CVIČENÍ 5

5.1

1. $\int 3\sqrt{x}\,dx = 3\int x^{\frac{1}{2}}dx = 3\frac{2}{3}\,x^{\frac{3}{2}} + c = 2\cdot\sqrt{x^3} + c$

2. $\int \frac{3}{4}\,dx = \frac{3}{4}x + c$

3. $\int 4x^{-3}\,dx = 4\int x^{-3}dx = -4\cdot\frac{1}{2}\,x^{-2} + c = -2x^{-2} + c = -2\frac{1}{x^2} + c$

4. $\int e^x\left(1 + \frac{e^x}{3}\right)dx = \int e^x + \frac{e^{2x}}{3}\,dx = \int e^x dx + \frac{1}{3}\int e^{2x}dx = e^x + \frac{1}{3}\cdot\frac{1}{2}e^{2x} + c = e^x + \frac{1}{6}e^{2x} + c$

5. $\int \frac{(2\sqrt{x}+1)^2}{x^2} + \cos^{-2}x\,dx = \int \frac{4x + 4\sqrt{x} + 1}{x^2}\,dx + \int \cos^{-2}x\,dx = \int \frac{4}{x}dx + \int 4\frac{\sqrt{x}}{x^2}\,dx + \int \frac{1}{x^2}dx + \int \cos^{-2}x\,dx$

$= 4\ln|x| + 4\int x^{\frac{1}{2}}\cdot x^{-2}\,dx + \int x^{-2}dx + \mathrm{tg}\,x + c$

$= 4\ln|x| + 4\int x^{-\frac{3}{2}}\,dx - x^{-1} + \mathrm{tg}\,x + c$

$= 4\ln|x| - 4\cdot 2\cdot x^{-\frac{1}{2}} - \frac{1}{x} + \mathrm{tg}\,x + c$

$= 4\ln|x| - \frac{8}{\sqrt{x}} - \frac{1}{x} + \mathrm{tg}\,x + c$

6. $\int(4x^5 + x^3 - 5)\,dx = \frac{4}{6}x^6 + \frac{1}{4}x^4 - 5x + c = \frac{2}{3}x^6 + \frac{1}{4}x^4 - 5x + c$

7. $\int \frac{x^4 - 10x^2 + 5}{x^2}\,dx = \int x^2 - 10 + \frac{5}{x^2}\,dx = \frac{1}{3}x^3 - 10x + 5\int x^{-2}dx = \frac{x^3}{3} - 10x - \frac{5}{x} + c$

8. $\int \frac{\sqrt{x}}{x^2}\,dx = \int x^{\frac{1}{2}}\cdot x^{-2}\,dx = \int x^{-\frac{3}{2}}\,dx = -2\,x^{-\frac{1}{2}} + c = -\frac{2}{\sqrt{x}} + c$

9. $\int \frac{5}{x^{2/7}}\,dx = 5\int x^{-2/7}dx = 5\cdot\frac{7}{5}\,x^{5/7} + c = 7x^{5/7} + c$

10. $\int \frac{3}{x^4} + \frac{1}{\sqrt{x}}\,dx = \int 3x^{-4} + x^{-1/2}\,dx = -3\frac{1}{3}\,x^{-3} + 2\,x^{\frac{1}{2}} + c = -\frac{1}{x^3} + 2\sqrt{x} + c$

5.2

1. $\int \sin(2x-5)\,dx = \begin{vmatrix} s = 2x-5 \\ ds = 2dx \end{vmatrix} = \int \frac{1}{2}\sin s\,ds = \frac{1}{2}(-\cos s) + k = -\frac{1}{2}\cos(2x-5) + C$

2. $\int \frac{3\ln^2 x}{x}\,dx = \begin{vmatrix} s = \ln x \\ ds = \frac{1}{x} \end{vmatrix} = 3\int s^2\,ds = 3\cdot\frac{1}{3}s^3 + k = \ln^3 x + C$

