce

5. BN: $x = 0$

6. $-1 + \frac{4}{x^2} = 0$

$\frac{4}{x^2} = 1$

$x^2 = 4 \ldots x = \pm 2$

$\ominus$ $\oplus$ $\oplus$ $-$; $-2 \quad 0 \quad 2$

7. $\left(-1 + \frac{4}{x^2}\right)' = 0$

$-2\frac{4}{x^3} = 0$

$-\frac{8}{x^3} = 0 \ldots x \neq 0$

$\nearrow$ $\oplus$, $\searrow$ $\ominus$; 0

8. $\left(-1 + \frac{4}{x^2}\right)'' = 0$

$\left(-\frac{8}{x^3}\right)' = 0$

$\frac{24}{x^4} = 0 \qquad x \neq 0$

$\cup$ $\oplus$, $\cup$ $\oplus$; 0

9. ABS: $\lim\limits_{x \to 0^+} -1 + \frac{4}{x^2} = \infty$

$\lim\limits_{x \to 0^-} -1 + \frac{4}{x^2} = \infty$

10. $f(0)$ neexistuje

$f(2) = 0$

$f(-2) = 0$

ASS: $\lim\limits_{x \to \infty} \frac{f(x)}{x} = \lim\limits_{x \to \infty} \frac{-1}{x} + \frac{4}{x^3} = \lim\limits_{x \to \infty} \frac{-x^2 + 4}{x^3} =$

$= \lim\limits_{x \to \infty} \frac{x^2\left(-1 + \frac{4}{x^2}\right)}{x^2 \cdot x} = \lim\limits_{x \to \infty} \frac{-1 + \frac{4}{x^2}}{x} = 0$

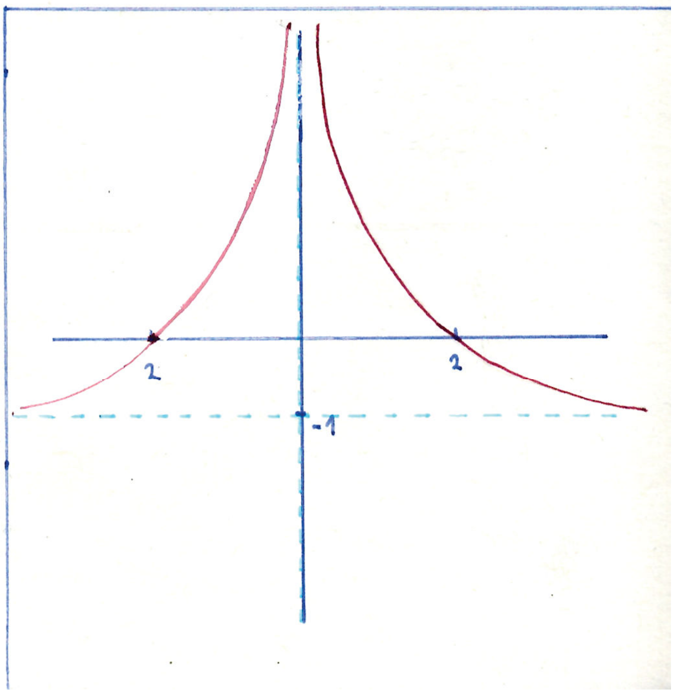
$a = 0$

$\lim\limits_{x \to \infty} f(x) - ax = \lim\limits_{x \to \infty} -1 + \frac{4}{x^2} = -1 \ldots b = -1$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \frac{-1 + \frac{4}{x^2}}{x} = 0 \ldots a = 0$

$\lim\limits_{x \to -\infty} f(x) - ax = \lim\limits_{x \to -\infty} -1 + \frac{4}{x^2} = -1 \ldots b = -1$

ASS: $y = -1$



3.4

$f(x) = \frac{2x}{x+3}$

1. $D(f) = \mathbb{R} \setminus \{-3\}$

2. $H(f)$: $\frac{2x}{x+3} = y$

$2x = y(x+3)$

$2x = yx + 3y$

$2x - yx = 3y$

$x(2-y) = 3y$

$x = \frac{3y}{2-y}$ $\quad y \neq 2$

Zk: $2 = \frac{2x}{x+3}$

$2x + 6 = 2x$

$6 = 0 \rightarrow y \neq 2$

$H(f) = \mathbb{R} \setminus \{2\}$

3. $f(-x) = -\frac{2x}{-x+3}$ $\quad f(-x) \neq f(x)$ … není sudá

$-f(x) = -\frac{2x}{x+3}$ $\quad f(-x) \neq -f(x)$… není lichá

4. neperiodická, neobsahuje periodickou funkci

5. BN: $x = -3$

6. $\frac{2x}{x+3} = 0$ $\quad x \neq -3$

$2x = 0$ $\quad x = 0$

$\oplus \quad -3 \quad \ominus \quad 0 \quad \oplus$

7. $\left(\frac{2x}{x+3}\right)' = 0$

$\frac{2(x+3) - 2x}{(x+3)^2} = 0$

$\frac{6}{(x+3)^2} = 0$

$\nearrow \oplus \quad -3 \quad \nearrow \oplus$

8. $\left(\frac{2x}{x+3}\right)'' = 0$

$\frac{3}{(x+3)^2} = 0$

$3 \cdot (-2)(x+3)^{-3} \cdot 1 = 0$

$\frac{-6}{(x+3)^3} = 0$

$\cup \oplus \quad -3 \quad \cap \ominus$

9. ABS: $\lim\limits_{x \to -3^+} \frac{2x}{x+3} = -\infty$

$\lim\limits_{x \to -3^-} \frac{2x}{x+3} = \infty$

10. $f(0) = 0$

$f(-3)$ neexistuje

ASS: $\lim\limits_{x \to \infty} \frac{f(x)}{x} = \lim\limits_{x \to \infty} \frac{2}{x+3} = 0$ … $a = 0$

