Introduction to AdS/CFT correspondence

1 Introduction

The goal of these lectures is to give a brief introduction to the AdS/CFT correspondence. We will mainly follow famous review paper that was published in 1999 soon after the formulation of given correspondence. Then the subject has developed in many directions and it is not possible to give an overview of all these directions.

As is well known string theories are well defined in strictly given number of dimensions. In fact, 10 dimensional string theory is described by a string with fermionic excitations on the world-sheet and the massless spectrum of given theories gives supergravity field theories. It is important that all closed string theories contain the graviton as the massless excitation and hence it is believed that string theory could be a candidate for a quantum theory of gravity. Since we know from every day experience that our world is three dimensional we should find mechanism how string theory describes phenomena in three dimensions. Usually we consider string theory on $R^4 \times M_6$, where M_6 is some six dimensional compact manifold. Then, the low energy interactions and physical content are determined by the geometry of M_6.

We also know that the strong interactions are described by Quantum Chromodynamics (QCD), that is the gauge theory based on the group $SU(3)$. QCD possesses an asymptotic freedom which means that the effective coupling constant decreases with increasing the energy scale. At low energies QCD is strongly coupled which means that standard perturbative treatment is not well suited. To overcome this difficulty it is possible to implement the numerical simulations when we defince QCD on the lattice. It was further suggested by 'tHooft [2] that the theory could simplify when the number of colors N is large, where N is the rank of the gauge group $SU(N)$. It was believed that we could solve exactly the theory with $N = \infty$ and then we could extrapolate these results to the case $1/N = 1/3$ in some way. It is also very interesting that the diagrammatic expansion of the field theory resembles string theory in the sense that the large N theory is a free string theory with the string coupling constant $1/N$. In this way the large N limit gives a connection between gauge theories and string theories.

It is very important that the string theory is not consistent in four flat dimensions. More precisely, it is possible to quantize string theory in four dimensions at least in principle, but the prise is that the anomaly appears that leads to the emergence of Liouville field. It is natural to interpret this field on the world-sheet as an additional coordinate so that it is natural to interpret string as moving in five dimensions.

It is also important to stress that the arguments that the gauge theories in the large N limit are related to string theory are very general and are valid for any gauge theory. Let us consider the gauge theory where the coupling does not run with the energy scale. Given theory is conformally invariant. A simple example is supersymmetric $SU(N)$ or $U(N)$ gauge theory in four dimensions with four spinor supercharges $\mathcal{N} = 4$. This theory contains gauge fields (gluons), four fermions and six scalar fields in the adjoint representation of the gauge field. The Lagrangian of such theories is completely determined by the supersymmetry. Further, this theory has a global $SU(4)$ R-symmetry that rotates the six scalar fields and the four fermions. The conformal group in four dimensions is $SO(4,2)$, that includes the Poincare transformations as well as scale transformations and special conformal transformations. By principle of duality we should expect that these symmetries should be symmetries of dual string theory. Natural idea is to find the five dimensional geometry that has these symmetries. It turns out that there is only one space with this isometry, five dimensional Anti-de-Sitter space (AdS_5), which is the maximally symmetric solution of Einstein's equations with a negative cosmological constant. Clearly we have to add five more dimensions to get ten dimensional space-time in order to have string theory defined in ten dimensions. As we said above the gauge theory has $SU(4) \simeq SO(6)$ global symmetry we search for five dimensional space that possesses this symmetry. It turns out that $SO(6)$ global symmetry is the symmetry of the five sphere S^5. In other words we should expect the duality between $\mathcal{N} = 4$ $U(N)$ Yang-Mills theory and ten dimensional superstring theory on $AdS_5 \times S^5$. Of course, we should be more precise with this statement.

The idea of the duality between these two theories was based on the analysis of the extended objects in string theories, which are D-branes and black holes. As we known D-branes are extended objects that are presented in all string theories. We use notation Dp-brane which means that this brane is extended in $p-$ spatial dimensions. Tensions of these objects are proportional to $1/g_s$, where g_s is a string coupling constant and we see that in case when the string coupling constant is small these objects are much heavier than ordinary strings. It is very important that unlike of other extended objects in string theories D-branes have very simple definitions: They are defined in string perturbation theory as surfaces where open strings can end [3]. Clearly these strings have infinite number of excitations as ordinary string theory but we are mainly interested in the massless modes that propagate on the world-volume of Dp-branes. They are modes that describe the transverse fluctutations of branes, gauge fields living on the world-volume of Dp-brane and their fermionic partners so that resulting theory is supersymmetric with 16 supercharges. If we have N coincident Dp-branes, now open strings can start and end on different branes so that it is natural that they carry indices that run from 1 to N. As a result we find that the low energy dynamics is described by $U(N)$ gauge theory. Further, Dp-branes are charged under $p+1$-form gauge potencials known as Ramond-Ramond fields. These massless modes are parts of closed string modes in corresponding closed string theory. If we now add D-branes together then due to the fact that they couple to Ramond-Ramond fields we find that they generate the flux of corresponding field strength, that in the end contribute to the stress energy tensor and consequently has an impact on the gravity so that the space is curved. More precisely, we can find solutions of the equations of motions where Dp-branes are sources and and they presence lead to the emergence of non-trivial gauge fields and curved spacetime, for some review, see [4].

As we show below when we consider set of N coincident D3-branes and study the near horizon geometry we find that it has the form of $AdS_5 \times S^5$. On the other hand we know that the low energy dynamics on their world-volume is governed by $\mathcal{N} = 4$ $U(N)$ SYM theory. These two descriptions are valid for different regimes in space of the coupling constants. Explicitly, perturbative field theory is valid when $g_s N$ is small while the low energy description is valid when the radius of curvature is much larger than the string scale that implies that $g_s N$ should be large. However then we can conjecture that SYM at strong coupling is described by the near horizon region of the brane whose geometry is $AdS_5 \times S^5$. The subject of this lecture is to give review of this conjecture.

1.1 Large N Gauge Theories as String Theories

Let us consider $U(N)$ Yang-Mills (YM) theory in four dimensions. The action of given theory has the form

$$S = -\frac{1}{g_{YM}^2} \int d^4x \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} \ , F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \ , \qquad (1)$$

where A_μ is hermitean field taking its value in the adjoin representation of $U(N)$. Explicitly, we have

$$A_\mu^\dagger = A_\mu \ , F_{\mu\nu}^\dagger = F_{\mu\nu} \ . \tag{2}$$

We know that A_μ has mass dimension $[A_\mu] = M$, where M is some mass scale. Further, $[\partial_\mu] = M$ so that $[F_{\mu\nu}] = M^2$. As a result we find that $[g_{YM}^2] = M^0$ so that the coupling constant is dimensionless. On the other hand we know that beta function for pure $SU(N)$ YM theory is

$$\mu \frac{dg_{YM}}{d\mu} = -\frac{11}{3} N \frac{g_{YM}^3}{16\pi^2} + O(g_{YM}^5) \tag{3}$$

Solving this equation we obtain

$$g_{YM}^2 = \frac{24}{11\pi^2} \frac{1}{\ln \frac{\mu}{\Lambda}} \ , \tag{4}$$

where Λ is some mass scale. and we see that the coupling constant decreases with mass scale which is the property known as asymptotic freedom. On the other hand we see that when the energy scale is decreases the coupling constant increases and the perturbative calculations will not be valid. Λ_{QCD} is the scale where the coupling constant formally diverges.

Our goal is to understand the limit $N \to \infty$ and its impact on the scaling of the coupling g_{YM}. Let us consider the beta function for pure $SU(N)$ YM given above, when we introduce the parameter

$$\lambda \equiv g_{YM}^2 N \tag{5}$$

so that

$$\mu \frac{dg_{YM}}{d\mu} = -\frac{11}{3} \frac{\lambda g_{YM}}{16\pi^2} \tag{6}$$

and we see that the left and right side are of the same order if we take the large N limit while λ is kept fixed. This limit is known as *'tHooft limit*. This limit is also valid when we include matter fields in the adjoint representation on condition that given theory is still asymptotic free. In case of the conformal theory as for example $\mathcal{N} = 4$ SYM theory we find that there is also possibility to consider the limit $\lambda \to \infty$.

Let us consider general theory with fields Φ_I^a, where a is index in the adjoint representation of $SU(N)$ and I is some label of field. We further presume that the 3−point vertices of all these fields ar proportional g_{YM}, the 4−point functions to g^2_{YM}. In other words we presume that the Lagrangian density has the form

$$\mathcal{L} \sim \mathrm{Tr}(d\Phi_I d\Phi_I) + g_{YM} c^{IJK} \mathrm{Tr}(\Phi_I \Phi_J \Phi_K) + g^2_{YM} d^{IJKL} \mathrm{Tr}(\Phi_I \Phi_J \Phi_K \Phi_L) \ , \quad (7)$$

where c^{IJK}, d^{IJKL} are constants. If we rescale the fields ar $\tilde{\Phi}_I \equiv g_{YM} \Phi_I$ we find that the Lagrangian density takes the form

$$\mathcal{L} = \frac{1}{g^2_{YM}} \left[\mathrm{Tr}(d\tilde{\Phi}_I d\tilde{\Phi}_I) + c^{IJK} \mathrm{Tr}(\tilde{\Phi}_I \tilde{\Phi}_J \tilde{\Phi}_K) + d^{IJKL} \mathrm{Tr}(\tilde{\Phi}_I \tilde{\Phi}_J \tilde{\Phi}_K \tilde{\Phi}_L) \right] \quad (8)$$

We would like to see what happens to correlation functions in the limit of large N when λ is constant. Clearly we could thought that it corresponds to the classical limit since by definition $g^2_{YM} = \frac{\lambda}{N} \to 0$. On the other hand the number of components of the fields goes to infinity as well so that we should be more careful. It is isefull to write Feynman diagrams of the theory (8) in the double line notation where the adjoint field Φ^a is represented as a direct product of a fundamental and an anti-fundamental field Φ^i_j so that we have for the propagator for $SU(N)$ theory (we omit space-time dependence)

$$\left\langle \Phi^i_j \Phi^k_l \right\rangle \propto (\delta^i_l \delta^k_j - \frac{1}{N} \delta^i_j \delta^k_l) \quad (9)$$

We see that there is small mixing term proportional to $\frac{1}{N}$ due to the fact that for $SU(N)$ theory we have the trace of the field to be equal to zero $\Phi^i_i = 0$. Then on the left side vanishes identically while the right side is equal to

$$\left\langle \Phi^i_i \Phi^k_l \right\rangle \propto (\delta^i_l \delta^k_i - \frac{1}{N} \delta^i_i \delta^k_l) = \delta^k_l - \delta^k_l = 0 \ . \quad (10)$$

However this term is proportional to $\frac{1}{N}$ so that it gives subleading contribution and can be ignored. Then we can see that any Feynman diagram of ajdoint fields has the form of the network of double lines. For example, in case of the vacuum diagram these double lines form the edges in triangulation of a surface. This surface is oriented since the lines have an orientation as follows from the fact that fundamental index points in one direction while anti-fundamental index in opposite direction. Further, when we compactify space by adding a point at infinity each diagram corresponds to a compact, closed, oriented surface.

The important question is the powers of N and λ associated with such a diagram. From (8) we see that the vertex has the factor $\frac{1}{g_{YM}^2} = \frac{N}{\lambda}$, while propagators are proportional to $\frac{\lambda}{N}$ as follows from the kinetic term in (8). There is also additional powers of N from the sum over indices in the loops which gives a factor of N for each loop in the diagram. In other words we find that a diagram with V vertices, E propagators corresponding E edges in simpicial decomposition and F-loops , corresponding faces in simplicial decomposition comes with following coefficient

$$N^{V-E+F}\lambda^{E-V} = N^{\chi}\lambda^{E-V} \ , \tag{11}$$

where $\chi = V - E + F$ is the Euler character of the surface corresponding to given diagram. On the other hand we know that for closed oriented surfaces $\chi = 2 - 2g$ where g is the genus of the surface. Finally we find that the perturbative expansion of any diagram in the field theory may be written as a double expansion

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} c_{g,i}\lambda^i = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \ , \tag{12}$$

where $f_g(\lambda)$ is some polynomial in λ. We see that in the large N limit the dominant contribution will come from surfaces of maximal χ that corresponds surface with minimal genus, which are surfaces with topology of sphere (plane) that will give contribution of N^2, while all other diagrams will come with powers of $1/N^2$. These leading order diagrams are known as *planar diagrams* and come with factor N^2.

It is remarkable that form of the expansion (12) has the form of the expansion as in case of closed oriented strings when we identify $1/N$ as the string coupling constant. This analogy is one of the strongest motivations for the belief that the string theories and field theories are related and it also suggests that this relation will be mostly easily seen in the large N limit where the dual string theory is weakly coupled. It is also important to stress that this analogy is based on the perturbative theory which generally does not converge so that we have to be careful with definite conclusions. We should rather consider this relation as the indication of this correspondence. Further, we know that field theory has effects as instantons that are non-perturbative in $1/N$ expansion. Then the exact matching will require that there should be found corresponding string theory interpretation.

It is also important to stress that this analogy can be extended in the direction where we correspondence between additional insertion on the field theory side and the vertex operators in the string theory interpretation. Further, given theory could be easily generalized when we incorporate matter in the fundamental representations. Matter in the fundamental representation has a single-line propagators in the diagrams which can be interpreted as the boundary of the corresponding surface. In other words now we have to sum over surfaces with boundaries as in case of open string theories.

The previous discussion suggests that gauge theories are dual to string theories with a coupling constant proportional to $1/N$ in the't Hooft limit. On the other there is no indication which string theory is dual to a particular gauge theory. It seems to be rather difficult to find corresponding string theory directly due to the fact that it is very difficult to formulate string theory in four dimensions or generally below the critical dimension. Explicitly, we have to add additional field on the string world-sheet in order to ensure consistency of given theory. This field is known as Liouville fied and can be interpreted as a fifth space-time dimensions. The idea that five dimensional string theory could be related to four dimensional gauge theories was firstly proposed by Polyakov in [5, 6], where the coupling of Liouville field to the other fields has some specific form. The AdS/CFT correspondence provides concrete realization of this idea where five additional dimensions, together with radial coordinate on AdS, leads to a standard ten dimensional string theory.

1.2 Black $p-$Branes

The first hint of the relation between large N field theories and string theory came from the study of $p-$branes in string theory. Originally the $p-$branes were found as classical solutions in supergravity that, as is well known, is the low energy limit of string theory. In 1995 J. Polchinski discovered that D-branes have full string theory description [3]. Then by comparing these two descriptions led to the discovery of AdS/CFT correspondence.

As we know, string theory has variety of classical solutions. We restrict ourselves to the examples of parallel Dp-branes. Let us consider type II string theory in ten dimensions. We search the solutions that carry electric charge with respect to Ramond-Ramond (R-R) $(p+1)$ form A_{p+1}, where p is even in IIA theory while p is odd in IIB theory. It can be also shown that these theories contain also magnetically charged $(6-p)$-branes that are electrically charged under the dual $dA_{7-p} = *dA_{p+1}$ potential, where $*$ is Hodge dual. Then we find that $R-R$ charges have to be quantized according to the Dirac procedure.