3. $\int \frac{1}{\sqrt{5-4x}}\,dx = \begin{vmatrix} s = 5-4x \\ ds = -4dx \end{vmatrix} = \int \frac{1}{\sqrt{s}}\cdot(-\frac{1}{4})ds = -\frac{1}{4}\int s^{-\frac{1}{2}}\,ds = -\frac{1}{4}\cdot 2\cdot s^{\frac{1}{2}} + k = -\frac{1}{2}s^{\frac{1}{2}} + k$

$= -\frac{1}{2}\sqrt{5-4x} + C$

4. $\int x e^{-x^2}\,dx = \begin{vmatrix} s = -x^2 \\ ds = -2x\,dx \end{vmatrix} = \int -\frac{1}{2}e^s\,ds = -\frac{1}{2}e^s + k = -\frac{1}{2}e^{-x^2} + C$

5. $\int \frac{1}{6}\left(1-\frac{x}{6}\right)^{-2}dx = \begin{vmatrix} s = 1-\frac{x}{6} \\ ds = -\frac{1}{6}dx \end{vmatrix} = -\int s^{-2}\,ds = -s^{-1} + k = -\left(1-\frac{x}{6}\right)^{-1} + C = \left(\frac{6-x}{6}\right)^{-1} + C =$

$= \frac{6}{6-x} + C$

6. $\int \frac{1}{\cos^2(1-x)}\,dx = \begin{vmatrix} s = 1-x \\ ds = -dx \end{vmatrix} = -\int \frac{1}{\cos^2 s}\,ds = -\tan s + k = -\tan(1-x) + C$

7. $\int 6x^2 e^{-2x^3}\,dx = \begin{vmatrix} s = -2x^3 \\ ds = -6x^2\,dx \end{vmatrix} = -\int e^s\,ds = -e^s + k = -e^{-2x^3} + C$

8. $\int \frac{\sin x}{2\sqrt{\cos^3 x}}\,dx = \begin{vmatrix} s = \cos x \\ ds = -\sin x\,dx \end{vmatrix} = \int \frac{-1}{2\sqrt{s^3}}\,ds = -\frac{1}{2}\int s^{-\frac{3}{2}}\,ds = -\frac{1}{2}(-2)\cdot s^{-\frac{1}{2}} + k = s^{-\frac{1}{2}} + k$

$= \cos x^{-\frac{1}{2}} + C = \frac{1}{\sqrt{\cos x}} + C$

9. $\int \frac{4\cos x}{\sqrt[3]{1+2\sin x}}\,dx = \begin{vmatrix} s = 1+2\sin x \\ ds = 2\cos x\,dx \end{vmatrix} = \int \frac{2}{\sqrt[3]{s}}\,ds = 2\int s^{-\frac{1}{3}}\,ds = 2\cdot\frac{3}{2}s^{\frac{2}{3}} + k = 3s^{\frac{2}{k}} + k =$

$= 3(1+2\sin x)^{\frac{2}{3}} + C = 3\sqrt[3]{(1+2\sin x)^2} + C$

10. $\int \sqrt{1+2x}\,dx = \begin{vmatrix} s = 1+2x \\ ds = 2dx \end{vmatrix} = \frac{1}{2}\int \sqrt{s}\,ds = \frac{1}{2}\int s^{\frac{1}{2}}\,ds = \frac{1}{2}\frac{2}{3}s^{\frac{3}{2}} + k = \frac{1}{3}(1+2x)^{\frac{3}{2}} + C = \frac{1}{3}\sqrt{(1+2x)^3} + C$

5.3

1. $\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{\frac{1}{2}}dx = 3\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \left[2\sqrt{x^3}\right]_1^4 = 2\sqrt{4^3} - 2\sqrt{1^3} = 2\sqrt{64} - 2\cdot 1 = 2\cdot 8 - 2 =$

$= 16 - 2 = 14$

2. $\int_2^5 \frac{4}{x}\,dx = 4\int_2^5 \frac{1}{x}dx = 4\left[\ln x\right]_2^5 = 4(\ln 5 - \ln 2) = 4\ln\frac{5}{2}$