$\lim\limits_{x \to \infty} f(x) - ax = \lim\limits_{x \to \infty} \frac{2x}{x+3} = \lim\limits_{x \to \infty} \frac{2x}{x(1+\frac{3}{x})} =$

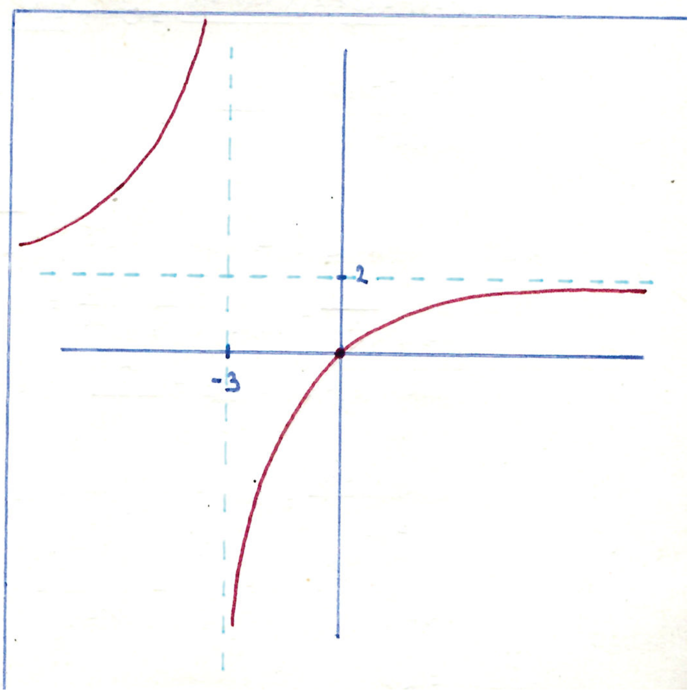
$= \lim\limits_{x \to \infty} \frac{2}{1+0} = 2$ … $b = 2$

$\lim\limits_{x \to -\infty} \frac{f(x)}{x} = \lim\limits_{x \to -\infty} \frac{2}{x+3} = 0$ … $a = 0$

$\lim\limits_{x \to -\infty} f(x) - ax = \lim\limits_{x \to -\infty} \frac{2x}{x+3} = \frac{2x}{x(1+\frac{3}{x})} = \frac{2}{1+0} = 2$

$b = 2$

ASS: $y = 2$



3.5

$f(x) = -\frac{10}{x^2+1} + 2$

1. $D(f) = \mathbb{R}$

2. $H(f)$: $-\frac{10}{x^2+1} + 2 = y$

$\frac{10}{x^2+1} = 2-y$

$10 = (2-y)(x^2+1)$

$10 = 2x^2 + 2 - x^2y - y$

$x^2(2-y) + (2-y) = 10$

$2 - y - 10 = x^2(y-2)$

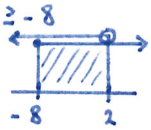
$x^2 = \frac{-y-8}{y-2}$

$x^2 = \frac{y+8}{2-y}$

$x = \pm\sqrt{\frac{y+8}{2-y}} \geq 0 \qquad y \neq 2$

1. $y+8 \geq 0 \ \wedge \ 2-y \geq 0$

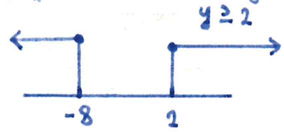
$y \geq -8 \qquad 2 \geq y$



2. $y+8 \leq 0 \ \wedge \ 2-y \leq 0$

$y \leq -8 \ \wedge \ 2 \leq y$

$y \geq 2$



$H(f) = \langle -8; 2)$

3. $f(-x) = -\frac{10}{(-x)^2+1} + 2 = -\frac{10}{x^2+1} + 2$ … $f(-x) = f(x)$ … sudá