In order to find explicit solution, let us consider low energy effective action in the string frame

$$S = \frac{1}{(2\pi)^7 l_s^8} \int d^{10}x\sqrt{-g}\left(e^{-2\phi}(R + 4g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi) - \frac{2}{(8-p)!}F_{p+2}^2\right) \ , \quad (13)$$

where l_s is the string length, F_{p+2} is the field strength of the $(p+1)-$form potential defined as $F_{p+2} = dA_{p+1}$. Note that for $p = 3$ we have $*F = F$ and given case is known as self-dual. This case is also problematic due to the lacking of the Lagrangian formulation as follows from the fact that schematically the kinetic term for the gauge field has the form $\int F \wedge *F$ which in case $F = *F$ gives $\int F \wedge F$ so that the equations of motion that follow from this action are $d * F = dF = d^2A_4 = 0$.

Then we are trying to find solution corresponding to the electric source of charge N for A_{p+1}. From the point of view of the transverse space we have point object so that it is natural to require that the metric is spherically symmetric in $(9-p)$ dimensions. Further, since Dp-brane is extended in p dimensions we require $SO(p)$ symmetry in these directions. In other words we presume that the metric has the form

$$ds^2 = ds^2_{10-p} + e^\alpha \sum_{i=1}^p dx^i dx_i \ , \quad (14)$$

where ds^2_{10-p} is a Lorentzian-signature metric in $(10-p)$ dimensions. Further, since this Dp-brane is source of the R-R field we have the condition

$$\int_{S^{8-p}} *F_{p+2} = N \ , \quad (15)$$

where S^{8-p} is $(8-p)-$sphere surrounding the source. The resulting metric in the string frame has the form

$$ds^2 = -\frac{f_+(\rho)}{\sqrt{f_-(\rho)}}dt^2 + \sqrt{f_-(\rho)}\sum_{i=1}^p dx^i dx_i + \frac{f_-(\rho)^{-\frac{1}{2}-\frac{5-p}{7-p}}}{f_+(\rho)}d\rho^2 + r^2 f_-(\rho)^{\frac{1}{2}-\frac{5-p}{7-p}}d\Omega^2_{8-p} \ , \quad (16)$$

with non-trivial profile of the dilaton field

$$e^{-2\phi} = g_s^{-2} f_-(\rho)^{-\frac{p-3}{2}} \ , \tag{17}$$

where the functions $f_\pm(\rho)$ have the form

$$f_\pm(\rho) = 1 - \left(\frac{r_\pm}{\rho}\right)^{7-p} \tag{18}$$

where g_s is the asymptotic string coupling constant. The parameters r_+ and r_- are expressed through the mass M (per unit volume) and the RR charge N of the solution by

$$M = \frac{1}{(7-p)(2\pi)^7 d_p l_P^8}((8-p)r_+^{7-p} - r_-^{7-p}) \ , N = \frac{1}{d_p g_p l_s^{7-p}}(r_+ r_-)^{\frac{7-p}{2}} \ , \tag{19}$$

where $l_P = g_s^{1/4} l_s$ is the 10$-$ dimensional Planck length and d_p is a numerical factor

$$d_p = 2^{5-p} \pi^{\frac{5-p}{2}} \Gamma\left(\frac{7-p}{2}\right) \ . \tag{20}$$

The metric in the Einstein frame $(g_E)_{\mu\nu}$ is defined by multiplying the string frame metric $g_{\mu\nu}$ by the factor $\sqrt{g_s e^{-\phi}}$ so that the action takes the standard Einstein-Hilbert form with the canonical kinetic term

$$S = \frac{1}{(2\pi)^7 l_P^8} \int d^{10}x \sqrt{-g_E} \left(R_E - \frac{1}{2} g_E^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \dots \right) \ . \tag{21}$$

This solution has the horizon at $\rho = r_+$ and curvature singularity at $\rho = r_-$, when we also presume that $r_+ > r_-$. On the other hand there is special solution when $r_+ = r_-$ and we find

$$M = \frac{1}{(2\pi)^7 g_s l_s^{p+1}} N \tag{22}$$

that is *extremal p-brane* Generally for $r_+ > r_-$ we find

$$M \geq \frac{1}{(2\pi)^7 g_s l_s^{p+1}} \tag{23}$$

that is known as *non-extremal black p-brane*. We call this brane black since there is an event horizon for $r_+ > r_-$ whose area goes to zero in the extremal limit $r_+ = r_-$. It is important to stress that the extremal solution with $p \neq 3$ has a singularity we find that the supergravity description breaks down near $\rho = r_+$ and hence the full description of given p-brane can be done in terms of superstring theory.

Let us now consider more explicitly the extremal limit $r_+ = r_-$ where the line element has the form

$$ds^2 = \sqrt{f_+(\rho)}\left(-dt^2 + \sum_{i=1}^{p} dx^i dx_i\right) + f_+(\rho)^{-\frac{3}{2}-\frac{5-p}{7-p}} d\rho^2 + \rho^2 f_+(\rho)^{\frac{1}{2}-\frac{5-p}{7-p}} d\Omega_{8-p}^2 . \tag{24}$$

We see that the original Euclidean group of black p-brane that was $SO(p)$ was extended to the Poincare group $SO(p,1)$. In order to investigate the geometry of the extremal solution near the horizon we define new coordinate r as

$$r^{7-p} = \rho^{7-p} - r_+^{7-p} \tag{25}$$

and introduce the isotropic coordinates

$$r^a = r\theta^a (a = 1, \ldots, 9-p), \sum_a (\theta^a)^2 = 1 . \tag{26}$$

Then the metric has the form

$$ds^2 = \frac{1}{\sqrt{H(r)}}\left(-dt^2 + \sum_{i=1}^{p} dx^i dx_i\right) + \sqrt{H(r)} \sum_{a=1}^{9-p} dr^a dr^a , \tag{27}$$

where

$$e^{\phi} = g_s H(r) \frac{3-p}{4} , \tag{28}$$

and where

$$H(r) \equiv \frac{1}{f_+(\rho)} = 1 + \frac{r_+^{7-p}}{r^{7-p}} , r_+^{7-p} = d_p g_s N l_s^{7-p} . \tag{29}$$

It is important to stress that previous solution is the solution of the supergravity equations of motion for any function $H(r)$ which is a harmonic function in the $(9-p)$ dimensions that are transverse to the $p-$brane. As a result we can consider more general solution

$$H(r) = 1 + \sum_{i=1}^{k} \frac{r_{(i)+}^{7-p}}{|\mathbf{r} - \mathbf{r}_i|^{7-p}} , r_{(i)+}^{7-p} , \tag{30}$$

where

$$r_{(i)+}^{7-p} = d_p g_s N_i l_s^{7-p} \ . \tag{31}$$

Physically this solution corresponds to the confinguration of parallel extremal p-branes located at k different positions $\mathbf{r} = \mathbf{r}_i$, where each carries N_i units of $R-R$ charge. It is very important to stress that these extremal D-branes are extremal. As a consequence of this fact it can be shown that when we have two separated D-branes that the gravity attraction is exactly canceled with the force of the gauge field since two D-branes repel. This is in sharp contrast with the configuration D-brane-D-anti-brane which is famous example of the unstable configuration. Returning back to D-branes the fact that there is no force between them leads to the possibility that these D-branes can be localized at different points $\mathbf{r}_i$.

Of course, the previous description of p-branes was given in terms of supergravity approximation. Clearly this description is suitable on condition when the curvature of p-brane geometry is small compared to the string scale so that we can neglect the string theory corrections. In other words, we have the condition $r_+ \gg l_s$. Further, in order to supress string loop corrections we have to demand that the effective string coupling e^{ϕ} should be small. In case of interest $p = 3$ the dilaton is constant and clearly this coupling constant could be small everywhere in the $3-$brane geometry when we demand that $g_s < 1$ and hence $l_P < l_s$. In other words the supergravity approximation is valid when

$$l_P < l_s \ll r_+ \ . \tag{32}$$

Since we know that

$$r_+^{7-p} = d_p g_s N l_s^{7-p} \tag{33}$$

we find that this expression is equivalent to

$$\begin{aligned} g_s^{1/4} l_s < l_s \ll d_p^{1/(7-p)} g_s^{\frac{1}{7-p}} N^{\frac{1}{7-p}} l_s \\ g_s^{1/4} < 1 \ll g_s^{\frac{1}{7-p}} N^{\frac{1}{7-p}} \Rightarrow \\ g_s^{\frac{7-p}{4}} < 1 \ll g_s N \end{aligned} \tag{34}$$

Clearly $1 > g_s^{\frac{7-p}{4}}$ and clearly $N > g_s N$ due to the fact that $g_s < 1$ and hence we can rewrite the upper non-equivality in the form

$$1 \ll g_s N < N \ . \tag{35}$$

1.3 D-Branes

As we said previously extremal Dp-brane can be also defined as a D-brane which is $(p+1)-$dimensional hyperplane in space-time where the open strings can end. Clearly open string theory contains the loop diagram with topology of cilinder that, from the closed string point of view corresponds to the propagation of the closed string. In other words loop diagram for the open string that ends on different D-branes corresponds to the propagation of the closed string between these D-branes. Using this fact we obtain that D-brane carry RR charge. Further, D-brane is invariant under half of the space-time supersymmetry and it can be shown that in type IIA (IIB) string theory one half of supersymmetry is preserved when p is even (odd). This fact is also consistent with the R-R fields that appear in given theory. Say differently, Dp-brane is BPS object that carries R-R charge.

We believe that the extremal p-brane in supergravity and Dp-brane are different descriptions of the same object. More precisely, there are descriptions that can be used in some specific situations. Explicitly, D-brane is defined using string world-sheet which is a well defined object in the string perturbation theory. When we have N D-branes at the same position that the effective loop expansion parameter for the open string is $g_s N$ rather than g_s since each open string boundary curve that ends on D-branes has Chan-Paton factor N and the string coupling g_s. Then the perturbative treatment is good on condition when $g_s N \ll 1$. We immediately see that this is complementary to the regime where the supergavity description can be used.

When we restrict to the massless modes in the spectrum of the states of the open string ending on Dp-brane we find that it forms $U(N)$ gauge theory in $(p+1)-$dimensions with $16-$ supercharges. There are $(9-p)$ scalar fields Φ^i in the adjoint representation of $U(N)$ that has following physical meaning. Let us presume that the vacuum expectation value $<\Phi^i>$ has k different eigenvalues with N_1 identical values ϕ^i_1, N_2 identical values with ϕ_2. Say differently, in matrix notation

$$<\Phi^i>=\left(\begin{array}{cccc} \phi^i_1 I_{N_1\times N_1} & 0 & 0 & 0 \\ 0 & \phi^i_2 I_{N_2\times N_2} & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \phi^i_k I_{N_k\times N_k} \end{array}\right) . \qquad (36)$$

This vacuum expectation value breaks the original symmetry of the theory that is $U(N)$ to the group that preserves (36) that is $U(N_1)\times\cdots\times U(N_k)$. Physically this situation describes configuration when N_1 Dp-branes are localized at $x_1^i = \phi_1^i l_s^2$, N_2 Dp-branes localized at $x_2^i = \phi_2^i l_s^2$ and so on. Then careful analysis of the fluctuations around this vacuum state shows that there are massive W-bosons in bi-fundamental representation of $U(N_i) \times U(N_j)$ that correspond to the open strings stretched between Dp-branes localized at $\vec{r}_i$ and $\vec{r}_j$. In fact, analysis of the open string with the Dirrichlet boundary conditions at $\vec{X}(\sigma = 0) = \vec{x}_i$, $\vec{X}(\sigma = \pi) = \vec{x}_j$ leads to the spectrum where the mass of the lightest states is proportional to the Euclidean distance $|\vec{r}_i - \vec{r}_j|$.

1.4 $\mathcal{N} = 4$ SYM theory

First of all we have to explain what means theory with $\mathcal{N}$ supersymetries. $\mathcal{N}$ is the number of irreducible spinors that are conserved charges that generate supersymmetry transformations. It turns out that $\mathcal{N} = 4$ is the maximal number that it is possible for interacting four dimensional field theory without gravity. In four dimension general spinor has four complex componets. Then the Weyl projection eliminates two of them so that Weyl spinor has two complex Weyl spinors. Since $\mathcal{N} = 4$ we have $4 \times 2 \times 2 = 16$ real superchages, where the first 2 means number of complex components and the second one means that one complex component has two real components. Further, we consider the gauge group $U(N)$ which means that all fields are $N \times N$ hermitean matrices.

The simplest way how to derive $\mathcal{N} = 4$ four dimensional gauge theory is to consider $\mathcal{N} = 1$ SYM theory in ten dimensions. (Note that in ten dimensions the dimension of spinor space is $2^{\frac{10}{2}} = 32$ complex components, however now we can impose Majorana and Weyl projection so that irreducible spinor has 16 real components which is exactly equal to number of supersymmetries of $\mathcal{N} = 4$ SYM theory in four dimensions.) This theory contains a ten-vector field A_M, a Majorana-Weyl spinor Ψ that both are in adjoint representation of the gauge group. Then we derive four dimensional theory by dimensional reduction which means that we presume that all fileds do not depend on six of the (Euclidean) spatial dimensions. As a result we obtain four dimensional theory with four-vector A_μ, six scalars ϕ^I and four Weyl spinors ψ^A. This four dimensional SYM theory has a global $SO(6) \sim SU(4)$ global symmetry that rotates six scalars. Due to this symmetry it also rotates four supercharges Q^A and it is known as $R-$symmetry.

Note About Kaluza-Klein compactification on a circle

Our goal is to compactify one space dimension on a circle S^1_R with radius R. It is convenient to decompose the coordinates x^M of $\mathbf{R}^D$ into a coordinate y on the circle and the remaining coordinates x^μ. For simplicity we consider the flat space-time metric so that the wave operator takes the form

$$\frac{1}{\sqrt{-g}}\partial_M[g^{MN}\partial_N] = \partial_\mu[\eta^{\mu\nu}\partial_\nu] + \frac{\partial^2}{\partial y^2} \ . \tag{37}$$

We are interested how various fields behave in the limit $R \to 0$.

We begin with a scalar field $\phi(X^M)$ that obeys periodic boundary conditions on S^1_R that has following Fourier decomposition

$$\phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}}\sum_{n\in\mathbf{Z}} \phi(x)e^{iny/R} \ . \tag{38}$$

Note that this field obeys the periodic boundary condition

$$\phi(x, y+2\pi) = \sum_{n\in\mathbf{Z}} \phi(x)e^{2\pi iny/R}e^{2\pi in} = \phi(x) \ . \tag{39}$$

Then $D-$dimensional kinetic term of a scalar field with mass m then decomposes as follows

$$\begin{aligned} &\frac{1}{2}\int d^Dx(-g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + m^2\phi\phi^*) = \\ &= \frac{1}{2\pi R}\int d^{D-1}x\int_0^{2\pi R} dy[-\sum_{n,m} e^{i(-n+m)y/R}(\partial_\mu\phi_n\eta^{\mu\nu}\partial_\nu\phi_m + \\ &+ \left(\frac{2\pi}{R}\right)^2 mn\phi_n\phi_m + m^2\phi_n\phi_m)] = \\ &= \sum_n \int d^{D-1}x[-\partial_\mu\phi_n\eta^{\mu\nu}\partial_\nu\phi_n + \left(\frac{2\pi n}{R}\right)^2 \phi_n^2 + m^2\phi_n^2] \end{aligned} \tag{40}$$

using the fact that

$$\begin{aligned} \int_0^{2\pi R} dye^{i(n+m)/Ry} = 0 \ , n \neq -m \ , \\ \int_0^{2\pi R} dye^{i(-n+m)/Ry} = 2\pi R \ , n = m \ . \end{aligned} \tag{41}$$

In other words in the process of compactification we obtain infinite number of scalar fields with masses $m_n^2 = m^2 + \left(\frac{2\pi n}{R}\right)^2$. We also see that in the limit $R \to 0$ the massive modes decouple and hence the dynamical modes are the zero modes only. Zero modes are modes ϕ_0 that do not carry momentum in y direction.