3. $\int_0^{\pi} 5\sin 4x\,dx = \left|\begin{matrix} t = 4x \\ dt = 4dx \end{matrix}\right| = 5\int_0^{4\pi}\frac{1}{4}\sin t\,dt = \frac{5}{4}\left[-\cos t\right]_0^{4\pi} = \frac{5}{4}(-\cos 4\pi + \cos 0) = \frac{5}{4}(-1+1) = 0$

$= \frac{5}{4}\left[-\cos 4x\right]_0^{\pi} = \frac{5}{4}(-\cos 4\pi + \cos 0) = \frac{5}{4}(-1+1) = 0$

4. $\int_0^{\frac{\pi}{2}} \sin x\,dx = \left[-\cos x\right]_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 = 0 + 1 = 1$

$\left|\begin{matrix} t = \ln x \\ dt = \frac{1}{x}dx \end{matrix}\right|$

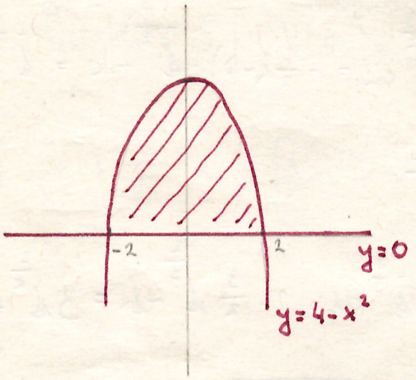
5. $\int_1^2 \frac{2(1+\ln x)}{x}\,dx = \int_1^2 \frac{2}{x} + \frac{2\ln x}{x}\,dx = 2\int_1^2 \frac{1}{x}dx + 2\int_1^2 \frac{\ln x}{x}dx = 2\left[\ln x\right]_1^2 + 2\int_{\ln 1}^{\ln 2} t\,dt =$

$= 2\left[\ln x\right]_1^2 + 2\left[\frac{1}{2}t^2\right]_{\ln 1}^{\ln 2} = 2(\ln 2 - \ln 1) + (\ln^2 2 - \ln^2 1) = 2\ln 2 - 0 + \ln^2 2 - 0$

$= 2\ln 2 + \ln^2 2 = \ln 4 + \ln^2 2$

5.4

1. Určete obsah rovinné plochy ohraničené křivkami $y = 4 - x^2$ a $y = 0$

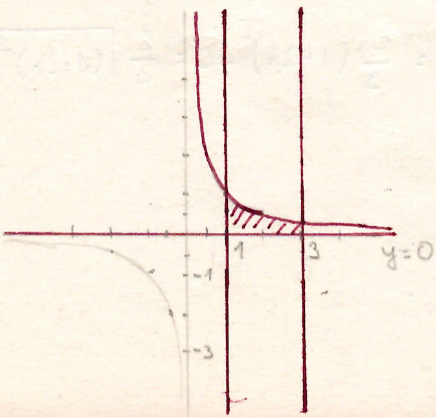


$\int_{-2}^{2} 4 - x^2 - 0\,dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \left(4\cdot 2 - \frac{1}{3}2^3\right) - \left(-8 - \frac{1}{3}(-2)^3\right)$

$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{16}{3} - \left(-\frac{16}{3}\right) = \frac{32}{3} = 10\frac{2}{3}$

$y = \frac{1}{x}$

2. Určete obsah rovinné plochy ohraničené křivkami $yx = 1$, $x = 1$, $x = 3$ a $y = 0$.



$\int_1^3 \frac{1}{x} - 0\,dx = \left[\ln x\right]_1^3 = \ln 3 - \ln 1 = \ln 3 - 0 = \ln 3$

CVIČENÍ 6

6.1

1. $y' = \frac{1}{x-2}$ $\quad x \neq 2$ $\quad x = 2 \rightarrow y = \ln|2-2| = \ln 0 \ldots$ nejde $\rightarrow x \neq 2.$

