rather than the cosine variation given in (5.2.7), and the steady-state sheaths have not fully formed due to ion transit timescale effects. However, we clearly see the sheath formation. The midpotential variation with time is shown on a short timescale in (*d*), illustrating its formation with $\Phi_{max} \sim \mathrm{T_e}$ as the sheaths form on the very fast electron timescale f_{pe}^{-1}, along with accompanying electron plasma oscillations, as noted previously for Figure 2.2.

6.3 THE HIGH-VOLTAGE SHEATH

Matrix Sheath

Sheath voltages are often driven to be very large compared to $\mathrm{T_e}$. The potential Φ in these sheaths is highly negative with respect to the plasma–sheath edge; hence $n_e \sim n_s\, e^{\Phi/\mathrm{T_e}} \to 0$ and only ions are present in the sheath. The simplest high-voltage sheath, with a uniform ion density, is known as a *matrix sheath*. Letting $n_i = n_s =$ const within the sheath of thickness s and choosing $x = 0$ at the plasma–sheath edge, then from (2.2.3),

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{en_s}{\epsilon_0} \tag{6.3.1}$$

which yields a linear variation of E with x:

$$E = \frac{en_s}{\epsilon_0} x \tag{6.3.2}$$

Integrating $\mathrm{d}\Phi/\mathrm{d}x = -E$, we obtain a parabolic profile

$$\Phi = -\frac{en_s}{\epsilon_0}\frac{x^2}{2} \tag{6.3.3}$$

Setting $\Phi = -V_0$ at $x = s$, we obtain the matrix sheath thickness

$$s = \left(\frac{2\epsilon_0 V_0}{en_s}\right)^{1/2} \tag{6.3.4}$$

In terms of the electron Debye length $\lambda_{Ds} = (\epsilon_0 \mathrm{T_e}/en_s)^{1/2}$ at the sheath edge, we see that

$$s = \lambda_{Ds}\left(\frac{2V_0}{\mathrm{T_e}}\right)^{1/2} \tag{6.3.5}$$

Hence the sheath thickness can be tens of Debye lengths.

Child Law Sheath

In the steady state, the matrix sheath is not self-consistent since it does not account for the decrease in ion density as the ions accelerate across the sheath. In the limit that the initial ion energy $\mathcal{E}_s$ is small compared to the potential, the ion energy and flux conservation equations (6.1.2) and (6.1.3) reduce to

$$\frac{1}{2}Mu^2(x) = -e\Phi(x) \tag{6.3.6}$$

$$en(x)u(x) = J_0 \tag{6.3.7}$$

where J_0 is the constant ion current. Solving for $n(x)$, we obtain

$$n(x) = \frac{J_0}{e}\left(-\frac{2e\Phi}{M}\right)^{-1/2} \tag{6.3.8}$$

Using this in Poisson's equation, we have

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = -\frac{J_0}{\epsilon_0}\left(-\frac{2e\Phi}{M}\right)^{-1/2} \tag{6.3.9}$$

Multiplying (6.3.9) by $\mathrm{d}\Phi/\mathrm{d}x$ and integrating from 0 to x, we have

$$\frac{1}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right)^2 = 2\frac{J_0}{\epsilon_0}\left(\frac{2e}{M}\right)^{-1/2}(-\Phi)^{1/2} \tag{6.3.10}$$

where we have chosen $\mathrm{d}\Phi/\mathrm{d}x = -E = 0$ at $\Phi = 0$ ($x = 0$). Taking the (negative) square root (since $\mathrm{d}\Phi/\mathrm{d}x$ is negative) and integrating again, we obtain

$$-\Phi^{3/4} = \frac{3}{2}\left(\frac{J_0}{\epsilon_0}\right)^{1/2}\left(\frac{2e}{M}\right)^{-1/4}x \tag{6.3.11}$$

Letting $\Phi = -V_0$ at $x = s$ and solving for J_0, we obtain

$$J_0 = \frac{4}{9}\epsilon_0\left(\frac{2e}{M}\right)^{1/2}\frac{V_0^{3/2}}{s^2} \tag{6.3.12}$$

Equation (6.3.12) is the well-known Child law of space-charge-limited current in a plane diode. With fixed spacing s it gives the current between two electrodes as a function of the potential difference between them, and has been traditionally used for electron diodes. However, with J_0 given explicitly as

$$J_0 = en_s u_B \tag{6.3.13}$$

in (6.3.12), we have a relation between the sheath potential, the sheath thickness, and the plasma parameters, which can be used to determine the sheath thickness s. Substituting (6.3.13) in (6.3.12) and introducing the electron Debye length at the sheath edge, we obtain

$$s = \frac{\sqrt{2}}{3} \lambda_{\mathrm{Ds}} \left(\frac{2V_0}{\mathrm{T_e}} \right)^{3/4} \tag{6.3.14}$$

Comparing this to the matrix sheath width, we see that the Child law sheath is larger by a factor of order $(V_0/\mathrm{T_e})^{1/4}$. The Child law sheath can be of order of 100 Debye lengths (~ 1 cm) in a typical processing discharge. Since there are no electrons within the sheath to excite the gas, the sheath region appears dark when observed visually.