Situation is more involved in case of gauge theory. Let us consider the action

$$S = -\int d^D x \int_0^{2\pi R} dy F_{MN} F^{MN} = -\int d^D x \int_0^{2\pi R} dy [F_{\mu\nu} F^{\mu\nu} + 2 F_{y\mu} F^{y\mu}] \ . \tag{42}$$

Let us expand the gauge field as

$$A_\mu = \frac{1}{\sqrt{2\pi R}} \sum_m a_\mu^m e^{iny/R} \ , \quad A_y = \frac{1}{\sqrt{2\pi R}} \sum_m \phi_m e^{iny/R} \ . \tag{43}$$

Then we have

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{\sqrt{2\pi R}} \sum_n [\partial_\mu a_\nu^n - \partial_\nu a_\mu^n] e^{iny/R} \equiv \sum_n f_{\mu\nu}^n e^{iny/R} \ , \\ F_{\mu\nu} \eta^{\mu\rho} \eta^{\nu\sigma} F_{\rho\sigma} &= \frac{1}{2\pi R} \sum_{m,n} f_{\mu\nu}^n \eta^{\mu\rho} \eta^{\nu\sigma} f_{\rho\sigma}^m e^{i(n-m)y/R} \ , \\ F_{\mu y} &= \frac{1}{\sqrt{2\pi R}} \sum_n (\partial_\mu \phi_n - \frac{in}{R} a_\mu^n) e^{iny/R} \ , \\ F_{\mu y} \eta^{\mu\rho} F_{\rho y} &= \frac{1}{2\pi R} \sum_{m,n} (\partial_\mu \phi_n - \frac{in}{R} a_\mu^n) \eta^{\mu\rho} (\partial_\rho \phi_m + \frac{im}{R} a_\rho^m) e^{i(m-n)y/R} \ . \end{aligned} \tag{44}$$

However we should remember that gauge theory possesses $U(1)$ symmetry. Let us fix part of this symmetry by fixing

$$\phi_n = 0 \ , n \neq 0 \ . \tag{45}$$

This gauge fixing can be achieved and fixed by the Fourier modes $\theta_k(x), k \neq 0$ when we perform the Fourier decomposition of the gauge function $\theta(x) = \frac{1}{\sqrt{2\pi R}} \sum_k e^{iky/L} \theta_k(x)$ where the gauge function $\theta(x)$ appears in the gauge transformations $A'_\mu = A_\mu + \partial_\mu \theta$. Using this fixing of the gauge symmetry we obtain the action in the form

$$S = -\int d^D x \sum_n [f_{\mu\nu}^n \eta^{\mu\rho} \eta^{\nu\sigma} f_{\rho\sigma}^n + 2 \partial_\mu \phi \eta^{\mu\nu} \partial_\nu \phi + 2 \left(\frac{n}{R}\right)^2 a_\mu^n \eta^{\mu\nu} a_\nu^n] \tag{46}$$

We see that there are infinite number of massive vector fields that again decouple in the limit $R \to 0$. Note that the zero mode of the gauge field a_μ possesses the gauge symmetry under $a'_\mu = a_\mu + \partial_\mu \theta_0$, since the Fourier mode $\theta_0(x)$ remained unfixed in the gauge fixing process given above.

Let us again return to the analysis of $\mathcal{N} = 4$ SYM theory. $\mathcal{N} = 4$ SYM theories are conformally invariant which implies that they are ultraviolet finite. If we combine space-time symmetries which are Lorentz invariance, translation invariance, supersymmetry, scale invariance, conformal invariance, $R-$symmetry gives the superconformal supergroup called $PSU(2,2|4)$, where supergroup has both bosonic and fermionic generators.

The parameters of $\mathcal{N} = 4$ with the gauge group $U(N)$ are YM coupling constant g, vacuum angle θ and the rank of the gauge group N. As we argued above for large N the natural parameter is 'tHooft parameter $\lambda = g^2 N$. Then for fixed λ the theory has $1/N$ expanssion for $N \gg 1$. There is remarkable symmetry of the leading term in the expanssion (the planar approximation) that is known as a dual conformal invariance. It turns out that this amount of symmetry is sufficient to make the theory completely integrable. Unfortunately it is not known whether this is property of the planar approximation or whether it can be extended to the complete theory. If we further introduce the complex parameter

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \ , \tag{47}$$

we find that $U(N)$ theory has an $SL(2,Z)$ duality symmetry where τ transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d} \ , \tag{48}$$

For $a = 0, b = 1, c = -1, d = 0$ we find $\tau \to -\frac{1}{\tau}$ which is known as $S-$duality. For $\theta = 0$ this relation gives $g \to \frac{4\pi}{g}$ which implies the relation between the strong coupling theory and weak coupling. It can be shown that S-duality is an exact non-abelian electric-magnetic equivalence.

$\mathcal{N}$ SYM theory has closed relation with the Dp-branes, for very nice review of this correspondence, see [7]. As we argued above a stack of coincident flat D3-branes has a world-volume theory which is $\mathcal{N} = 4$ SYM $U(N)$ theory. As we also shown above the configuration with D-branes at different transverse positions is given by the vacuum expectation value (36) which leads to the emergence of massive bosons with the mass

$$m_{ij} = |\vec{r}_i - \vec{r}_j| T \tag{49}$$

where T is fundamental string tension. We see that this is the similar situation as in case of Higgs mechanism. On the other hand there is a difference since the gauge fields become massive when the scalar fields in the adjoint representation of the original gauge group are eaten while in case of the Standard model the $SU(2)$ Higgs doublet is in the fundamental representation. This difference between these two descriptions is emphasized in the names of these two phenomena, Coulomb branch in case of our case while the second case is known as Higgs branch.

Let us consider $\mathcal{N} = 4, d = 4$ SYM with the gauge group $U(N)$. Usually we say that free $U(1)$ multiplet decouples leading to interacting $SU(N)$ theory. In D3-brane intepretation this decoupling multiplet describes the dynamics of the center of mass of the collection of D3-branes.

1.4.1 The Probe Approximation

Let us consider a D3-brane embedded in ten dimensional target space-time. The probe approximation means that we neglect the back reaction of the brane on the geometry and the other background fields. Since the brane is source for one unit of flux we have to require that the background flux N is large so that $N + 1 \approx N$. As we know the world-volume of a single D-brane contains $U(1)$ gauge field strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Since we presume that the action does not contain derivatives of F we can say that these fields vary sufficiently slowly so that their derivatives can be neglected. Similar restriction we also apply for other world-volume fields.

In more details, the world-volume effective action for D3-brane is famous Dirac-Born-Infeld action

$$S = -T_{D3} \int d^4\sigma e^{-\Phi} \sqrt{-\det(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} + S_{CHS} \ , \tag{50}$$

where

$$G_{\alpha\beta} = g_{MN} \partial_\alpha x^M \partial_\beta x^N \tag{51}$$

where Chern-Simons term has the form

$$S_{CHS} = \mu_3 \int_\Sigma P[\sum C^{(n)} e^B] e^{2\pi\alpha' F} \ , \tag{52}$$

where $\sum C^{(n)}$ is sum over Ramond-Ramond fields where μ_3 is Ramond-Ramond charge of given brane. Further $P[\dots]$ means pullback of given field to the world-volume of D3-brane. Note that there are integrations of forms over world-volume of D3-brane where the wedge products between forms is implicitly presumed.

Now we would like to see how from DBI action the YM action can be derived. Let us presume that D3-brane is embedded in the flat target space-time $g_{MN} = \eta_{MN} = \mathrm{diag}(-1, 1, \dots, 1)$. Further we presume static gauge where

$$x^\mu = \sigma^\mu \ , \mu = 0, 1, 2, 3 \ . \tag{53}$$

Then we obtain

$$g_{\alpha\beta} = g_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu + g_{IJ}\partial_\alpha x^I \partial_\beta x^J = \eta_{\alpha\beta} + \partial_\alpha x^I \partial_\beta x_I \tag{54}$$

Then we obtain

$$\begin{aligned} & \sqrt{-\det(g_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})} = \sqrt{-\det\eta}\sqrt{\det(\det(\delta^\mu_\nu + \partial^\mu\phi^I\partial_\nu\phi_I + 2\pi\alpha' F^\mu_{\ \nu})} = \\ = \ & \sqrt{-\det\eta}(1 + \frac{1}{2}\partial^\mu\phi^I\partial_\mu\phi_I - \frac{(2\pi\alpha')^2}{4}F_{\mu\nu}F^{\mu\nu}) \ , \end{aligned} \tag{55}$$

using the fact that

$$\det(I + \mathbf{M}) = \exp\mathrm{Tr}\ln(I + \mathbf{M}) \approx \exp\mathrm{Tr}(\mathbf{M} - \frac{1}{2}\mathbf{M}^2 \approx 1 + \mathrm{Tr}\mathbf{M} - \frac{1}{2}\mathrm{Tr}\mathbf{M}^2 \ . \tag{56}$$

In summary we find that the DBI action reduces to the YM action in the limit of slowly varying fields. The generalization to non-abelian configuration is given by non-abelian form of DBI action.

2 Conformal Field Theories and AdS Spaces

2.1 Conformal Field Theories

As we know symmetry plays fundamental role in the particle physics, where Lorentz and Poincare symmetries are basic principles for the constructions of corresponding actions. It is natural to ask the question whether there exists non-trivial generalization of these symmetries. An interesting generalization of Poincare invariance is the addition of a scale invariance symmetry that implies that the physics on the different scales are the same. On the other hand this symmetry is non-consistent wit the existence of S-matrix since the S-matrix is based on the definition of the asymptotic states that are defined at far infinity. Clearly in theory, that is invariant under scaling transformation, the meaning of asymptotic distance does not make sense. Many interesting field theories, as for example Yang-Mills theory in four dimensions, are scale invariant, at least classically. In fact, generally scale invariance is broken by quantum theory, since it requires cutoff that breaks scale invariance explicitly. On the other hand $\mathcal{N} = 4$ SYM theory in four dimension is well defined quantum field theory where the scaling symmetry is not broken. We also know that quantum field theories are characterized by renormalization group flow from some scale invariant UV fixed point, that could be free theory, to some scale invariant IR fixed point, that could be also free.

Very important question is whether unitary interacting scale-invariant theories are also invariant under full conformal group. The definite answer to this question exists in two dimensions where it was proved that scale invariant theories are always conformaly invariant. In the case of four dimensions the situation is more involved with recent extensive discussion of this point where however no definitive conclusion has been reached yet. In any case we now review the main properties of conformal group.

2.1.1 Conformal Group and Conformal Algebra

Conformal group is the group of transformations that preserve the form of the metric up to scale factor

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \ . \tag{57}$$

The set of conformal transformations forms a group where the Poincare group is subgroup of given group since it corresponds to the case $\Lambda(x) = 1$. In order to investigate conformal group in more details let us consider an infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \epsilon(x) \ . \tag{58}$$

Since this can be considered as a diffeomorphism transformation we find that the metric transforms as

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x) \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \tag{59}$$

that for (58) gives

$$g'_{\mu\nu}(x+\epsilon) = g_{\mu\nu} + g_{\mu\rho}\partial_{\nu}\epsilon^{\rho} + \partial_{\mu}\epsilon^{\sigma} g_{\sigma\nu} \tag{60}$$

Now the requirement that the transformations are conformal implies

$$g'_{\mu\nu} = g_{\mu\nu} + fg_{\mu\nu} = g_{\mu\nu} + \partial_\mu \epsilon^\rho g_{\rho\nu} + g_{\mu\rho}\partial_\nu \epsilon^\rho \tag{61}$$

and hence

$$\partial_\mu \epsilon^\rho g_{\rho\nu} + g_{\mu\rho}\partial_\nu \epsilon^\rho = fg_{\mu\nu} \tag{62}$$

where the factor f can be determined by the trace of given expression

$$f = \frac{2}{d}\partial_\mu \epsilon^\mu \tag{63}$$

In what follows we presume that the metric $g_{\mu\nu}$ is flat $g_{\mu\nu} = \eta_{\mu\nu}$. Then if we apply additional derivative on (62) ∂_ρ we find

$$\partial_\rho \partial_\mu \epsilon_\nu + \partial_\rho \partial_\nu \epsilon_\mu = \partial_\rho f \eta_{\mu\nu} \tag{64}$$

At the same time taking partial derivative with ∂_μ of equation (62) now with indices ρ, ν we find

$$\partial_\mu \partial_\rho \epsilon_\nu + \partial_\mu \partial_\nu \epsilon_\rho = \partial_\mu f \eta_{\rho\nu} \tag{65}$$

If we combine these two equations we obtain

$$\begin{aligned} 2\partial_\rho \partial_\mu \epsilon_\nu + \partial_\nu [\partial_\rho \epsilon_\mu + \partial_\mu \epsilon_\rho] = \partial_\rho f \eta_{\mu\nu} + \partial_\mu f \eta_{\rho\nu} \Rightarrow \\ 2\partial_\rho \partial_\mu \epsilon_\nu = \partial_\rho f \eta_{\mu\nu} + \partial_\mu f \eta_{\rho\nu} - \partial_\nu f \eta_{\mu\rho} \end{aligned} \tag{66}$$

where in the last step we again used e.q. (62) with μ, ρ indices. Now contracting this expression with $\eta^{\rho\mu}$ we obtain

$$2\partial^\nu \partial_\nu \epsilon_\mu = (2-d)\partial_\mu f \ . \tag{67}$$

Then we apply ∂_ν on this expression and ∂^2 on (62) we obtain

$$\begin{aligned} 2\partial^2 \partial_\nu \epsilon_\mu = (2-d)\partial_\mu \partial_\nu f \Rightarrow \\ 2\partial^2 \partial_\mu \epsilon_\nu = (2-d)\partial_\mu \partial_\nu f \Rightarrow \\ \partial^2(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = (2-d)\partial_\mu \partial_\nu f \Rightarrow \\ (2-d)\partial_\mu \partial_\nu f = \eta_{\mu\nu}\partial^2 f \ , \end{aligned} \tag{68}$$

where in the first step we used $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$ and than the first equation again. Finally we contract this expression with $\eta^{\mu\nu}$ and we obtain

$$(d-1)\partial^2 f = 0 \ . \tag{69}$$

Now using these equations we derive the explicit form of the conformal transformations.