$f(-x) \neq -f(x)$… není lichá

$-f(x) = \frac{10}{x^2+1} + 2$

4. neperiodická, neskládá se z periodické funkce

5. BN nemá

6. $-\frac{10}{x^2+1} + 2 = 0$

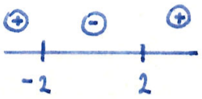
$\frac{10}{x^2+1} = 2$

$10 = 2x^2 + 2$

$2x^2 = 8$

$x^2 = 4$

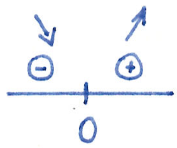
$x = \pm 2$



7. $\left(-\frac{10}{x^2+1} + 2\right)' = 0$

$-10\,(x^2+1)^{-2}(-1)2x = 0$

$\frac{20x}{(x^2+1)^2} = 0$



8. $\left(-\frac{10}{x^2+1} + 2\right)'' = 0$

$\left(\frac{20x}{(x^2+1)^2}\right)' = 0$

$\frac{20(x^2+1)^2 - 20x\,2(x^2+1)\cdot 2x}{(x^2+1)^4} = 0$

$\frac{20(x^2+1)^2 - 80x^2(x^2+1)}{(x^2+1)^4}$

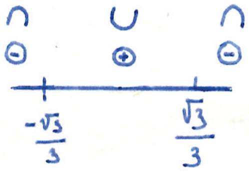
$\frac{20x^2 + 20 - 80x^2}{(x^2+1)^3} = 0$

$\frac{-60x^2 + 20}{(x^2+1)^3} = 0$

$\frac{20(1-3x^2)}{(x^2+1)^3} = 0$

$3x^2 = 1 \qquad x = \pm\sqrt{\frac{1}{3}}$

$x^2 = \frac{1}{3} \qquad x = \pm\frac{\sqrt{3}}{3}$



10. $f(0) = -8$

$f(-2) = 0$

$f(2) = 0$

$f\left(\frac{\sqrt{3}}{3}\right) = \frac{-10}{4/3} + 2 = \frac{-10\cdot 3}{4} + \frac{8}{4}$

$= \frac{-30+8}{4} = \frac{-22}{4} = -\frac{11}{2}$

9. ABS: nemá BN → nemá ABS

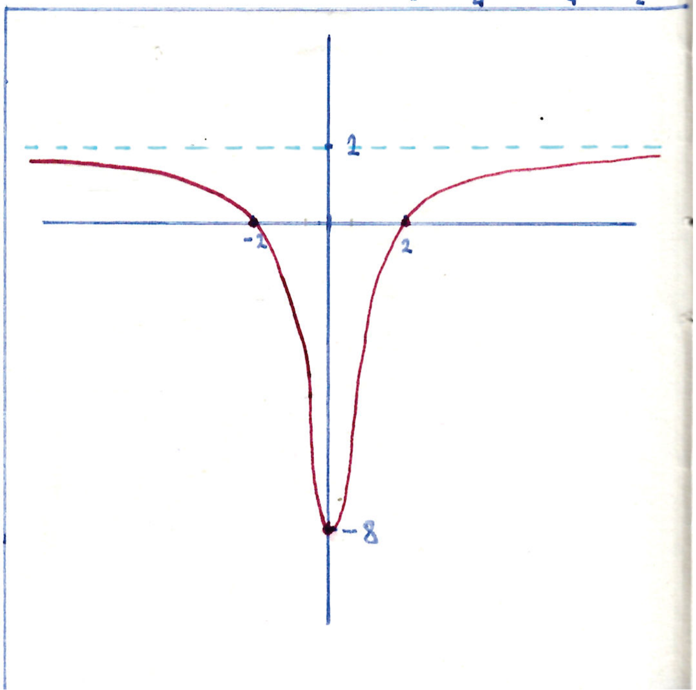
ASS: $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \lim\limits_{x\to\infty} \frac{-10}{x^3+x} + \frac{2}{x} = 0$ … $a = 0$

$\lim\limits_{x\to\infty} f(x) - ax = \lim\limits_{x\to\infty} -\frac{10}{x^2+1} + 2 = 0 + 2 = 2$ … $b = 2$

$\lim\limits_{x\to-\infty} \frac{f(x)}{x} = \lim\limits_{x\to-\infty} -\frac{10}{x^3+x} + \frac{2}{x} = 0$ … $a = 0$

$\lim\limits_{x\to-\infty} f(x) - ax = \lim\limits_{x\to-\infty} -\frac{10}{x^2+1} + 2 = 2$ … $b = 2$

ASS: $y = 2$



3.6

$f(x) = \frac{(x+3)^2}{x^2}$

1. $D(f) = \mathbb{R} \setminus \{0\}$

2. $H(f): \frac{(x+3)^2}{x^2} = y$

$x^2 + 6x + 9 = yx^2$

$x^2(1-y) + 6x + 9 = 0$

$D: b^2 - 4ac$

$36 - 4 \cdot 9(1-y)$

$36 - 36 + 36y$

$x = \frac{-6 \pm \sqrt{36y}}{2-2y} = \frac{-6 \pm 6\sqrt{y}}{2(1-y)}$ $\quad y \geq 0$, $y \neq 1$