$\frac{dy}{dx} = \frac{1}{x-2}$

$dy = \frac{1}{x-2}\,dx$

$\int dy = \int \frac{1}{x-2}\,dx$ $\quad \left| \begin{array}{l} t = x-2 \\ dt = dx \end{array} \right| = \int \frac{1}{t}\,dt = \ln t + k = \ln x-2 + c$

$y = \ln|x-2| + c \quad , \quad x \neq 2, \; c \in \mathbb{R}$

2. $3y^2 y' = 2\cos\frac{x}{2}$

$3y^2 \frac{dy}{dx} = 2\cos\frac{x}{2}$

$3y^2 dy = 2\cos\frac{x}{2}\,dx$

$3\int y^2 dy = 2\int \cos\frac{x}{2}\,dx$ $\quad \left| \begin{array}{l} t = \frac{x}{2} \\ dt = \frac{1}{2}dx \\ 2dt = dx \end{array} \right| = 2\int 2\cos t\,dt = \sin t + k = \sin\frac{x}{2} + c$

$3 \cdot \frac{1}{3} y^3 = 4\sin\frac{x}{2} + c$

$y^3 = 4\sin\frac{x}{2} + c$

$y = \sqrt[3]{4\sin\frac{x}{2} + c} \qquad x \in \mathbb{R}, \; c \in \mathbb{R}$

3. $e^{-y}(1 + y') = 0$

$e^{-y} + e^{-y}\frac{dy}{dx} = 0$

$e^{-y}\,dy = -e^{-y}\,dx$

$dy = -dx$

$\int dy = -\int dx$

$y = -x + C \qquad x \in \mathbb{R}, \; c \in \mathbb{R}$

4. $xy' = -(x+1)y$ $\qquad x = 0 \to y = K\cdot\frac{1}{0}e^{0}$... nejde $\to x \neq 0.$

$x\frac{dy}{dx} = -(x+1)y$

$\frac{1}{y}dy = \frac{-(x+1)}{x}dx \qquad x \neq 0$

$\int \frac{1}{y}dy = \int -1-\frac{1}{x}dx$

$\ln y = -x - \ln x + c \quad /\cdot e$

$e^{\ln y} = e^{-x-\ln x + c}$

$y = e^{-x}\cdot e^{-\ln x}\cdot e^{c}$

$y = e^{-x}\cdot\left(\frac{1}{x}\right)\cdot K$

$y = K\frac{1}{x}e^{-x}, \quad x \neq 0, \quad K \in \mathbb{R}$

5. $\frac{1}{y}y' = \frac{2}{x} \qquad y \neq 0$ $\qquad x = 0$ nebo $K = 0 \to y = K\cdot 0^2 = 0 \to \frac{1}{0}y' = \frac{2}{0}$...nejde $\to x \neq 0.$

$y = 0\cdot x^2 = 0 \to \frac{1}{0}y' = \frac{1}{x}$...nejde $\to K \neq 0.$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} \qquad x \neq 0$

$\frac{1}{y}dy = \frac{2}{x}dx$

$\int \frac{1}{y}dy = \int \frac{2}{x}dx$

$\ln y = 2\ln x + c \quad /\cdot e$

$e^{\ln y} = e^{2\ln x + c}$

$y = e^{\ln x^2}\cdot e^{c}$

$y = x^2 K$

$y = Kx^2, \quad x \neq 0, \quad K \neq 0$

6. $e^{y}y' = 1$ $\qquad x = -c \to y = \ln|-c+c| = \ln|0|$ ... nejde $\to x \neq -c$