We see that for $d = 1$ there are no constraints on function f and hence any smooth transformation is conformal in one dimension. Of course, this is trivial result since the notion of angle does not exist in one dimension. The case $d = 2$ corresponds to the two dimensional conformal field theory which is analyzed in case of string theory so that it is not interesting for us. For that reason we restrict ourselves to the case $d \geq 3$. In this case we find $\partial^2 f = 0$ from (69) and then from (68) we find $\partial_\mu \partial_\nu f = 0$, which however implies that f is at most linear in the coordinates

$$f(x) = A + B_\mu x^\mu \ , A, B_\mu = \text{const} \ . \tag{70}$$

Now inserting this expression into (66) we find that $\partial_\mu \partial_\nu \epsilon_\rho$ is constant which means that ϵ_μ is at most quadratic in coordinates, so that we can write

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \ , c_{\mu\nu\rho} = c_{\mu\rho\nu} \ . \tag{71}$$

Since the constraints equations hold for all x we can treat each power of coordinate separately. First of all it is clear that the constant term a_μ is not restricted by these equations and it corresponds to the well known infinitesimal translation. Focusing on the linear term we insert it into the equation (62) and we obtain

$$b_{\nu\rho}\delta^\rho_\mu + b_{\mu\rho}\delta^\rho_\nu = \frac{2}{d} b^\lambda{}_\lambda \eta_{\mu\nu} \tag{72}$$

Since $b_{\mu\nu}$ can be expressed as linear combination of the symmetric, antisymmetric tensor and trace we find that the previous equation can be solved with

$$b_{\mu\nu} = \alpha \eta_{\mu\nu} + m_{\mu\nu} \ , m_{\mu\nu} = -m_{\nu\mu} \tag{73}$$

since then $b_{\mu\nu}\eta^{\nu\mu} = d\alpha (m_{\mu\nu}\eta^{\nu\mu} = 0)$ so that we really find that (72) is obeyed by (73). Now the pure trace corresponds to the infinitesimal scale transformations while antisymmetric part is an infinitesimal rigid rotation. Finally we consider the quadratic term. First of all we have

$$f = \frac{2}{d}\partial_\mu \epsilon^\mu = \frac{4}{d} c^\mu{}_{\nu\rho}\delta^\rho_\mu x^\rho = \frac{4}{d} c^\mu{}_{\mu\rho} x^\rho \ . \tag{74}$$

Then we obtain

$$c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu \ , \tag{75}$$

where

$$b_\mu = \frac{1}{d} c^\sigma{}_{\sigma\mu} \ . \tag{76}$$

Corresponding *special conformal transformation* has the form

$$x'^\mu = x^\mu + 2(x^\rho b_\rho) x^\mu - b^\mu (x^\nu x_\nu) \ . \tag{77}$$

Now we would like to discuss how fields transform under these transformations. Let us consider general transformation in the form

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \ , \tag{78}$$

where ω_a are parameters of these transformations. Physical fields transform under these transformations as

$$\Phi'(x') = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \tag{79}$$

where the functional $\mathcal{F}$ follows from the transformation properties of these fields. Let us define *generator* G_a of symmetry transformation from following relation

$$\delta_\omega \Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i\omega_a G_a \Phi(x) \ . \tag{80}$$

Now from previous definitions we have

$$\begin{aligned}
\Phi'(x + \omega_a \frac{\delta x^\mu}{\delta \omega_a}) = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a} \Rightarrow \\
\Phi'(x) + \partial_\mu \Phi \omega_a \frac{\delta x^\mu}{\delta \omega_a} = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a} \Rightarrow \\
\delta_\omega \Phi(x) = \Phi'(x) - \Phi(x) = -\omega_a \frac{\delta x^a}{\delta \omega_a} \partial_\mu \Phi + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}
\end{aligned} \tag{81}$$

and hence we have

$$iG_a\Phi = \frac{\delta x^\mu}{\delta\omega_a}\partial_\mu\Phi - \frac{\delta\mathcal{F}}{\delta\omega_a} \ . \tag{82}$$

For simplicity we consider the case where $\mathcal{F}(\Phi) = \Phi$. Now we have $\epsilon^\mu = a^\mu$ and the generator has the form

$$P_\mu = -i\partial_\mu \ , \text{(translation)} \tag{83}$$

In case of dilation we have $\frac{\delta x^\mu}{\delta\alpha} = x^\mu$ and hence the generator has the form

$$D = -ix^\mu\partial_\mu \ . \tag{84}$$

In case of special conformal transformation we have

$$\frac{\delta x^\mu}{\delta b^\nu} = 2x^\nu x^\mu - \delta^\mu_\nu(x^\rho x_\rho) \tag{85}$$

and hence the generator has the form

$$K_\mu = -i(2x_\mu x^\nu\partial_\nu - (x^\rho x_\rho)\partial_\mu) \tag{86}$$

In the same way we can determine the generator of Lorentz rotation

$$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{87}$$

Now we can calculate the algebra of these generators in the standard way as the commutators of these operators acting on the test function and we find

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \ , \\
[D, K_\mu] &= -iK_\mu \ , \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}P_\mu) \ , \\
[P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \ , \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \ .
\end{aligned} \tag{88}$$

Now we would like to show that this algebra is isomorphic to the algebra of $SO(d,2)$ and can have the standard form of $SO(d,2)$ alegra with signature $(-,+,+,\ldots,+,-)$. We define these generators as

$$J_{\mu\nu} = L_{\mu\nu} \ , J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu) \ , J_{\mu(d+1)} = \frac{1}{2}(K_\mu + P_\mu) \ , J_{(d+1)d} = D \ . \tag{89}$$

These generators obey $SO(d+1,1)$ commutation relations

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \ . \tag{90}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, \dots, 1, -1)$.

It is also very useful to consider conformal field theory in Euclidean space where the conformal group is $SO(d+1,1)$. Now R^d is conformally equivalent to S^d and hence the field theory on R^d (with appropriate boundary conditions at infinity) is isomorphic to the theory on S^d.

2.1.2 Conformal Invariance in Classical Field Theory

We say that given field theory is conformal field theory at the classical level if its action is invariant under conformal transformations. In order to formulate such a theory we have to give a prescription of the conformal transformations on the classical fields.

Let us now consider an infinitesimal conformal transformation parameterized by ω_g. We would like to find a matrix representation T_g such that a multicomponent field $\Phi(x)$ transforms as

$$\Phi'(x') = (1 - i\omega_g T_g)\Phi(x) \ , \tag{91}$$

where the generator T_g has to be added to the space-time part that was given above in order to find the full generators of symmetry. In order to find such a generator we consider a subgroup of the Poincare group that leaves the point $x = 0$ invariant which is the Lorentz group. Then we introduce a matrix representation $S_{\mu\nu}$ in order to define the action of infinitesimal Lorentz transformations on the field $\Phi(0)$

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0) \ , \tag{92}$$

where $S_{\mu\nu}$ is spin operator whose explicit form is given by nature of the field Φ. Then in order to calculate the action of this operator on the field at the space-time point x we have to perform the translation of given operator as

$$e^{ix^\rho P_\rho} L_{\mu\nu} e^{-ix^\rho P_\rho} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \tag{93}$$

where we used the Hausdorff formula for two A and B operators

$$e^{-A} B e^{B} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \dots \tag{94}$$

so that we generally obtain

$$P_\mu \Phi(x) = -i\partial_\mu \Phi(x) \ ,$$
$$L_{\mu\nu}\Phi(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x) \ . \tag{95}$$

In the same way we proceed with the conformal group. The subgroup that leaves the origin $x = 0$ invariant is generated by rotations, dilations and special conformal transformations. If we look on the algebra of generators of corresponding symmetry we obtain an algebra that has similar structure as Poincare algebra with the dilatation, where now K_μ play the similar roles as P_μ. We denote $S_{\mu\nu}, \bar{\triangle}$ and κ_μ as the respective values of the generators $L_{\mu\nu}, D$ and K_μ at $x = 0$. Clearly they have to form a matrix representation of the reduced algebra

$$\begin{aligned}
[\bar{\triangle}, S_{\mu\nu}] &= 0 \ , \\
[\bar{\triangle}, \kappa_\mu] &= -i\kappa_\mu \ , \\
[\kappa_\nu, \kappa_\mu] &= 0 \ , \\
[\kappa_\rho, S_{\mu\nu}] &= i(\eta_{\rho\mu}\kappa_\nu - \eta_{\rho\nu}\kappa_\mu) \ , \\
[S_{\mu\nu}, S_{\rho\sigma}] &= i(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}) \ .
\end{aligned} \tag{96}$$

Then we translate given operators, or more precisely, calculate the action of these operators on the field at the point x when we calculate

$$\begin{aligned}
e^{ix^\mu P_\mu} D e^{-ix^\mu P_\mu} &= D + x^\nu P_\nu \ , \\
e^{ix^\rho P_\rho} K_\mu e^{-ix^\rho P_\rho} &= K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu} + 2x_\mu(x^\nu P_\nu) - x^2 P_\mu \ ,
\end{aligned} \tag{97}$$

and hence we obtain the action of the operators on the physical fields

$$\begin{aligned}
D\Phi(x) &= (-ix^\nu \partial_\mu + \hat{\triangle})\Phi(x) \ , \\
K\Phi(x) &= (\kappa_\mu + 2x_\mu \hat{\triangle} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu)\Phi(x) \ .
\end{aligned} \tag{98}$$

Now we demand that $\Phi(x)$ belong to an irreducible representation of the Lorentz group and then by Schur's lemma, any matrix that commutes with all the generators $S_{\mu\nu}$ must be a multiple of the identity. Consequently the matrix $\hat{\triangle}$ is a multiple of the identity and from the algebra of the matrix we find that κ_μ has to vanish. $\hat{\triangle}$ is then number equal to $-i\triangle$ where $\triangle$ is the scaling dimension of the field Φ which is the scaling dimension of the field Φ defined as

$$\begin{aligned} x' &= \lambda x \ , \\ \Phi'(\lambda x) &= \lambda^{-\triangle}\Phi(x) \ . \end{aligned} \tag{99}$$

It is important to stress that previous form of the operators correspond to the conformal algebra realized on the space-time functions as differential operators. On the other hand in the quantum field theories these generators should correspond to the operators in Heisenberg picture, where they act on the local fields (again, local operators) in form of the commutators

$$\begin{aligned} [P_\mu, \Phi(x)] = i\partial_\mu \Phi(x) \ , \\ [L_{\mu\nu}, \Phi(x)] = (i(x_\mu\partial_\nu - x_\nu\partial_\mu) + S_{\mu\nu})\Phi(x) \ , \\ [D, \Phi(x)] = i(-\triangle + x^\mu\partial_\mu)\Phi(x) \ , \\ [K_\mu, \Phi(x)] = (i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu + 2x_\mu\triangle) - 2x^\nu S_{\mu\nu})\Phi(x) \ , \end{aligned} \tag{100}$$

where of course $S_{\mu\nu}$ are matrices of the finite dimensional representation of the Lorentz group that act on the indices of the field Φ. Further, $\triangle$ is the scaling dimension matrix that acts on the collection of the local operators Φ (with fixed spin) that may not be digagonalizable in general.

Further, the commutation relations given above imply that P_μ raises the scaling dimension of the field while K_μ lowers it. In unitary field theories there is a lower bound on the dimension of fields, for example in case of the scalar field we have $\triangle \geq (d-2)/2$. Then clearly each representation of the conformal group contains operators of lowest dimension that has to be annihilated by K_μ at $x = 0$. These operators ar called as *primary operators*. Then non-primary operators may be obtained by acting P_μ further.

We define the vacuum state as the state that obeys

$$P_\mu |0\rangle = J_{\mu\nu} |0\rangle = D |0\rangle = K_\mu |0\rangle = 0 \ . \tag{101}$$

Space-time symmetry and energy-momentum tensor

We introduced the symmetry of given system as an algebra of conserved charges that act on the Hilbert space. On the other hand in field theories we usually postulate that these symmetries are given as integrals of local conserved charges. In other words, if the current is conserved

$$\partial_\mu j^\mu = 0 \tag{102}$$

then we can construct the conserved charge

$$Q = \int d^{d-1}x j_0 \ . \tag{103}$$

Let us now presume that the action under infinitesimal translation $x'^\mu = x^\mu + \epsilon^\mu, \epsilon^\mu = \text{const}$. Then the corresponding conserved current has the form

$$T_c^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\partial^\nu\Phi \ , \partial_\mu T^{\mu\nu} = 0 \tag{104}$$

that by Nöether assumption obeys the conserved law

$$\partial^\mu T^c_{\mu\nu} = 0 \ . \tag{105}$$

Generally the energy-momentum tensor that is derived by Nöether procedure is not symmetric

$$T^c_{\mu\nu} \neq T^c_{\nu\mu} \tag{106}$$

but it can be made symmetric by Belinfante prescription.

Let us further presume that the action is invariant under scale transformation

$$x' = (1+\alpha)x \ , \mathcal{F}(\Phi) = (1-\alpha\triangle)\Phi \ . \tag{107}$$

Note that for general variation we have following conserved current

$$j_a^\mu = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta^\mu_\nu\mathcal{L}\right)\frac{\delta x^\nu}{\delta\omega_a} - \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\frac{\delta\mathcal{F}}{\delta\omega_a} \ , \tag{108}$$

where

$$\begin{aligned} x'^\mu &= x^\mu + \omega_a\frac{\delta x^\mu}{\delta\omega_a} \ , \\ \Phi'(x') &= \Phi(x) + \omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}(x) \ . \end{aligned} \tag{109}$$

Then in case of the scale invariance we find

$$j^\mu_D = -\mathcal{L}x^\mu + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi)}x^\nu\partial_\nu\Phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\triangle\Phi =$$
$$= T^\mu_{c\nu}x^\nu + \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi}\triangle\Phi \ , \tag{110}$$

Since by presumption the current j^μ_D is conserved we find

$$\partial_\mu j^\mu_D = \partial_\mu T^\mu{}_\nu x^\nu + T^\mu_{c\mu} + \triangle\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi)}\right) = 0 \ . \tag{111}$$

Let us now define virial of the field Φ

$$V^\mu = \frac{\delta\mathcal{L}}{\delta\partial^\rho\Phi}(\eta^{\mu\rho}\triangle + iS^{\mu\rho})\Phi \ , \tag{112}$$

where $S^{\mu\nu}$ is the spin operator of the field Φ. Let us also presume that virial is the divergence of another tensor $\sigma^{\nu\mu}$

$$V^\mu = \partial_\sigma\sigma^{\sigma\mu} \ , \tag{113}$$

where the last condition is obeyed in a large class of physical theories. Then we define

$$\sigma^{\mu\nu}_+ = \frac{1}{2}(\sigma^{\mu\nu} + \sigma^{\nu\mu}) \ ,$$
$$X^{\lambda\rho\mu\nu} = \frac{2}{d-2}\left(\eta^{\lambda\rho}\sigma^{\mu\nu}_+ - \eta^{\lambda\mu}\sigma^{\rho\nu}_+ + \eta^{\mu\nu}\sigma^{\lambda\rho}_+ + \frac{1}{d-1}(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\rho\nu})\sigma^{\alpha}_{+\alpha}\right) \ , \tag{114}$$

and we consider the following modified energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_c + \partial_\rho B^{\rho\mu\nu} + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} \ , \tag{115}$$

where the first two terms constitute the Belinfante tensor defined as

$$T^{\mu\nu}_B = T^{\mu\nu}_c + \partial_\rho B^{\rho\mu\nu} \ , B^{\rho\mu\nu} = -B^{\mu\rho\nu} \ . \tag{116}$$

Note that the presence of the tensor $T^{\rho\mu\nu}$ does not affect the conserved law

$$\partial_\mu T_B^{\mu\nu} = \partial_\mu T_c^{\mu\nu} + \partial_\mu \partial_\rho B^{\rho\mu\nu} = \partial_\mu T^{\mu\nu} \ . \tag{117}$$