Zk: $1 = \frac{x^2+6x+9}{x^2}$

$x^2 = x^2 + 6x + 9$

$6x = -9$

$x = -\frac{9}{6} = -\frac{3}{2}$

$\Rightarrow y = 1$

$H(f) = \langle 0; \infty)$

3. $f(-x) = \frac{(-x+3)^2}{(-x)^2} = \frac{(3-x)^2}{x^2}$

$-f(x) = -\frac{(x+3)^2}{x^2}$

$f(-x) \neq f(x)$ ... není sudá

$f(-x) \neq f(x)$ ... není lichá

4. neperiodická, neobsahuje periodickou funkci

5. BN: $x = 0$

6. $\frac{(x+3)^2}{x^2} = 0 \quad x \neq 0$

$(x+3)^2 = 0 \quad x = -3$

$\oplus \quad \oplus \quad \oplus$

$-3 \quad 0$

7. $((x+3)^2 \cdot x^{-2})' = 0$

$2(x+3) \cdot 1 x^{-2} + (x+3)^2 \cdot (-2) x^{-3}$

$\frac{2(x+3)}{x^2} - \frac{2(x+3)^2}{x^3} = 0$

$\frac{2x^2 + 6x - 2(x^2+6x+9)}{x^3} = 0$

$\frac{2x^2 + 6x - 2x^2 - 12x - 18}{x^3} = 0$

$\frac{-6x-18}{x^3} = 0 \quad x \neq 0$

$6x = -18$

$x = -3$

$\searrow \quad \nearrow \quad \searrow$

$\ominus \quad \oplus \quad \ominus$

$-3 \quad 0$

8. $\left(\frac{(x+3)^2}{x^2}\right)'' = 0$

$\left(\frac{-6(x+3)}{x^3}\right)' = 0$

$\frac{-6x^3 + 6(x+3)3x^2}{x^6} = 0$

$\frac{-6x^3 + 18x^3 + 54x^2}{x^6} = 0$

$\frac{12x^3 + 54x^2}{x^6} = 0$

$\frac{6x^2(2x+9)}{x^6} = 0$

$\frac{6(2x+9)}{x^4} = 0$

$6(2x+9) = 0 \quad x \neq 0$

$2x = -9$

$x = -4.5$

$\cap \quad \cup \quad \cup$

$\ominus \quad \oplus \quad \oplus$

$-4.5 \quad 0$

9. ABS: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{(x+3)^2}{x^2} = \lim_{x \to 0^+} \frac{x^2+6x+9}{x^2} =$

$= \lim_{x \to 0^+} 1 + \underbrace{\frac{6}{x}}_{\infty} + \underbrace{\frac{9}{x^2}}_{\infty} = \infty$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{(x+3)^2}{x^2} = \lim_{x \to 0^-} 1 + \frac{6}{x} + \frac{9}{x^2} = \infty$ $\quad -\infty \quad \infty$ rychlejší konvergence

10. $f(0)$ neex.

$f(-3) = 0$

ASS: $\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^2+6x+9}{x^3} = \lim_{x \to \infty} \frac{1}{x} + \frac{6}{x^2} + \frac{9}{x^3} = 0$

$a = 0$

$\lim_{x \to \infty} f(x) - ax = \lim_{x \to \infty} \frac{x^2+6x+9}{x^2} = \lim_{x \to \infty} 1 + \frac{6}{x} + \frac{9}{x^2} = 1$

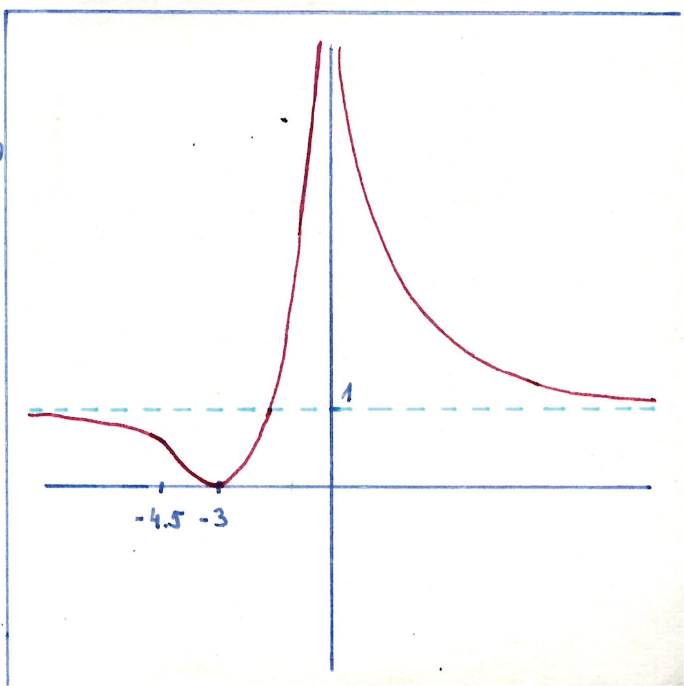
$b = 1$

$\lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \frac{x^2+6x+9}{x^3} = \lim_{x \to -\infty} \frac{1}{x} + \frac{6}{x^2} + \frac{9}{x^3} = 0$

$a = 0$

$\lim_{x \to -\infty} f(x) - ax = \lim_{x \to -\infty} 1 + \frac{6}{x} + \frac{9}{x^2} = 1$ ... $b = 1$

ASS: $y = 1$



CVIČENÍ 4

4.1

1. $\frac{\partial}{\partial x}(2xy + \ln y - \cos x) = 2y + \sin x$

$\frac{\partial}{\partial y}(2xy + \ln y - \cos x) = 2x + \frac{1}{y}$

2. $\frac{\partial}{\partial x}(\sin(x-y)) = \cos(x-y)$

$\frac{\partial}{\partial y}(\sin(x-y)) = \cos(x-y)(-1) = -\cos(x-y)$

3. $\frac{\partial}{\partial x}\left(\frac{1}{xy} + \ln x\right) = \frac{\partial}{\partial x}\left(\frac{1}{y}x^{-1} + \ln x\right) = -\frac{1}{yx^2} + \frac{1}{x} = -\frac{1-yx}{yx^2} = \frac{xy-1}{x^2y}$