Inserting (6.3.12) into (6.3.11) yields the potential within the sheath as a function of position

$$\Phi = -V_0 \left(\frac{x}{s} \right)^{4/3} \tag{6.3.15}$$

The electric field $E = \mathrm{d}\Phi/\mathrm{d}x$ is

$$E = \frac{4}{3} \frac{V_0}{s} \left(\frac{x}{s} \right)^{1/3} \tag{6.3.16}$$

and the ion density $n = (\epsilon_0/e)\, \mathrm{d}E/\mathrm{d}x$ is

$$n = \frac{4}{9} \frac{\epsilon_0}{e} \frac{V_0}{s^2} \left(\frac{x}{s} \right)^{-2/3} \tag{6.3.17}$$

We see that n is singular as $x \to 0$, a consequence of the simplifying assumption in (6.3.6) that the initial ion energy $\mathcal{E}_\mathrm{s} = 0$. The analysis can be carried through for a finite $e\mathcal{E}_\mathrm{s} = \frac{1}{2} M u_\mathrm{B}^2$, using (6.1.2), resolving the singularity and yielding $n \to n_\mathrm{s}$ as $x \to 0$ (Problem 6.1).

The ion motion within the sheath can be determined using conservation of energy (6.3.6). Assuming that an ion enters the sheath with initial velocity $u(0) = 0$, we insert (6.3.15) into (6.3.6) and solve for $u = \mathrm{d}x/\mathrm{d}t$ to obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_0 \left(\frac{x}{s} \right)^{2/3} \tag{6.3.18}$$

with

$$v_0 = \left(\frac{2eV_0}{M} \right)^{1/2} \tag{6.3.19}$$

the characteristic ion velocity in the sheath. Integrating (6.3.18) yields

$$\frac{x(t)}{s} = \left(\frac{v_0 t}{3s}\right)^3 \tag{6.3.20}$$

Setting $x = s$ in (6.3.20), we obtain the ion transit time across the sheath:

$$\tau_{\mathrm{i}} = \frac{3s}{v_0} \tag{6.3.21}$$

The Child law solution is valid if the sheath potentials are large compared to the electron temperature. It is therefore not appropriate for use where the sheath potential is the potential between a plasma and a floating electrode. However, with some modification, we shall see in Chapter 12 that it is useful in determining the sheath width of an rf-driven discharge. Because the ion motion was assumed collisionless, it is also not appropriate for higher-pressure discharges. We shall treat collisional formulations of the sheath region in Section 6.5.

6.4 GENERALIZED CRITERIA FOR SHEATH FORMATION

Using a kinetic treatment without ion collisions, the Bohm criterion for a stable sheath can be generalized to arbitrary ion and electron distributions. First formulated by Boyd and Thompson (1959), a more rigorous and complete treatment in the limit $\lambda_{\mathrm{De}} \to 0$ can be found in Riemann (1991). The result is

$$\frac{e\mathrm{T_e}}{M}\int_0^\infty \frac{1}{v^2} f(v)\,\mathrm{d}v \le \mathrm{T_e}\frac{\mathrm{d}(n_{\mathrm{e}} + n_{-})}{\mathrm{d}\Phi}\bigg|_{\Phi=0} \tag{6.4.1}$$

where $f(v)$ is the one-dimensional speed distribution of the positive ions, $n_{\mathrm{e}} + n_{-}$ is the sum of the densities of the negatively charged species, and Φ is the potential, with $\Phi = 0$ at the sheath–presheath edge. For our previous case of cold ions and Maxwellian electrons, (6.4.1) becomes

$$\frac{e\mathrm{T_e}}{M}\int_0^\infty \frac{1}{v^2}\,\delta(v - u_{\mathrm{s}})\,\mathrm{d}v \le \mathrm{T_e}\frac{\mathrm{d}}{\mathrm{d}\Phi}\left(\mathrm{e}^{\Phi/\mathrm{T_e}}\right)\bigg|_{\Phi=0} \tag{6.4.2}$$

where $\delta(v - u_{\mathrm{s}})$ is the Dirac δ function. Evaluating the integral on the left and taking the derivative on the right, we have

$$\frac{e\mathrm{T_e}}{M}\frac{1}{u_{\mathrm{s}}^2} \le 1$$