We search for $B^{\rho\mu\nu}$ such that the new tensor $T_B^{\mu\nu}$ is symmetric. Then we can certainly construct another conserved current

$$J_\rho^{J\mu\nu} = x^\mu T_{B\rho}^{\ \ \nu} - x^\nu T_{B\rho}^{\ \ \mu} \tag{118}$$

so that

$$\begin{aligned} \partial^\rho J_\rho^{J\mu\nu} = \partial^\rho (x^\mu T_{B\rho}^\nu - x^\nu T_{B\rho}^\mu) = \\ = \eta^{\mu\rho} T_{B\rho}^\nu - \eta^{\rho\nu} T_{B\rho}^\mu + x^\mu \partial^\rho T_{B\rho}^\nu - x^\nu \partial^\rho T_{B\rho}^\mu = \\ = T_B^{\mu\nu} - T_B^{\nu\mu} = 0 \end{aligned} \tag{119}$$

Returning to (115) we claim that the last term is needed in order to make $T^{\mu\nu}$ traceless. Due to the symmetry properties of the tensor X we see that it does not spoil the conserved property since

$$\partial_\mu \partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} = 0 \ . \tag{120}$$

It can be also shown that it does not spoil the property of the Belinfante tensor since

$$X^{\lambda\rho\mu\nu} - X^{\lambda\rho\nu\mu} = \frac{2}{(d-2)(d-1)} \sigma_{+\alpha}^{\alpha} (\eta^{\lambda\mu}\eta^{\rho\nu} - \eta^{\lambda\nu}\eta^{\rho\mu}) \tag{121}$$

that vanishes when contracted with $\partial_\lambda \partial_\rho$. Finally the trace of the new term is

$$\frac{1}{2} \partial_\lambda \partial_\rho X^{\lambda\rho\mu}_{\ \ \ \ \mu} = \partial_\lambda \partial_\rho \sigma_+^{\lambda\rho} = \partial_\mu V^\mu \ . \tag{122}$$

From definition of $B^{\rho\mu\nu}$ we find

$$\partial_\rho B^{\rho\mu}_{\ \ \ \mu} = \frac{1}{2} i \partial_\rho \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi)} S^{\mu\rho} \Phi \right) \tag{123}$$

and then from the definition of the virial we obtain the final result

$$T^\mu_{\ \mu} = \partial_\mu j_D^\mu \ , \tag{124}$$

and therefore the scale invariance implies that the modified stress energy tensor is traceless. Then we can construct the special conformal current

$$K^{\mu}_{(\rho)} = [\rho_\nu x^2 - 2x_\nu(\rho_\sigma x^\sigma)]T^{\nu\mu} \tag{125}$$

for constant vector ρ. Note that this current obeys the equation

$$\begin{aligned}\partial_\mu K^{\mu}_{(\rho)} = [2\rho_\nu x_\mu - 2\eta_{\nu\mu}(\rho_\sigma x^\sigma) - 2x_\nu\rho_\mu]T^{\nu\mu} + \\ +[\rho_\nu x^2 - 2x_\nu(\rho_\sigma x^\sigma)]\partial_\mu T^{\nu\mu} = 0\end{aligned} \tag{126}$$

where the expression in the first bracket vanished due to the symmetry of $T^{\mu\nu}$ and also from the fact that $T^{\mu\nu}$ is traceless and where the last term vanished due to the conservation of $T^{\mu\nu}$. In this way we are able to construct the current corresponding to the special conformal transformations. Note however that we have to made crucial presumption about the existence of the virial current.

3 Supersymmetry and Gauge Theories

In order to understand the spectrum of operators in the QFT living on the boundary we have to say few basic facts considering SUSY. We start with Supersymmetry algebra in $3+1$ dimensions.

3.0.1 Supersymmetry Algebra in $3+1$ dimensions

We know that the Poincare symmetry of the flat space-time is generated by P_μ and $L_{\mu\nu}$. Supersymmetry enlarges the Poincare algebra by including spinor supercharges

$$\begin{aligned} Q^a_\alpha \ , \alpha = 1,2 \text{ left Weyl spinor } , \\ \bar{Q}_{\dot{\alpha}a} = (Q^a_\alpha)^\dagger \ , \text{right Weyl spinor } , \end{aligned} \tag{127}$$

where $\mathcal{N}$ is the number of independent supersymmetries of the algebra. We use two-component spinor notation $\alpha = 1,2$. The supercharges transform as Weyl spinors of $SO(1,3)$ and commute with translations. The remaining susy structure relations are

$$\left\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b \ , \left\{Q^a_\alpha, Q^b_\beta\right\} = 2\epsilon_{\alpha\beta} Z^{ab} \tag{128}$$

By construction the generators Z^{ab} are anti-symmetric in the indices a and b and commute with all generators of the supersymmetry algebra. Sometimes Z^{ab} are called as *central charges* so that we have

$$Z^{ab} = -Z^{ba} \ , [Z^{ab}, \dots] = 0 \ . \tag{129}$$

Clearly in case $\mathcal{N} = 1$ we have $Z = 0$. The supersymmetry algebra is invariant under a global phase rotation of all supercharges Q^a_α which is a group $U(1)_R$. Further, in case $\mathcal{N} > 1$, the different supercharges may be rotated into one another under the unitarity group $SU(\mathcal{N})_R$. Thes symmetrise of the supersymmetry algebra are called $R-$ symmetries. Generally these symmetries could be broken by quantum mechanical effects.

We are mainly interested in $\mathcal{N} = 4$ SYM theory where the Lagrangian density has the form

$$\begin{aligned}\mathcal{L} = \mathrm{Tr} \Bigg(& -\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \sum_a i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i + \\ & + \sum_{a,b,i} g C_i^{ab} \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g^2}{2} \sum_{i,j} [X^i, X^j]^2 \Bigg) \ ,\end{aligned} \tag{130}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \ , D_\mu X^i = \partial_\mu X^i + i[A_\mu, X^i] \ , \tag{131}$$

and where the constants C_i^{ab} and C_{iab} are related to the Clifford Dirac matrices for $SO(6)_R \sim SU(4)_R$ which can shown when we consider $\mathcal{N} = 4$ SYM in four dimension as the dimensional reduction on T^6 of $D = 10$ SYM. By construction this Lagrangian is invariant under $\mathcal{N} = 4$ Poincare supersymmetry.

It is instructive to determine the equations of motion that follows from the Lagrangian density given above. For simplicity we restrict to the case of the bosonic variables only. Varying with respect to X^i we obtain

$$\begin{aligned} & 2\partial_\mu (D^\mu X^i) + 2i(A_\mu D^\mu X^i - iD^\mu X^i A_\mu) + \\ & +2g^2(X_j [X^i, X^j] - [X^i, X^j] X_j) = \\ & = 2D_\mu D^\mu X^i + 2g^2 [X_j, [X^i, X^j]] = 0 \end{aligned} \tag{132}$$

and varying the action with respect to A_μ we obtain

$$\frac{2}{g^2}[\partial_\nu F^{\nu\mu} + iA_\mu F^{\nu\mu} - iF^{\nu\mu}A_\nu] + 2i[X_i, D^\mu X^i] = \\ = \frac{2}{g^2}D_\nu F^{\nu\mu} + 2i[X_i, D^\mu X^i] = 0 \ . \tag{133}$$

Note that A_μ, X^i belong to the adjoint representation of $U(N)$ so that

$$A^\dagger = A \ , (X^i)^\dagger = X^i \ . \tag{134}$$

It is also useful to expand A_μ, X^i with the help of the generators of the Lie group $T_\alpha, T_\alpha^\dagger = T_\alpha$ so that

$$A_\mu = A_\mu^\alpha T_\alpha \ , X^i = X^{i\alpha}T_\alpha \ . \tag{135}$$

so that

$$[X^i, X^j] = X^{i\alpha}X^{j\beta}[T_\alpha, T_\beta] = X^{i\alpha}X^\beta f_{\alpha\beta}^{\ \ \gamma}T_\gamma \ , \\ D_\mu X^i = (\partial_\mu X^{i\alpha} + iA_\mu^\gamma X^{i\delta}f_{\gamma\delta}^{\ \ \alpha})T_\alpha \ , \tag{136}$$

where $f_{\alpha\beta}^{\ \ \gamma}$ are structure constants of $U(n)$ algebra defined by the relation

$$[T_\alpha, T_\beta] = f_{\alpha\beta}^{\ \ \gamma}T_\gamma \ , (f_{\alpha\beta}^{\ \ \gamma})^* = -f_{\alpha\beta}^{\ \ \gamma} \ . \tag{137}$$

Then the equations of motion have the form

$$D_\mu(\partial^\mu X^{i\alpha} + iA^{\mu\gamma}X^{i\delta}f_{\gamma\delta}^\alpha)T_\alpha + [X_j, (X^{i\gamma}X^{j\delta}f_{\gamma\delta}^{\ \ \omega}T_\omega)] = \\ = \partial_\mu(\partial^\mu X^{i\alpha} + iA^{\mu\gamma}X^{i\delta}f_{\gamma\delta}^\alpha)T_\alpha + iA_\mu^\sigma(\partial^\mu X^\omega + iA^{\mu\gamma}X^i\delta f_{\gamma\delta}^{\ \ \omega})f_{\sigma\omega}^{\ \ \alpha}T_\alpha + X_j^\beta X^{i\gamma}X^{j\delta}f_{\gamma\delta}^{\ \ \omega}f_{\beta\omega}^{\ \ \alpha}T_\alpha = 0 \tag{138}$$

which has to be valid for all T_α and hence we obtain

$$\partial_\mu(\partial^\mu X^{i\alpha} + iA^{\mu\gamma}X^{i\delta}f_{\gamma\delta}^\alpha) + iA_\mu^\sigma(\partial^\mu X^\omega + iA^{\mu\gamma}X^i\delta f_{\gamma\delta}^{\ \ \omega})f_{\sigma\omega}^{\ \ \alpha} + X_j^\beta X^{i\gamma}X^{j\delta}f_{\gamma\delta}^{\ \ \omega}f_{\beta\omega}^{\ \ \alpha} = 0 \tag{139}$$

In the same way we obtain the equation of motion for A^α_μ

$$\begin{aligned}
&\partial_\nu(\partial^\nu A^{\mu\alpha}-\partial^\mu A^{\nu\alpha}+iA^{\nu\gamma}A^{\mu\delta}f_{\gamma\delta}^{\ \ \alpha})T_\alpha+\\
&+[X_i,(\partial^\mu X^{i\beta}+iA^{\mu\gamma}X^{i\delta}f_{\gamma\delta}^{\ \ \beta})T_\beta]=\\
&=\partial_\nu(\partial^\nu A^{\mu\alpha}-\partial^\mu A^{\nu\alpha}+iA^{\nu\gamma}A^{\mu\delta}f_{\gamma\delta}^{\ \ \alpha})T_\alpha+iX_i^\sigma(\partial^\mu X^{i\beta}+iA^{\mu\gamma}X^{i\delta}f_{\gamma\delta}^{\ \ \beta})f_{\sigma\beta}^{\ \ \alpha}T_\alpha=0
\end{aligned} \tag{140}$$

and hence we again find

$$\partial_\nu(\partial^\nu A^{\mu\alpha}-\partial^\mu A^{\nu\alpha}+iA^{\nu\gamma}A^{\mu\delta}f_{\gamma\delta}^{\ \ \alpha})+iX_i^\sigma(\partial^\mu X^{i\beta}+iA^{\mu\gamma}X^{i\delta}f_{\gamma\delta}^{\ \ \beta})f_{\sigma\beta}^{\ \ \alpha}=0\ . \tag{141}$$

Note that by definition we have

$$A^\mu=\eta^{\mu\nu}A_\nu\ ,\Phi^i=\delta^{ij}\Phi_j\ ,\partial^\mu=\eta^{\mu\nu}\partial_\nu\ , \tag{142}$$

where $\eta^{\mu\nu}=\mathrm{diag}(-1,1,1,1)\ ,\delta^{ij}=\mathrm{diag}(1,1,1)$.

Given Lagrangian is also scale invariant. Explicitly, we assign the standard mass dimensions to the fields and couplings

$$[A_\mu]=[X^i]=1\ ,[\lambda_a]=\frac{3}{2}\ ,[g]=0\ . \tag{143}$$

Clearly we see that the Lagrangian density has the scaling dimension $[\mathcal{L}]=4$ which, together with the scaling dimension $[d^4x]=-4$ implies that the action is invariant under scaling transformation. As we know this scaling symmetry combines with Poincare invariance into larger conformal symmetry $SO(2,4)\sim SU(2,2)$. This symmetry combines with $\mathcal{N}=4$ Poincare supersymmetry into superconformal symmetry given by the supergroup $SU(2,2|4)$.

It is very remarkable that this theory, upon perturbative quantization, does not exhibit ultraviolet divergences in the correlation functions of its canonical fields. Nonperturbatively there could be instanton corretions but they give finite contributions too and hence it is believed that the theory is UV finite. As a result the renormalization group $\beta-$function of the theory vanishes identically which implies that the theory is exactly scale invariant at the quantum level and the superconformal group $SU(2,2|4)$ is symmetry that is preserved in quantum theory too.

As we know there are various phases of given theory whose nature is determined by the potential of the theory. The potential energy term in the action of $\mathcal{N}=4$ SYM has the form

$$g^2\sum_{i,j}\int\mathrm{Tr}[X^i,X^j]^2 \tag{144}$$

whose equations of motion have the form

$$[X^i,[X^i,X^j]]=0 \tag{145}$$

that has the solution

$$[X^i,X^j]=0\ ,i,j=1,\dots,6\ . \tag{146}$$

[1] Now we show that there are two solutions of the equations (146)

- *The superconformal phase*: This phase is characterized by the conditions

$$\left\langle X^i\right\rangle=0\ ,i=1,\dots,6\ . \tag{147}$$

  We see that the gauge algebra $\mathcal{G}$ is unbroken and the superconformal symmetry $SU(2,2|4)$ is also unbroken. The physical states and operators are gauge invariant (say differently, $\mathcal{G}-$singlets) and transform under unitarity representations of $SU(2,2|4)$.

- *The spontaneously broken or Coulomb phase* that is characterized by the condition

$$\left\langle X^i\right\rangle\neq 0 \text{for at least one} i \tag{148}$$

  The nature of the resulting theory depends on the degree of residual symmetry. Generally we have $\mathcal{G}\rightarrow U(1)^r$ where r is the rank of the group $\mathcal{G}$ and the low energy behavior is that of r copies of $\mathcal{N}=4$ $U(1)$ theory. Superconformal symmetry is spontaneously broken too due to the non-zero vacuum expectation value of $\langle X^i\rangle$ which also set the scale of the theory.

3.0.2 Superconformall symmetry

The global continous symmetry group of $\mathcal{N}=4$ SYM is given supergroup $SU(2,2|4)$ with following ingredients [2] :

[1] We should also consider the solution that obeys $[X^i,X^j]=I_{N\times N}k\ ,k=\text{constant}$. However this is not valid for any finite X^i since taking the trace of this equation we find that the left side vanishes identically while the right side implies that $k=0$.

[2] We shall explain the meaning of the world supergroup. As the name suggests the supergroup is generalization of ordinary group in the sense that it possesses the same group structure. The Lie algebra of supergroups is superalgebra that is $\mathbf{Z}_2-$graded algebra. The element of supergroup has the form $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, where A,B are Grassmann even matrices, while B and C are Grassmann odd elements.

- *Conformal Symmetry* forming the group $SO(2,4) \sim SU(2,2)$ that is generated by translations P^μ, Loretnz transformations $L_{\mu\nu}$ dilatations D and special conformal transformations K^μ ,

- $R-$ symmetry that forms the group $SO(6)_R \sim SU(4)_R$ that is generated by $T^A, A = 1, \dots, 15$

- *Poincare supersymmetries* generated by the supercharges Q^a_α and their complex conjugates $\bar{Q}_{\dot{\alpha}a}$, $a = 1, \dots, 4$. These charges generate the $\mathcal{N} = 4$ Poincare supersymmetry.