$\frac{\partial}{\partial y}\left(\frac{1}{xy} + \ln x\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}y^{-1} + \ln x\right) = -\frac{1}{xy^2}$

4. $\frac{\partial}{\partial x}\ln(x^2-y^2) = \frac{1}{x^2-y^2}2x = \frac{2x}{x^2-y^2}$

$\frac{\partial}{\partial y}\ln(x^2-y^2) = \frac{1}{x^2-y^2}(-2y) = \frac{-2y}{x^2-y^2}$

5. $\frac{\partial}{\partial x}\frac{x^2}{y^2} = \frac{2x}{y^2}$

$\frac{\partial}{\partial y}\frac{x^2}{y^2} = -\frac{2x^2}{y^3}$

6. $\frac{\partial}{\partial x}\cos xy = -\sin xy \cdot y = -y\sin xy$

$\frac{\partial}{\partial y}\cos xy = -\sin xy \cdot x = -x\sin xy$

7. $\frac{\partial}{\partial x}e^{(1-x^2)y} = e^{(1-x^2)y} \cdot -2xy = -2xy\,e^{(1-x^2)y}$

$\frac{\partial}{\partial y}e^{(1-x^2)y} = e^{(1-x^2)y}(1-x^2) = (1-x^2)e^{(1-x^2)y}$

4.2

1. $f(x,y) = 2xy - \ln y - \cos x$ $\qquad \frac{\partial}{\partial x} = 2y + \sin x \qquad \frac{\partial}{\partial y} = 2x + \frac{1}{y}$

$$\frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x}(2y + \sin x) = \cos x$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y}\left(2x + \frac{1}{y}\right) = -\frac{1}{y^2}$$

$$\frac{\partial^2}{\partial xy} f(x,y) = \frac{\partial}{\partial y}(2y + \sin x) = 2$$

$$\frac{\partial}{\partial yx} f(x,y) = \frac{\partial}{\partial x}\left(2x + \frac{1}{y}\right) = 2$$

2. $f(x,y) = \sin(x+y)$ $\qquad \frac{\partial}{\partial x} = \cos(x-y) \qquad \frac{\partial}{\partial y} = -\cos(x-y)$

$$\frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x}(\cos(x-y)) = -\sin(x-y)$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y}(-\cos(x-y)) = \sin(x-y)(-1) = -\sin(x-y)$$

$$\frac{\partial^2}{\partial xy} f(x,y) = \frac{\partial}{\partial y}(\cos(x-y)) = -\sin(x-y)(-1) = \sin(x-y)$$

$$\frac{\partial^2}{\partial yx} f(x,y) = \frac{\partial}{\partial x}(-\cos(x-y)) = \sin(x-y)$$

3. $f(x,y) = \frac{1}{xy} + \ln x$ $\qquad \frac{\partial}{\partial x} = \frac{xy-1}{x^2y} \qquad \frac{\partial}{\partial y} = -\frac{1}{xy^2}$

$$\frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x}\left(\frac{xy-1}{x^2y}\right) = \frac{yx^2y - 2xy(xy-1)}{x^4y^2} = \frac{x^2y^2 - 2x^2y^2 + 2xy}{x^4y^2} = \frac{2xy - x^2y^2}{x^4y^2}$$

$$= \frac{2 - xy}{x^3y}$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y}\left(-\frac{1}{xy^2}\right) = \frac{\partial}{\partial y}\left(-\frac{1}{x}y^{-2}\right) = \frac{2}{xy^3}$$

$$\frac{\partial^2}{\partial xy} f(x,y) = \frac{\partial}{\partial y}\left(\frac{xy-1}{x^2y}\right) = \frac{xx^2y - x^2(xy-1)}{x^4y^2} = \frac{x^3y - x^3y + x^2}{x^4y^2} = \frac{x^2}{x^4y^2} = \frac{1}{x^2y^2}$$

$$\frac{\partial^2}{\partial yx} f(x,y) = \frac{\partial}{\partial x}\left(-\frac{1}{xy^2}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{y^2}x^{-1}\right) = \frac{1}{x^2y^2}$$

4. $f(x,y) = \ln(x^2-y^2)$ $\qquad \frac{\partial}{\partial x} = \frac{2x}{x^2-y^2} \qquad \frac{\partial}{\partial y} = \frac{-2y}{x^2-y^2}$

$$\frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2-y^2}\right) = \frac{2(x^2-y^2)-2x\,2x}{(x^2-y^2)^2} = \frac{2x^2-2y^2-4x^2}{(x^2-y^2)^2} = \frac{-2y^2-2x^2}{(x^2-y^2)^2} = \frac{-2(x^2+y^2)}{(x^2-y^2)^2}$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y}\left(\frac{-2y}{x^2-y^2}\right) = \frac{-2(x^2-y^2)-(-2y)(-2y)}{(x^2-y^2)^2} = \frac{-2x^2+2y^2-4y^2}{(x^2-y^2)^2} = \frac{-2x^2-2y^2}{(x^2-y^2)^2} = \frac{-2(x^2+y^2)}{(x^2-y^2)^2}.$$

$$\frac{\partial^2}{\partial xy} f(x,y) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2-y^2}\right) = \frac{-2x(-2y)}{(x^2-y^2)^2} = \frac{4xy}{(x^2-y^2)^2}$$

$$\frac{\partial^2}{\partial yx} f(x,y) = \frac{\partial}{\partial x}\left(\frac{-2y}{x^2-y^2}\right) = \frac{-(-2y)(2x)}{(x^2-y^2)^2} = \frac{4xy}{(x^2-y^2)^2}$$