$c \neq -x$

$e^{y}\frac{dy}{dx} = 1$

$e^{y}dy = dx$

$\int e^{y}dy = \int dx$

$e^{y} = x + c \quad /\ln$

$\ln e^{y} = \ln|x+c|$ $\qquad x + c > 0$

$y = \ln|x+c|, \quad ; x \in \mathbb{R}, \quad c \neq -x$ $\qquad c > -x$

7. $y^4 y' = 2x^4$

$y^4 \frac{dy}{dx} = 2x^4$

$y^4 dy = 2x^4 dx$

$\int y^4 dy = 2\int x^4 dx$

$\frac{1}{5} y^5 = 2\frac{1}{5} x^5 + c$

$y^5 = 2x^5 + c$

$y = \sqrt[5]{2x^5 + c}$, $x \in \mathbb{R}$, $c \in \mathbb{R}$

6.2

1. $y' = \frac{1}{x}$, $y(0) = 5$

$dy = \frac{1}{x} dx$ $x \neq 0$

$\int dy = \int \frac{1}{x} dx$

$y = \ln|x| + c$, $x \neq 0$, $c \in \mathbb{R}$

Partikulární řešení pro $y(0) = 5$ neexistuje, protože $x \neq 0$.

2. $y' = \frac{1}{x}$, $y(1) = 5$

$\vdots$

$y = \ln|x| + c$, $x \neq 0$, $c \in \mathbb{R}$

P. Ř. pro $y(1) = 5$: $5 = \ln 1 + c$

$5 = 0 + c \Rightarrow c = 5$

Partikulární řešení je $y = \ln|x| + 5$, $x \neq 0$

3. $y' = x^6 - 2x$, $y(1) = 0$

$dy = (x^6 - 2x) dx$

$\int dy = \int x^6 - 2x \, dx$

$y = \frac{1}{7} x^7 - x^2 + c$, $x \in \mathbb{R}$, $c \in \mathbb{R}$

$y(1) = 0$: $0 = \frac{1}{7} \cdot 1 - 1 + c$

$0 = \frac{1}{7} - 1 = -\frac{6}{7} + c$

$c = \frac{6}{7}$

Partikulární řešení je $y = \frac{1}{7} x^7 - x^2 + \frac{6}{7}$, $x \in \mathbb{R}$

4. $y' = \frac{1}{x-2}$, $y(-3) = \ln 5$

$\vdots$

$y = \ln|x-2| + c \qquad x \neq 2 \qquad c \in \mathbb{R}$

Part. řešení pro $y(-3) = \ln 5$:
$$\ln 5 = \ln|-3-2| + c$$
$$\ln 5 = \ln|-5| + c$$
$$\ln 5 = \ln 5 + c$$
$$c = 0$$

Partikulární řešení pro $y(-3) = \ln 5$ je $y = \ln|x-2|$, $x \neq 2$.

5. $3y^2 y' = 2\cos\frac{x}{2}$, $y(\pi) = 2$

$\vdots$

$y = \sqrt[3]{4\sin\frac{x}{2} + c}$, $x \in \mathbb{R}$, $c \in \mathbb{R}$

Part. řešení pro $y(\pi) = 2$:
$$2 = \sqrt[3]{4\sin\frac{\pi}{2} + c}$$
$$2^3 = 4\sin\frac{\pi}{2} + c$$
$$8 = 4 + c$$
$$c = 4$$

Partikulární řešení pro $y(\pi) = 2$ je $y = \sqrt[3]{4\sin\frac{\pi}{2} + 4}$, $x \in \mathbb{R}$.

6. $xy' = -(x+1)y$, $y(-1) = 1$

$\vdots$

$y = K\frac{1}{x}e^{-x}$, $x \neq 0$, $K \in \mathbb{R}$

Part. řešení pro $y(-1) = 1$:
$$1 = K \cdot \frac{1}{-1} \cdot e^{-(-1)}$$
$$1 = -Ke$$
$$-\frac{1}{e} = K$$

Partikulární řešení pro $y(-1) = 1$ je $y = -\frac{1}{ex}e^{-x}$, $x \neq 0$.

7. $\frac{1}{y} y' = \frac{2}{x}$, $y(0) = \frac{3}{2}$

$\vdots$

$y = Kx^2$, $K \neq 0$, $x \neq 0$

Part. řešení pro $y(0) = \frac{3}{2}$: $\frac{3}{2} = K \cdot 0^2$

$\frac{3}{2} = 0$... spor $\rightarrow$ Partikulární řešení pro $y(0) = \frac{3}{2}$ neexistuje.