- *Conformal symmetries* that are generated by the supercharges $S_{\alpha a}$ and their complex conjugates $\bar{S}^a_{\dot{\alpha}}$. The presence of these supercharges follows from the fact that Poincare supersymmetries and the special conformal transformations K_μ do not commute. Since both are symmetries their commutator has to be also symmetry that we denote as S generators.

The two bosonic subalgebras $SO(2,4)$ and $SU(4)_R$ commute. The supercharges Q^a_α and $S^a_{\dot{\alpha}}$ transform under **4** of $SU(4)_R$ while $\bar{Q}_{\dot{\alpha}a}$ and $S_{\alpha a}$ transform under $\mathbf{4}^*$. Then we can fit these generators in the the supermatrix that represents the corresponding superalgebra

$$\begin{pmatrix} P_\mu \ K_\mu \ L_{\mu\nu} \ D & Q^a_\alpha \ \bar{S}^a_{\dot{\alpha}} \\ \bar{Q}_{\dot{\alpha}a} \ S_{\alpha a} & T^A \end{pmatrix} \tag{149}$$

To proceed further note that we have following natural grading of the algebra that is given by the scaling dimension of the generators

$$\begin{aligned} [D] = [L_{\mu\nu}] = [T^A] = 0 \ , [P^\mu] = 1 \ , [K_\mu] = -1 \ , \\ [Q] = 1/2 \ , [S] = -1/2 \ . \end{aligned} \tag{150}$$

It is also important to stress that the superconformal algebras exist for $d \leq 6$ only. Schematically the comutation relations of the superconformal algebra include in addition to the previous relations, the relations

$$\begin{aligned} [D, Q] = -\frac{i}{2} Q \ , [D, S] = \frac{i}{2} S \ , [K, Q] \simeq S \ , [P, S] \simeq Q \ , \\ \{Q, Q\} \simeq P, \{S, S\} \simeq K \ , \{Q, S\} \simeq M + D + R \ . \end{aligned} \tag{151}$$

The exact form of the commutation relations depend on the dimensions of the space-time where given conformal field theory lives as follows from the fact that the spinorial representations of the conformal group are different for different space-time dimensions.

3.1 Superconformal Multiplets of Local Operators

We are interested in the construction of all local, gauge invariant operators which are polynomial in the canonical fields. As we know that canonical fields are X^i, λ_a and A_μ that have classical dimensions $1, 3/2$ and 1, respectively. It is clear that the gauge invariant operators can be constructed from the covariant objects $X^i, \lambda_a, F^{\pm}_{\mu\nu}$ and the covariant derivative D_μ with following canonical dimensions

$$[X^i] = [D_\mu] = 1 \ , [F^{\pm}_{\mu\nu}] = 2 \ , [\lambda_a] = \frac{3}{2} \ , \tag{152}$$

where $F^{\pm}_{\mu\nu}$ are (anti) self-dual gauge field strength. We see that the classical dimension of the composite operators is positive and that the number of operators whose dimension is less than a given number is finite. The only operator with dimension 0 is the unit operator.

Very important object in the theory are *superconformal primary operators*. Since the conformal supercharges S have dimension $-1/2$ we see that successive application of S to any operator of definite dimension leads to the operator of dimension 0. On the other hand we have to end there since otherwise we would generate operators with negative dimension which is impossible in a unitary representation. Then we define a *superconformal primary operator* $\mathcal{O}$ as a operator that is non-vanishing operator that obeys

$$[S, \mathcal{O}]_{\pm} = 0 \tag{153}$$

In other words we say that superconformal primary operator is the lowest dimension operator in given superconformal mutiplet. We should stress that there is an important difference from a conformal primary operator which is the operator that is annihilated by special conformal generators K^μ. In fact, every superconformal primary operator is conformal primary operator as follows from the fact that

$$[K^\mu, \mathcal{O}] \sim [\{S, S\}, \mathcal{O}] = 0 \tag{154}$$

since $[S, \mathcal{O}] = 0$. Finally an operator $\mathcal{O}$ is called a *superconformal descant operator* of an operator $\mathcal{O}'$, when it is of the form

$$\mathcal{O} = [Q, \mathcal{O}'] \ , \tag{155}$$

for some well-defined local polynomial gauge invariant operator $\mathcal{O}'$. Clearly if $\mathcal{O}$ is a descendant of $\mathcal{O}'$ then these two operators belong to the same superconformal multiplet. As follows from the commutation relation above we have

$$\begin{aligned} & [D, \mathcal{O}] = [D, [Q, \mathcal{O}']] = [Q, [D, \mathcal{O}']] + [[D, Q], \mathcal{O}'] = \\ = \ & \triangle_{\mathcal{O}'} [Q, \mathcal{O}'] + \frac{1}{2} [Q, \mathcal{O}'] = \\ = \ & (\triangle_{\mathcal{O}'} + \frac{1}{2}) [Q, \mathcal{O}'] = (\triangle_{\mathcal{O}'} + \frac{1}{2}) \mathcal{O} \ , \end{aligned} \tag{156}$$

using the fact that

$$[D, \mathcal{O}'] = \triangle_{\mathcal{O}'} \mathcal{O} \ , \quad [D, Q] = \frac{1}{2} Q \ . \tag{157}$$

and we see that the dimension of the operator $\mathcal{O}$ is $\triangle_{\mathcal{O}} = \triangle_{\mathcal{O}'} + \frac{1}{2}$. Further we see that the operator $\mathcal{O}$ cannot be a conformal primary operator because there is in the same multiplet at least one operator $\mathcal{O}'$ with dimension lower than $\mathcal{O}$. In other words in given irreducible superconformal multiplet there is a single superconformal primary operator which is the operator of lowest dimension and all others are superconformal descendants of this primary.

Let us explicitly construct superconformal primary operator in $\mathcal{N} = 4$ SYM. First of all we know that superconformal primary operator cannot be given as $Q-$commutator of another operator. In fact, let us presume that we have an operator $\mathcal{O} = [Q, X]$ for some operator X and calculate the commutator of this operator with S

$$[S, [Q, X]] = [Q, [X, S]] + [X, [S, Q]] = \cdots + [X, M + D + R] \tag{158}$$

where the last expression is non-zero since X has definite dimension and $R-$charge. It turns out that we have to know how Q transforms the canonical fields. Schematically we have

$$\begin{aligned} \{Q, \lambda\} = F^+ + [X, X] \ , \quad [Q, X] = \lambda, \\ \{Q, \bar{\lambda}\} = DX \ , \quad [Q, F] = D\lambda \end{aligned} \tag{159}$$

From these relations we see that local polynomial operator that contains any of the elements on the right side of the above structure relations cannot be primary since then it can be expressed as the commutator with Q with another operator. In other words chiral primary operators cannot involve neither the gauginos λ nor the gauge field strengths $F^{\pm}$. Since it has to be function of the scalars X only it cannot involve neither derivatives nor commutators of X. As a result we find that the superconformal primary operators are gauge invariant scalars that involve X only in a symmetrized way. The symplest operators are the single trace operators of the form

$$\mathrm{str}(X^{i_1}X^{i_2}\dots X^{i_n}) \ , \tag{160}$$

where $i_j, j = 1, \dots, n$ are $SO(6)_R$ fundamental representation indices and where str means the symmetrized trace over the gauge algebra so that the resulting operator is totally symmetric in the $SO(6)_R$ indices i_j. Generally given operator transforms under a reducible representation of $SO(6)$ which, in our case, is the symmetrized product of fundamentals. Irreducible operators may be obtained by isolating the traces over $SO(6)_R$ indices. Since $\mathrm{Tr}X^i = 0$ in $SU(N)$ the simplest operators are

$$\begin{aligned} \sum_i \mathrm{Tr}X^iX_i \text{....Konishi multiplet} \\ \mathrm{Tr}(X^iX^j + X^jX^i) \sim \text{supergravity multiplet} \end{aligned} \tag{161}$$

These operators have own image in the AdS space as we will see in following sections.

Of course, it is also possible to construct more complicated *multiple trace operators* that are obtained as products of single trace operators.

3.1.1 $\mathcal{N} = 4$ Chiral or BPS Multiplets of Operators

It turns out that there are special representations that are very important in the discussion of AdS/CFT correspondence. These representations correspond the the situation when at least one supercharge Q commutes with the primary operator. Such representations are usually referred as *chiral multiplets* or BPS mutiplets. These representations are shortened and hence their dimension cannot be renormalized or cannot receive quantum corrections. It turns out that these half-BPS operators play an important role in the AdS/CFT correpondence. The simplest series is given by single-trance operators of the form

$$\mathcal{O}_k(x) \equiv \frac{1}{n_k}\mathrm{str}\left(X^{\{i_1}(x)\dots X^{i_k\}}(x)\right) \ , \tag{162}$$

where $\{i_1, \dots, i_k\}$ means $SO(6)_R$ traceless part of the tensor and n_k means an overall normalization of the operator that is fixed by normalizing its $2-$point function.

3.2 Anti-de Sitter Space

3.2.1 Geomery of Anti-de Sitter Space

In order to understand the AdS/CFT correspondence we have to know some basic geometric facts about Anti-de Sitter space time. Very important is the relation between conformal compactifications of AdS and of flat space. In the case of the Euclidean signature metric we can compactify the flat space R^n to the $n-$sphere S^n by *adding a point at infinity* so that the conformal field theory is naturally defined on S^n. On the other hand $(n+1)$ dimensional hyperbolic space which is the Euclidean version of AdS space can be conformally mapped into the $(n+1)$ dimensional disk D_{n+1}. Therefore the boundary of compactified hyperbolic space is the compactified Euclid space. It turns out that a similar correspondence holds in case of the Lorentizan signature metric as well.

Let us say few words about **Conformal Structure of Flat Space**. Let us start with the simplest case $R^{1,1}$ which is two-dimensional Minkowski space with the metric

$$ds^2 = -dt^2 + dx^2 \ , (-\infty < t, x < +\infty) \tag{163}$$

Let us introduce two coordinates

$$u_\pm = t \pm x \ , x = \frac{u_+ - u_-}{2} \ , t = \frac{u_+ + u_-}{2} \tag{164}$$

so that the line element has the form

$$ds^2 = -du_+ du_- \ . \tag{165}$$

Finally we introduce two coordinates τ, θ

$$u_\pm = \tan\left(\frac{\tau \pm \theta}{2}\right) \tag{166}$$

so that

$$du_+ = \frac{1}{2\cos^2\left(\frac{\tau+\theta}{2}\right)}(d\tau + d\theta)\ , \quad du_- = \frac{1}{2\cos^2\left(\frac{\tau-\theta}{2}\right)}(d\tau - d\theta) \qquad (167)$$

so that the line element has the form

$$ds^2 = \frac{1}{4\cos^2\left(\frac{\tau+\theta}{2}\right)\cos^2\left(\frac{\tau-\theta}{2}\right)}(-d\tau^2 + d\theta^2)\ , \qquad (168)$$

Now we see that the flat Minkowski space is conformally mapped into the interior of the compact region

$$|\frac{\tau \pm \theta}{2}| < \pi/2\ . \qquad (169)$$

This space has four corners. The corners at $\tau, \theta = (0, \pm\pi)$ corresponds o the points at spatial infinity $x = \pm\infty$ in the original coordinates. Since light ray trajectories are invariant under a conformal rescaling of the metric this mapping gives the causal structure of $R^{1,1}$. Note also that the new coordinates (τ, θ) are well defined at the asymptotic regions of the flat space. In other words the conformal compactification is useful for a rigorous definition of asymptotic flatness of space-time when we say that the space-time is asymptotically flat if it has the same boundary structure as that of the flat space after conformal transformation.

$R^{1,p}\ , p \geq 2$

Let us consider higher dimensional Minkowski space-time

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2_{p-1}\ , \qquad (170)$$

where $d\Omega_{p-1}$ is the line element on the unit sphere S^{p-1}. Introduce variables $u_\pm$ defined as

$$u_\pm = t \pm r\ , du_\pm = dt \pm dr \qquad (171)$$

we find that the metric element has the form

$$ds^2 = -du_+ du_- + \frac{1}{4}(u_+ - u_-)^2 d\Omega^2_{p-1}\ . \qquad (172)$$

As the next step we introduce variables $\tilde{u}_\pm$ defined as

$$u_\pm = \tan\tilde{u}_\pm\ , du_\pm = \frac{1}{\cos^2\tilde{u}_\pm} du_\pm \qquad (173)$$

so that

$$ds^2 = \frac{1}{\cos^2\tilde{u}_+ \cos^2\tilde{u}_-}\left(-d\tilde{u}_+ d\tilde{u}_- + \frac{1}{4}\sin^2(\tilde{u}_+ - \tilde{u}_-)d\Omega_{p-1}^2\right) \tag{174}$$

Finally we introduce variables $\tilde{u}_\pm = \frac{1}{2}(\tau \pm \theta)$ so that we obtain the final form of the metric

$$ds^2 = \frac{1}{4\cos^2\tilde{u}_+ \cos^2\tilde{u}_-}\left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{p-1}^2\right) \ . \tag{175}$$

Note that we have

$$\tau = \tan^{-1}(t+r) + \tan^{-1}(t-r) \ , \theta = \tan^{-1}(t+r) - \tan^{-1}(t-r) \ . \tag{176}$$

Now the points $(t,r) = (\infty, 0), (-\infty, 0), (0, \infty)$ are mapped to the points $(\tau, \theta) = (\pi, 0), (-\pi, 0), (0, \pi)$ so that the the original half-plane $t \in (-\infty, \infty), r \leq 0$ is mapped to the triangular region in τ, θ plane defined these three points. However the conformally scaled metric

$$ds'^2 = -d\tau^2 + d\theta^2 \sin^2\theta d\Omega_{p-1}^2 \tag{177}$$

can be analytically continued outside of the triangle and the maximally extended space with

$$0 \leq \theta \leq \pi, \quad -\infty < \tau < +\infty \tag{178}$$

has the geometry $R \times S^p$ which is Einstein static universe, where the points $\theta = 0, \pi$ correspond to the north and south poles of S^p.