5. $f(x,y) = \frac{x^2}{y^2}$ $\qquad \frac{\partial}{\partial x} = \frac{2x}{y^2} \qquad \frac{\partial}{\partial y} = -\frac{2x^2}{y^3}$

$$\frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x}\left(\frac{2x}{y^2}\right) = \frac{2}{y^2}$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y}\left(-\frac{2x^2}{y^3}\right) = \frac{6x^2}{y^4}$$

$$\frac{\partial^2}{\partial xy} f(x,y) = \frac{\partial}{\partial y}\left(\frac{2x}{y^2}\right) = \frac{-4x}{y^3}$$

$$\frac{\partial^2}{\partial yx} f(x,y) = \frac{\partial}{\partial x}\left(\frac{-2x^2}{y^3}\right) = \frac{-4x}{y^3}$$

6. $f(x,y) = \cos xy$ $\qquad \frac{\partial}{\partial x} = -y\sin xy \qquad \frac{\partial}{\partial y} = -x\sin xy$

$$\frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x}(-y\sin xy) = -y\cos xy \cdot y = -y^2\cos xy$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y}(-x\sin xy) = -x\cos xy \cdot x = -x^2\cos xy$$

$$\frac{\partial^2}{\partial xy} f(x,y) = \frac{\partial}{\partial y}(-y\sin xy) = -\sin xy - y\cos xy \cdot x = -\sin xy - xy\cos xy$$

$$\frac{\partial^2}{\partial yx} f(x,y) = \frac{\partial}{\partial x}(-x\sin xy) = -\sin xy - x\cos xy \cdot y = -\sin xy - xy\cos xy$$

7. $f(x,y) = e^{(1-x^2)y}$ $\qquad \frac{\partial}{\partial x} = -2xy\, e^{(1-x^2)y}$ $\qquad \frac{\partial}{\partial y} = (1-x^2)e^{(1-x^2)y}$

$$\frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x}\left(-2xy\, e^{(1-x^2)y}\right) = -2y e^{(1-x^2)y} - 2xy\, e^{(1-x^2)y}(-2xy) =$$

$$= -2ye^{(1-x^2)y} + 4x^2y^2e^{(1-x^2)y} = 2y\, e^{(1-x^2)y}(2x^2y - 1)$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y}(1-x^2)e^{(1-x^2)y} = (1-x^2)e^{(1-x^2)y}\cdot(1-x^2) = (1-x^2)^2e^{(1-x^2)y}$$

$$\frac{\partial^2}{\partial xy} f(x,y) = \frac{\partial}{\partial y}\left(-2xy\, e^{(1-x^2)y}\right) = -2x\, e^{(1-x^2)y} - 2xy\, e^{(1-x^2)y}(1-x^2) = -2xe^{(1-x^2)y}\left(1 + (1-x^2)y\right)$$

$$\frac{\partial^2}{\partial yx} f(x,y) = \frac{\partial}{\partial x}\left((1-x^2)\, e^{(1-x^2)y}\right) = -2xe^{(1-x^2)y} + (1-x^2)e^{(1-x^2)y}\cdot -2xy = -2xe^{(1-x^2)y} - 2xy(1-x^2)e^{(1-x^2)y} =$$

$$= -2xe^{(1-x^2)y}\left(1 + y(1-x^2)\right)$$

$$= -2xe^{(1-x^2)y}(1 + y - x^2y) = 2xe^{(1-x^2)y}(x^2y - y - 1)$$

4.3

1. $f(x,y) = 2x^3 + 3y^2 - 6xy$

$\frac{\partial}{\partial x} f(x,y) = 6x^2 - 6y = 0$ $\qquad \frac{\partial}{\partial y} f(x,y) = 6y - 6x = 0$

$6x^2 - 6x = 0$ $\qquad 6y = 6x$

$6x(x-1) = 0 \rightarrow x = 0$ $\qquad y = x$

$x = 1$

$y = x = 0 \longrightarrow SB[0,0]$

$y = x = 1 \longrightarrow SB[1,1]$

$\frac{\partial^2}{\partial x^2} = 12x$ $\qquad \frac{\partial^2}{\partial y^2} = 6$ $\qquad \frac{\partial^2}{\partial xy} = -6$ $\qquad \frac{\partial^2}{\partial yx} = -6$ $\Rightarrow$ Hessova matice: $\begin{pmatrix} 12x & -6 \\ -6 & 6 \end{pmatrix}$