Since

$$\begin{aligned} \frac{\partial}{\partial\tau} &= \frac{\partial}{\partial u_+}\frac{\partial u_+}{\partial\tau} + \frac{\partial}{\partial u_-}\frac{\partial u_-}{\partial\tau} = \\ &= \frac{\partial}{\partial u^+}\frac{1}{2\cos^2\tilde{u}_+} + \frac{\partial}{\partial u_-}\frac{1}{2\cos^2\tilde{u}_-} = \\ &= \frac{1}{2}(1+u_+^2)\frac{\partial}{\partial u_+} + \frac{1}{2}(1+u_-^2)\frac{\partial}{\partial u_-} \ . \end{aligned} \tag{179}$$

using the fact that $\cos^2\tilde{u}_+ = \frac{1}{1+u_+^2}$. Now note that in the original variables we have

$$\begin{aligned}
\frac{\partial}{\partial u_+} + \frac{\partial}{\partial u_-} = \frac{\partial}{\partial t} \ , \\
u_+^2 \frac{\partial}{\partial u_+} + u_-^2 \frac{\partial}{\partial u_-} = \\
= (t^2 + r^2)(\frac{\partial}{\partial u_+} + \frac{\partial}{\partial u_-} + 2tr(\frac{\partial}{\partial u^+} - \frac{\partial}{\partial u^-}) = \\
= (t^2 + r^2)\frac{\partial}{\partial t} + 2tr\frac{\partial}{\partial r} = \\
= (-t^2 + r^2)\frac{\partial}{\partial t} + 2t(r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t}) \ .
\end{aligned} \tag{180}$$

Note that we have

$$P_\mu = -i\partial_\mu \ , K_\mu = -i(2x_\mu x^\nu \partial_\nu - (x^\rho x_\rho)\partial_\mu) \tag{181}$$

as the generators of translations and special conformal transformations on $R^{1,p}$. If we further identify $-i\frac{\partial}{\partial \tau} = H$, where H is the Hamiltonian corresponding to the evolution in $\tau-$coordinate so that we have relation

$$H = \frac{1}{2}(P_0 + K_0) = J_{0,p+2} \ . \tag{182}$$

In other words the generator $H = J_{0,p+2}$ corresponds to the $SO(2)$ part of the maximally compact subgroup $SO(2) \times SO(p+1)$ of $SO(2, p+1)$. Thus the subgroup $SO(2) \times SO(p+1)$ of the conformal group $SO(2, p+1)$ can be identified with the isometry of the Einstein static universe $R \times S^p$.

Anti-de Sitter Space The $(p+2)$-dimensional anti-de Sitter space AdS_{p+2} can be represented by the hyperboloid

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = R^2 \tag{183}$$

in the flat $(p+3)-$dimensional space with the metric

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2 \ . \tag{184}$$

By construction we see that this space has an isometry $SO(2,p+1)$ and it is homogeneous and isotropic. Equation (183) can be solved by setting

$$\begin{aligned} X_0 = R\cosh\rho\cos\tau \ , X_{p+2} = R\cosh\rho\sin\tau \ , \\ X_i = R\sinh\rho\Omega_i \ , (i=1,\dots,p+1; \sum_i \Omega_i^2 = 1) \ . \end{aligned} \tag{185}$$

It is easy to see that this ansatz solves (183

$$\begin{aligned} X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = \\ = R^2\cosh^2\rho\cos^2\tau + R^2\cosh^2\rho\sin^2\tau - \sum_{i=1}^{p+1} R^2\sinh^2\rho\Omega_i^2 = \\ = R^2\cosh^2\rho - R^2\sinh^2\rho = R^2 \end{aligned} \tag{186}$$

Then with the help of these variables we find that the line element has the form

$$\begin{aligned} ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2 = \\ = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2) \ . \end{aligned} \tag{187}$$

If we take $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$ we find that the solution (185) covers the whole hyperbolic space. Then (τ, ρ, Ω_i) are callet the global coordinates of AdS. Since for ρ near 0 we have

$$\cosh^2\rho \sim 1 \ , \sinh^2\rho \sim \rho^2 \mathrm{for} \rho \ll 1 \tag{188}$$

we find that the metric has the form in this region

$$ds^2 = R^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega^2) \tag{189}$$

we see that hyperboloid has the topology $S^1 \times R^{p+1}$, where S^1 is parameterized by τ so that it is closed time-like curve, while R^{p+1} is parameterized by ρ, Ω_i. In order to obtain causal space-time we simply unwrap the circle S^1 so that we take $-\infty < \tau < \infty$ with no identification. In following when we consider AdS_{p+2} space we will implicitly presume its universal covering space.

The isometry group $SO(2, p+1)$ of AdS_{p+2} has the maximal compact subgroup $SO(2) \times SO(p+1)$ where it is clear that $SO(2)$ represents the constant translations in the $\tau-$direction and $SO(p+1)$ gives rotation of S^p.

For some purposes it is useful to introduce the coordinate θ that is related to ρ by $\tan\theta = \sinh\rho$, $0 \le \theta \le \pi/2$ so that $d\rho^2 = \frac{d\theta^2}{\cos^2\theta}$ so that the line element has the form

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2) \ . \tag{190}$$

The causal structure of the space-time does not change by a conformal rescaling on the metric. In other words multiplying the metric by $R^{-2}\cos^2\theta$ it becomes

$$ds'^2 = -d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2 \ . \tag{191}$$

This is the metric of the Einstein static universe that appeared with the dimension lower by one in the conformal compactification of $R^{1,p}$. However it is important to stress that now the coordinate takes value in $0 \le \theta < \pi/2$ rather than $0 \le \theta < \pi$ that appears in the compactification of $R^{1,p}$. In other words AdS_{p+2} can be conformally mapped into one half of the Einstein static universe. In general, if a space-time can be conformally compactified into a region which has the same boundary structure as one half of the Einstein static universe the space-time is called asymptotically AdS.

The boundary of AdS is extended in time-like direction τ so that we need to specify boundary condition on the $R \times S^p$ at $\theta = \pi/2$ which defines the boundary of space-time given as points at $\rho \to \infty$. It is really important that the boundary of AdS_{p+2} or the boundary of the conformally compactified AdS_{p+2} is the same as the conformal compactification of the $(p+1)-$ dimensional Minkowski space-time.

In addition to the global parametrization given above there is also another set of coordinates $(u, t, \vec{x})(0 < u, \vec{x} \in R^p)$ that are defined by

$$X_0 = \frac{1}{2u}(1 + u^2 R^2 (1 + \vec{x}^2 - t^2)) \; , X_{p+2} = Rut \; ,$$
$$X^i = Rux^i \; , (i = 1, \dots, p) \; ,$$
$$X^{p+1} = \frac{1}{2u}(1 - u^2 R^2 (1 - \vec{x}^2 + t^2)) \; . \tag{192}$$

so that

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 =$$
$$= \frac{1}{4u^2}(1 + 2u^2 R^2 (1 + \vec{x}^2 - t^2) + u^4 R^4 (1 + \vec{x}^2 - t^2)^2) +$$
$$+ R^2 u^2 t^2 - R^2 u^2 \vec{x}^2 - \frac{1}{4u^2}(1 - 2u^2 R^2 (1 - \vec{x}^2 + t^2) + u^4 R^4 (1 - \vec{x}^2 + t^2)^2) = R^2 \tag{193}$$

Then the metric has the form

$$ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2) \right) \; . \tag{194}$$

These coordinates cover one half of the hyperboloid. These coordinates are called the Poincare coordinates. In this form of the metric the subgroup $ISO(1,p)$ and $SO(1,1)$ of the $SO(2, p+1)$ isometry are manifest where $ISO(1,p)$ is the Poincare transformation on $t(t, \vec{x})$ and $SO(1,1)$ is

$$(t, \vec{x}, u) \rightarrow (ct, c\vec{x}, c^{-1}u) \; , c > 0 \tag{195}$$

In the AdS/CFT correspondence this is identified with the dilatation D in the conformal symmetry group of $R^{1,p}$.

Consider again the metric (187) and (194). We clearly see that the norm of the vector ∂_τ is always non-zero in (187) and hence it is natural to call τ as the global time coordinate of AdS. On the other hand the time-like vector ∂_t in (194) becomes null at $u = 0$. This point is call as Killing horizon.

Euclidean Rotation As it is clear from the metric (187) the metric is static with respect to the time coordinate τ which is also global coordinate. Then it is possible to perform Wick rotation in the quantum field theory on

AdS_{p+2}.Explicitly, we perform Wick rotation $e^{i\tau H} \to e^{-\tau_E H}$, which means that $\tau_E = -i\tau$. Inserting this form to the relation between original variables X and τ we find

$$X_0 = R\cosh\rho\cosh\tau_E \ , \quad X_{p+2} = iR\cosh\rho\sinh\tau_E \ , \tag{196}$$

Let us define coordinate X_E as

$$X_E = -iX_{p+2} = R\cosh\rho\sinh\tau_E \ . \tag{197}$$

Then the original metric has the form

$$ds^2 = -dX_0^2 + dX_E^2 + \sum_{i=1}^{p+1} dX_i^2 \tag{198}$$

and the equation for the hyperboloid has the form

$$X_0^2 - X_E^2 - \sum_{i=1}^{p+1} X_i^2 = 0 \ . \tag{199}$$

Note also that the same space is obtained by rotating the time coordinate t of the Poincare coordiantes as $t_E = -it$ even though the Poincare coordinates cover only a part of the entire AdS.

In the coordinates $(\rho, \tau_E, \vec{\Omega}_p)$ and $(u, t_E, \vec{x})$ the Euclidean metric is expressed as

$$\begin{aligned} ds_E^2 = R^2(\cosh^2\rho d\tau_E^2 + d\rho^2 + \sinh^2\rho d\Omega_p^2) = \\ = R^2(\frac{du^2}{u^2} + u^2(dt_E^2 + d\vec{x}^2)) \ . \end{aligned} \tag{200}$$

It is also very convenient to use another form of the metric when we set $u = 1/y$, $du = -\frac{1}{y^2}dy$ so that we obtain the metric in the form

$$ds^2 = R^2\left(\frac{dy^2 + dx_1^2 + \dots dx_{p+1}^2}{y^2}\right) \tag{201}$$

This Euclidean form of the metric is very useful for analysis of the correlation functions in field theory. As we know from the standart textbook of quantum field theory the correlation functions $< \phi_1 \dots \phi_n >$ defined on the Euclidean space are related by the Wick rotation to the T-ordered correlation functions $< 0|T(\phi_1 \dots \phi_n|0 >$ in the Minkowski space. The same is true in the anti-de Sitter space on condition when the theory has a positive definite Hamiltonian with respect to the global time coordinate τ.

4 The Correspondence

4.1 The Maldacena limit

The space-time metric of N coincident D3-branes has following form

$$ds^2 = \left(1 + \frac{L^4}{y^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{L^4}{y^4}\right)^{1/2} (dy^2 + y^2 d\Omega_5^2) \ , \tag{202}$$

where the radius L of the D3-brane is given by

$$L^4 = 4\pi g_s N (\alpha')^2 \ . \tag{203}$$

No we would like to analyze the geometry in more details. For $y \gg L$ we have that the metric has the form of the flat space-time R^{10}. When $y < L$ the geometry is referred as the *throat* and naively we should say that it is singular for $y \ll L$. However in this regime it is natural to perform redefinition

$$u \equiv \frac{L^2}{y} \ , \tag{204}$$

where the limit $y \ll L$ corresponds to the large $u \gg L$ limit. Then we have $du = -\frac{L^2}{y^2} du$ and the metric has the form

$$\begin{aligned} ds^2 = \frac{1}{\sqrt{1 + \frac{u^4}{L^4}}} \eta_{\mu\nu} dx^\mu dx^\nu + \sqrt{1 + \frac{u^4}{L^4}} L^4 (\frac{du^2}{u^4} + \frac{1}{u^2} d\Omega_5^2) \xrightarrow{(u \gg L)} \\ = L^2 \left(\frac{1}{u^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\Omega_5^2 \right) \end{aligned} \tag{205}$$

that corresponds to the product geometry. Once component is the five-sphere S^5 with the metric $L^2 d\Omega_5^2$. The remaining component is the hyperbolic space

AdS_5 with constant negative curvature metric

$$ds^2_{AdS_5} = \frac{L^2}{u^2}(du^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \ . \tag{206}$$

In other words, the geometry close the brane ($y \sim 0, u \sim \infty$) is regular na highly symmetrical and corresponds to the space $AdS_5 \times S^5$ where both components have identical radius L.

The Maldacena limit corresponds to keeping fixed g_s and N as well as all physical length scales while letting $\alpha' \to 0$. It is very nice that this limit of string theory exists and is very interesting. In this limit only $AdS_5 \times S^5$ region of the D3-brane geometry survives and contributes to tht string dynamics of physical processes while the dynamics in the asymptotically flat region decouples from the theory.

To see this more explicitly consider the Maldacena limit directly on the string theory non-linear sigma model in the D3-brane background. We restrict ourselves to the metric part only so that we ignore the contributions from the tensor field F_5^+. We denote $D = 10$ coordinates by $x^M, M = 0, 1, \dots, 9$ and the metric by $G_{MN}(x)$. The first 4 coordinates coincide with x^μ of the Poincare invariant D3-brane world-volume while the coordinates on the 5$-$sphere are x^M for $M = 5, \dots, 9$ and $x^4 = u$. The full D3-brane metric has the form

$$\begin{aligned} ds^2 &= \left(1 + \frac{L^4}{y^4}\right)^{-1/2} \eta_{\mu\nu}dx^\mu dx^\nu + \left(1 + \frac{L^4}{y^4}\right)^{1/2} (dy^2 + y^2 d\Omega_5^2) = \\ &= L^2 \left[\left(1 + \frac{L^4}{u^4}\right)^{1/2} \left(\frac{du^2}{u^2} + d\Omega_5^2\right) + \left(1 + \frac{L^4}{u^4}\right)^{-1/2} \frac{1}{u^2}\eta_{\mu\nu}dx^\mu dx^\nu\right] \equiv \\ &\equiv L^2 \bar{G}_{MN}(x, L) dx^M dx^N \end{aligned} \tag{207}$$

Let us consider non-linear sigma model

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-\gamma}\gamma^{\alpha\beta} G_{MN}(x) \partial_\alpha x^M \partial_\beta x^N \ , \tag{208}$$

where $\gamma_{\alpha\beta}$ is two dimensional world-sheet metric with inverse $\gamma^{\alpha\beta}$ and where $\partial_\alpha \equiv \frac{\partial}{\partial\sigma^\alpha}$ where $\sigma^\alpha, \alpha = 0, 1$ are world-sheet coordinates. Finally $x^M(\sigma)$ describe an embedding of the string-world sheet into the target space-time. Now for the metric given in (207) we obtain

$$S = \frac{L^2}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-\gamma}\gamma^{\alpha\beta} \bar{G}_{MN}(x, L) \partial_\alpha x^M \partial_\beta x^N \ . \tag{209}$$

We see that the overall coupling for the sigma model dynamics is given by

$$\frac{L^2}{4\pi\alpha'} = \sqrt{\frac{\lambda}{4\pi}} \ , \lambda = g_s N \ . \tag{210}$$

Keeping N and g_s fixed but leting $\alpha' \to 0$ implie $L \to 0$ as follows from previous expression. We also see that under this limit the sigma model action has a smooth limit given by

$$S_G = \sqrt{\frac{\lambda}{4\pi}} \int_\Sigma d^2\sigma \sqrt{-\gamma}\gamma^{\alpha\beta} \bar{G}_{MN}(x,0)\partial_\alpha x^M \partial_\beta x^N \ . \tag{211}$$

where the metric $\bar{G}_{MN}(x,0)$ is the metric on $AdS_5 \times S^5$

$$\bar{G}_{MN} dx^M dx^N = \frac{1}{u^2}\eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\Omega_5^2 \ , \tag{212}$$

where the metric has unit radius. We also see that the coupling $1/\sqrt{\lambda}$ replaced α'. Now we proceed to the explicit statement of AdS/CFT correspondence. This conjecture states the equivalence between the following theories

- Type IIB superstring theory on $AdS_5 \times S^5$ where both AdS_5 and S^5 have the same radius L, where the $5-$form F_5^+ has integer flux $N = \int_{S^5} F_5^+$, where the string coupling is g_s
- $\mathcal{N} = 4$ SYM in $4-$dimensions with the gauge group $SU(N)$ and Yang-Mills coupling g_{YM} in its superconformal phase

with the following identifications between the parameters of both theories

$$g_s = g_{YM}^2 \ , L^4 = 4\pi g_s N (\alpha')^2 \ . \tag{213}$$

We should carefully explain what the equivalence or duality means. Briefly, equivalence includes a precise map between the states and fields on the superstring side and the local gauge invariant operators on $\mathcal{N} = 4$ SYM side as well as a correspondence between correlators in both theories.