$[0,0]$: $\left|\begin{pmatrix} 0 & -6 \\ -6 & 6 \end{pmatrix}\right| = 0\cdot 6 - 36 = -36 < 0 \rightarrow H < 0 \longrightarrow SB[0,0]$

$[1,1]$: $\left|\begin{pmatrix} 12 & -6 \\ -6 & 6 \end{pmatrix}\right| = 72 - 36 = 36 > 0 \rightarrow H > 0 \longrightarrow [1,1]$ je lokální extrém

$\oplus \rightarrow \cup$

$\frac{\partial^2}{\partial x^2} f(x,y)\Big|_{[1,1]} = 12 > 0 \rightarrow$ konvexní tvar $f(x,y) \longrightarrow m[1,1]$

2. $f(x,y) = x^2 + 4xy + 6y^2 - 2x + 8y - 5$

$\frac{\partial}{\partial x} f(x,y) = 2x + 4y - 2 = 0$

$x + 2y - 1 = 0$

$-2 - 3y + 2y - 1 = 0$

$-y - 3 = 0$

$y = -3$

$\frac{\partial}{\partial y} f(x,y) = 4x + 12y + 8 = 0$

$x + 3y + 2 = 0$

$x = -2 - 3y$

$x = -2 - 3(-3)$

$x = -2 + 9$

$x = 7 \quad \longrightarrow \quad SB\,[7,-3]$

$\frac{\partial^2}{\partial x^2} = 2 \qquad \frac{\partial^2}{\partial y^2} = 12 \qquad \frac{\partial^2}{\partial xy} = 4 \qquad \frac{\partial^2}{\partial yx} = 4 \quad \longrightarrow$ Hessova matice: $\begin{pmatrix} 2 & 4 \\ 4 & 12 \end{pmatrix}$

$[7,-3]: \left|\begin{pmatrix} 2 & 4 \\ 4 & 12 \end{pmatrix}\right| = 2\cdot 12 - 4\cdot 4 = 24 - 16 = 8 > 0 \longrightarrow H > 0 \longrightarrow [7,-3]$ je lokální extrém

$\oplus \to \cup$

$\frac{\partial^2}{\partial x^2} f(x,y)\Big|_{[7,-3]} = 2 > 0 \longrightarrow$ konvexní tvar $f(x,y) \longrightarrow m\,[7,-3]$

3. $f(x,y) = 5 + 6x - 4x^2 - 3y^2$

$\frac{\partial}{\partial x} f(x,y) = 6 - 8x = 0$

$-8x = -6$

$8x = 6$

$x = \frac{6}{8} = \frac{3}{4} \quad \longrightarrow \quad SB\,[\frac{3}{4}, 0]$

$\frac{\partial}{\partial y} f(x,y) = -6y = 0$

$y = 0$

$\frac{\partial^2}{\partial x^2} = -8 \qquad \frac{\partial^2}{\partial y^2} = -6 \qquad \frac{\partial^2}{\partial xy} = 0 \qquad \frac{\partial^2}{\partial yx} = 0 \quad \longrightarrow$ Hessova matice: $\begin{pmatrix} -8 & 0 \\ 0 & -6 \end{pmatrix}$

$[\frac{3}{4}, 0]: \left|\begin{pmatrix} -8 & 0 \\ 0 & -6 \end{pmatrix}\right| = 48 - 0 = 48 > 0 \longrightarrow H > 0 \longrightarrow [\frac{3}{4}, 0]$ je lokální extrém

$\ominus \to \cap$

$\frac{\partial^2}{\partial x^2} f(x,y)\Big|_{[\frac{3}{4},0]} = -8 < 0 \longrightarrow$ konkávní tvar $f(x,y) \longrightarrow M\,[\frac{3}{4}, 0]$

4

4. $f(x,y) = 8x^3 + y^3 - 6xy + 4$

$$\frac{\partial}{\partial x} f(x,y) = 24x^2 - 6y = 0$$
$$4x^2 - y = 0$$
$$y = 4x^2$$

$$\frac{\partial}{\partial y} f(x,y) = 3y^2 - 6x = 0$$
$$y^2 - 2x = 0$$
$$(4x^2)^2 - 2x = 0$$
$$16x^4 - 2x = 0$$
$$8x^4 - x = 0$$
$$x = 0 \leftarrow x(8x^3 - 1) = 0$$
$$8x^3 - 1 = 0 \leftarrow$$
$$x^3 = \frac{1}{8}$$
$$x = \sqrt[3]{\frac{1}{8}}$$
$$x = \frac{1}{2}$$

$SB[0,0] \leftarrow y = 4 \cdot 0^2 = 0$

$SB[\frac{1}{2}, 1] \leftarrow y = 4 \cdot (\frac{1}{2})^2 = 1$

$$\frac{\partial^2}{\partial x^2} = 48x \qquad \frac{\partial^2}{\partial y^2} = 6y \qquad \frac{\partial^2}{\partial xy} = -6 \qquad \frac{\partial^2}{\partial yx} = -6 \longrightarrow \text{Hessova matice: } \begin{pmatrix} 48x & -6 \\ -6 & 6y \end{pmatrix}$$

$$[0,0]: \left| \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} \right| = 0 - 36 = -36 < 0 \to H < 0 \longrightarrow S[0,0]$$

$$[\tfrac{1}{2}, 1]: \left| \begin{pmatrix} 24 & -6 \\ -6 & 6 \end{pmatrix} \right| = 24 \cdot 6 - 36 = 6(24-6) = 6 \cdot 18 = 108 > 0 \longrightarrow H > 0 \longrightarrow [\tfrac{1}{2}, 1] \text{ je lokální extrém}$$

$\oplus \to \cup$

$$\frac{\partial^2}{\partial x^2} f(x,y) \Big|_{[\frac{1}{2},1]} = 24 > 0 \longrightarrow \text{konvexní tvar } f(x,y) \longrightarrow m[\tfrac{1}{2}, 1]$$

5. $f(x,y) = x(x-6) + y(y-9) + xy$

$$\frac{\partial}{\partial x} = x - 6 + x + y = 0$$
$$y = 6 - 2x$$
$$y = 6 - 2 \cdot 1 = 6 - 2 = 4$$
$$\Downarrow$$
$$SB[1,4]$$

$$\frac{\partial}{\partial y} f(x,y) = y - 9 + y + x = 0$$
$$2y + x - 9 = 0$$
$$2(6 - 2x) + x - 9 = 0$$
$$12 - 4x + x - 9 = 0$$
$$-3x + 3 = 0$$
$$3x = 3$$
$$x = 1$$

$$\frac{\partial^2}{\partial x^2} = 2 \qquad \frac{\partial^2}{\partial y^2} = 2 \qquad \frac{\partial^2}{\partial xy} = 1 \qquad \frac{\partial^2}{\partial yx} = 1 \longrightarrow \text{Hessova matice: } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$[1,4]: \left| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right| = 4 - 1 = 3 > 0 \to H > 0 \longrightarrow [1,4] \text{ je lokální extrém}$$

$\oplus \to \cup$

$$\frac{\partial^2}{\partial x^2} f(x,y) \Big|_{[1,4]} = 2 > 0 \longrightarrow \text{konvexní tvar } f(x,y) \longrightarrow m[1,4]$$

6. $f(x,y) = x^3 + xy^2 + 6xy$

$$\frac{\partial}{\partial x} f(x,y) = 3x^2 + y^2 + 6y = 0$$

$$\frac{\partial}{\partial y} f(x,y) = 2xy + 6x = 0$$

$$xy + 3x = 0$$

$$x = 0 \leftarrow x(y+3) = 0$$

$$y = -3 \leftarrow$$

$$3\cdot 0^2 + y^2 + 6y = 0$$

$$y^2 + 6y = 0$$

$$SB[0,0] \leftarrow y = 0 \leftarrow y(y+6) = 0$$

$$SB[0,-6] \leftarrow y = -6 \leftarrow$$

$$3x^2 + (-3)^2 + 6(-3) = 0$$

$$3x^2 + 9 - 18 = 0$$

$$3x^2 = 9$$

$$x^2 = 3$$

$$x = \pm\sqrt{3} \longrightarrow SB[\sqrt{3}, -3]$$

$$\longrightarrow SB[-\sqrt{3}, -3]$$

$$\frac{\partial^2}{\partial x^2} = 6x \qquad \frac{\partial^2}{\partial y^2} = 2x \qquad \frac{\partial^2}{\partial xy} = 2y + 6 \qquad \frac{\partial^2}{\partial yx} = 2y + 6 \longrightarrow \text{Hessova m: } \begin{pmatrix} 6x & 2y+6 \\ 2y+6 & 2x \end{pmatrix}$$

$$[0,0]: \left|\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}\right| = 0 - 36 = -36 < 0 \longrightarrow H < 0 \longrightarrow S[0,0]$$

$$[0,-6]: \left|\begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}\right| = 0 - 36 = -36 < 0 \longrightarrow H < 0 \longrightarrow S[0,-6]$$

$[\sqrt{3},-3]: \left|\begin{pmatrix} 6\sqrt{3} & 0 \\ 0 & 2\sqrt{3} \end{pmatrix}\right| = 6\sqrt{3} \cdot 2\sqrt{3} - 0 = 12 \cdot 3 = 36 > 0 \longrightarrow H > 0 \longrightarrow [\sqrt{3},-3]$ je lokální extrém

$\oplus \rightarrow \cup$

$\frac{\partial^2}{\partial x^2} f(x,y)\Big|_{[\sqrt{3},-3]} = 6\sqrt{3} > 0 \longrightarrow$ konvexní tvar $f(x,y) \longrightarrow m[\sqrt{3},-3]$

$[-\sqrt{3},-3]: \left|\begin{pmatrix} -6\sqrt{3} & 0 \\ 0 & -2\sqrt{3} \end{pmatrix}\right| = -6\sqrt{3} \cdot (-2\sqrt{3}) - 0 = 12 \cdot 3 = 36 > 0 \longrightarrow H > 0 \longrightarrow [-\sqrt{3},-3]$ je lokální extrém

$\ominus \rightarrow \cap$

$\frac{\partial^2}{\partial x^2} f(x,y)\Big|_{[-\sqrt{3},-3]} = -6\sqrt{3} < 0 \longrightarrow$ konkávní tvar $f(x,y) \longrightarrow M[-\sqrt{3},-3]$