We should also ask the question about validity of various approximations. The loop diagrams in field theory can be trust when

$$g_{YM}^2 N = g_s N \sim \frac{L^4}{(\alpha')^2} \ll 1 \ . \tag{214}$$

On the other hand the classical gravity description can be used when the radius of curvature L of AdS and S^5 becomes large compared to the string length,

$$\frac{L^4}{(\alpha')^2} \sim g_s N \sim g_{YM}^2 N \gg 1 \ . \tag{215}$$

We see that the gravity regime and the perturbative field theory regime are incompatible which is nice since we can avoid the clear contradiction since these theories are very different. For that reason we call this correspondence as duality in the sense that these two theories are conjectured that these theories are the same but when one theory is weakly coupled while the second one is strongly coupled. However this fact implies that it is really difficult to prove this duality since all known calculations tools are defined in the weakly perturbative regime. On the other hand it makes this duality very useful since when we presume it validity we can calculate expressions, that cannot be reached by the perturbative calculus in SYM theory, by calculations in the weakly coupled dual theory.

The formulation of the conjecture given above is known as the *strong form* since it is presumed that it holds for all values of N and of $g_s = g_{YM}^2$. On the other hand string theory quantization on a general curved manifold (including $AdS_5 \times S^5$) seems to be very difficult and for that reason it is not completely clear what the strong equivalence means. On the other hand we will see that $AdS_5 \times S^5$ background has very special property which leading to the integrability of the theory on given background. Before we proceed to this fundamental property of the theory which was discovered four years after original formulation of this conjecture we will follow original historical development and discuss limits in which the Maldacena conjecture becomes more tractable but still remains non-trivial.

4.1.1 The 't Hooft Limit

The 't Hooft limit consists in keeping the 't Hooft coupling $\lambda = g_{YM}^2 N = g_s N$ fixed and letting $N \to \infty$. As we know this limit is well defined in Yang-Mills theory and corresponds to a topological expansion of the field theory's Feynman diagrams. In case of AdS space we can proceed as follows. The string coupling can be expressed from the 't Hooft coupling as $g_s = \lambda/N$. Since λ is being kept fixed we see that 't Hooft limit corresponds to weak coupling string perturbatiion theory.

This form of the conjecture is weaker than the original version but still provides powerful correspondence between classical string theory and the large N limit of gauge theories.

4.1.2 The Large λ limit

When we discussed 't Hooft limit we kept $\lambda = g_s N$ fixed while $N \to \infty$. Since $N \to \infty$ we find that the only parameter that is left is λ. Quantum field perturbation theory corresponds to $\lambda \ll 1$. On the other side of the correspondence it is natural to take $\lambda \gg 1$ instead. Let us now discuss in more details the meaning of the expanssion around λ large. Let us perform an expansion in powers of α' the effective action so that we have

$$\mathcal{L} = a_1 \alpha' R + a_2 (\alpha')^2 R^2 + a_3 (\alpha')^3 R^3 + \dots \ . \tag{216}$$

We are interested at distance scales that are typical of the throat whose scale is set by the AdS radius L. In other words the scale of the Riemann tensor is set by

$$R \sim \frac{1}{L^2} = \frac{(g_s N)^{-1/2}}{\alpha'} = \frac{\lambda^{-1/2}}{\alpha'} \tag{217}$$

and we see that the expansion of the effective action in powers of α' effectively corresponds in the expansion in powers of $\lambda^{-1/2}$

$$\mathcal{L} = a_1 \lambda^{-1/2} + a_2 \lambda^{-1} + a_3 (\alpha')^3 \lambda^{-3/2} + \dots \tag{218}$$

and we see that the expansion in the small α' is effective replaced expansion in large λ where the parameter of expansion is $\lambda^{-1/2} \ll 1$.

$\mathcal{N} = 4$ conformal SYM all N, g_{YM} $\bullet g_s = g_{YM}^2$	$\Leftrightarrow$	Full Quantum Type IIB string theory on $AdS_5 \times S^5$ $\bullet L^4 = 4\pi g_g N \alpha'^2$
• 't Hooft limit of $\mathcal{N} = 4$ SYM $\lambda = g_{YM}^2 N$ fixed, $N \to \infty$ $\bullet 1/N$ expansion	$\Leftrightarrow$	• Classical Type IIB string theory on $AdS_5 \times S^5$ $\bullet g_s$ string loop expansion
• Large λ limit of $\mathcal{N} = 4$ SYM (for $N \to \infty$) $\bullet \lambda^{-1/2}$ expansion	$\Leftrightarrow$	• Classical Type IIB supergravity on $AdS_5 \times S^5$ $\bullet \alpha'$ expansion

4.2 Relation between Global Symmetries

It is clear that in order to be duality between boundary theory and the bulk theory to be valid the global unbroken symmetries of the two theories to be identical. The continuous global symmetry of $\mathcal{N} = 4$ SYM in its conformal phase (which is the phase where the vacuum expectation value of scalar fields is zero) is superconformal group $SU(2,2|4)$ with the maximal bosonic subgroup $SU(2,2) \times SU(4)_R \sim SO(2,4) \times SO(6)_R$, where $SO(2,4)$ is conformal group in 4−dimensions while $SU(4)_R$ is automorphism group of the $\mathcal{N} = 4$ Poincare supersymmetry algebra. We clearly see that the bosonic group corresponds to the isometry group of the $AdS_5 \times S^5$ background. As we argued before this group can be completed to the full supergroup in case of SYM theory. In case of the AdS side we find that 16 of 32 Poincare supersymmetries are preserved by the array of N parallel D3-branes. Further when we take the near horizon limit of this geometry we find that this AdS limit is supplemented by another 16 conformal supersymmetries that are broken in the full D3-brane geometry. Thus the global symmetry $SU(2,2|4)$ matches on both sides of the AdS/CFT correspondence.

$\mathcal{N} = 4$ SYM theory also obeys Montonen-Olive or S-duality symmetry that is realized on the complex constant τ by Möbius transformations in $SL(2,\mathbf{Z})$. On the other hand we known that this symmetry is a global discrete symmetry of Type IIB string theory. This symmetry is unbroken by D3-brane solution as well that it maps non-trivially the dilaton and axion values that are constant for given type of solution. On the other hand S-duality is a useful symmetry in case of the strongest for of the AdS/CFT correspondence only since when we consider the 't Hooft limit $N \to \infty$ while keeping $\lambda^2 = g_{YM}^2 N$ fixed we find that S-duality does not have the right meaning. Explicitly, under S-duality we have $g_{YM} \to \frac{1}{g_{YM}}$ and hence $\lambda \to \frac{N^2}{\lambda}$ which means that in the limit $N \to \infty$ lambda is no longer fixed and goes to infinity as well.

4.3 Mapping Type IIB Fields and CFT Operators

Since we showed that the global symmetry groups on both sides of the duality coincide we now have to show that the representations of the group $SU(2,2|4)$ also coincide on both sides. We already made remarks considering operators in SYM theory. We also shown that the significant role is played by the *single color trace operators* since all higher trace operators can be construct from these operators with the help of operator product expansion. Then we should expect that single trace operators correspond to single particle states on the AdS side. Multiple trace operators should be interpreted as bound states of these one particle states. Multiple trace BPS operators have the property that their dimension on the AdS side is simply the sum of the dimensions of the BPS components.

In order to identify the contents of irreducible representations of $SU(2,2|4)$ on the AdS side, we describe all Type IIB massless supergravity and massive string degrees of freedom by fields ϕ living on $AdS_5 \times S^5$. We introduce coordinates $z^\mu, \mu = 0, 1, \ldots, 4$ for AdS_5 and $y^u, u = 1, \ldots, 5$ and decompose the metric as

$$ds^2 = g^{AdS}_{\mu\nu} dz^\mu dz^\nu + g^S_{uv} dy^u dy^v \ . \tag{219}$$

Then it is convenient to decompose the field $\phi(z, y)$ in series on S^5

$$\phi(z, y) = \sum_{k=0} \phi_k(z) Y_k(y) \ , \tag{220}$$

where Y_k means the function that belongs to the basis of spherical harmonics on S^5. For scalars for example, Y_k are labelled by the rank k of the totally symmetric traceless representations of $SO(6)$.

4.3.1 Spherical Harmonics

Here we review few basic facts considering spherical harmonics. The set of scalar functions on S^D form a vector space that is an infinite dimensional reducible representation of $SO(D + 1)$. Scalar spherical harmonics form a complete basis of this space.

It is convenient to regard a function on S^D as a restriction of functions on the $\mathbf{R}^{D+1}$ in which S^D is embedded. An arbitrary C^∞ function on $\mathbf{R}^{D+1}$ may be expanded in polynomials in the Cartesian coordinates x^i, so that it is sufficient to consider separately functions on $\mathbf{R}^{D+1}$ that are homogeneous in x^i of degree k. Clearly when we restrict these functions on S^D then these functions are not all independent. Consider for example $r^2 x^{i_1} \ldots x^{i_k}$ where $r^2 = \sum_{i=1}^{D+1} x_i^2$. This is a function of degree $k + 2$ but when we restrict to the sphere we find that it is identical to $x^{i_1} \ldots x^{i_k}$ (due to the fact that $r^2 = 1$ on sphere) which is a function of degree k. Further, if we wish to restrict ourselves to functions linearly independent of those of lower degree, we must consider only functions

$$C_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k} \tag{221}$$

where the coeficients $C_{i_1\dots i_k}$ obey the condition $C_{i_1\dots i_k}\delta^{i_m i_n}=0$ for any $1\leq m,n\leq k$. It is also clear we could demand that $C_{i_1\dots i_k}$ is symmetric in $i_1,\dots i_k$.

In other worlds each independent component of a totally symmetric traceless tensor of rank k defines spherical harmonic by

$$Y^I=C^I_{i_1\dots i_k}x^{i_1}\dots x^{i_k}\ . \tag{222}$$

Now we would like to find the eigenvalues of $(\nabla)^2_{S^D}$ on the sphere. Note that harmonics as polynomials in $\mathbf{R}^{D+1}$ obey the equation

$$\begin{aligned}(\nabla^2)_{D+1}f_k=\frac{\partial}{\partial x^j}\frac{\partial}{\partial x_j}\sum C_{i_1\dots i_k}x^{i_1}\dots x^{i_k}=\\=k\frac{\partial}{\partial x^j}\sum C_{i_1\dots i_k}\delta^{i_1}_j x^{i_2}\dots x^{i_k}=\\=k(k-1)\sum C_{i_1 i_2 i_3\dots i_k}C_{i_1\dots i_k}\delta^{i_1}_j\delta^{i_2 j}x^{i_3}\dots x^{i_k}=k(k-1)\sum C_{i_1\dots i_k}\delta^{i_1 i_2}x^{i_3}\dots x^{i_k}=0\end{aligned} \tag{223}$$

where we firstly used the symmetry of $C_{i_1\dots i_k}$ and then the property that it is a traceless tensor. Further note that we can write

$$(\nabla^2)_{D+1}=\frac{1}{r^D}\partial_r[r^D\partial_r]+\frac{1}{r^2}(\nabla^2)_{S^D}\ . \tag{224}$$

Note also that when we express all variables using spherical variables we find that f_k has to depend on r through the overall prefactor r^k. Then we find

$$\begin{aligned}(\nabla^2)_{D+1}f_k=0=\frac{1}{r^D}\partial_r[r^D\partial_r f_k]+\frac{1}{r^2}(\nabla^2)_{S^D}f_k=\\=\frac{k}{r^D}\partial_r[r^{D-1}f_k]+\frac{1}{r^2}(\nabla^2)_{S^D}f_k=\\\frac{k(D+k-1)}{r^D}r^{D-2}f_k+\frac{1}{r^2}(\nabla^2)_{S^D}f_k=0\rightarrow\\(\nabla^2)_{S^D}f_k=-k(D+k-1)f_k\end{aligned} \tag{225}$$

The previous discussion was performed in case of the scalar spherical harmionics. This discussion can be generalized to the case of the vector and tensor spherical harmonics.

Returning to our analysis we find that the field in the AdS receives contribution from the momentum along S^5 since we can certainly decompose 10-dimensional Laplacian as

$$\frac{1}{\sqrt{-g_{MN}}}\partial_M[g^{MN}\sqrt{-g_{MN}}\partial_N] =$$
$$= \frac{1}{\sqrt{-g^{AdS}_{\mu\nu}}}\partial_\mu[\sqrt{-g^{AdS}_{\mu\nu}}g^{\mu\nu}_{AdS}\partial_\nu] + \frac{1}{\sqrt{g^S_{uv}}}\partial_u[\sqrt{g^S_{uv}}g^{uv}\partial_v] \ . \tag{226}$$

In the same way as in the case of Kaluza-Klein compactification we find that the field compactified on S^5 receives contribution to the mass where the mass term is given by the expression

$$m^2 = -\frac{1}{\sqrt{g^S_{uv}}}\partial_u[\sqrt{g^S_{uv}}g^{uv}\partial_v] \ . \tag{227}$$

For the eigenvalues of the Laplacion on S^5 for various spins we find following relations between mass and scaling dimension

$$\begin{aligned}
&\text{scalars} \quad m^2 = k(k+4) \ , \\
&\text{spin}1/2, 3/2 \quad |m| = 2 + k \ , \\
&p-\text{form} \quad m^2 = (4+k-p)(k+p) \ , \\
&\text{spin } 2 \quad m^2 = (4+k)k \ .
\end{aligned} \tag{228}$$

The question is how these modes are related to the operators on the dual side.

4.4 The Field↔ Operator Correspondence

As we said above conformal field theory does not have asymptotic states or an S-matrix so that the natural objects that we have to consider are operators. For example, in $\mathcal{N} = 4$ SYM we have marginal operator that changes the values of the coupling constant. As we know the field theory coupling constant is related to the coupling constant in string theory which, as we know, is related to the expectation value of the dilaton, where the expectation value of the dilaton is set by the boundary condition for the dilaton at infinity.

In other words changing the gauge theory coupling constant corresponds to the changing the boundary value of the dilaton. To be more explicitly, let us denote by $\mathcal{O}$ the corresponding operator. Then we consider that we add the additional term

$$\int d^4x \phi_0(x)\mathcal{O}(x) \tag{229}$$

to the Lagrangian. Let us presume that such a term was not present in the original Lagrangian. In opposite situation we could consider $\phi_0(x)$ to be equal to some coefficient of $\mathcal{O}(x)$ in the Lagrangian. Then it is natural to assume that this will change the boundary condition of the dilaton at the boundary of AdS, where in the Poincare coordinate system it is $\phi(x,z)|_{z=0} = \phi_0(x)$. To be more explicit, according to the famous paper by E. Witten [8], it is natural to propose

$$\left\langle e^{\int d^4x \phi_0(x)\mathcal{O}(x)} \right\rangle_{CFT} = \mathcal{Z}_{string}[\phi(x,z)|_{z=0} = \phi_0(x)] \ , \tag{230}$$

where the left hand side is the generating function of correlation functions in the field theory, ϕ_0 is an arbitrary function , where we calculate the correlation functions of / by taking the functional derivatives with respect to ϕ_0 and then setting $\phi_0 = 0$. The right side is the full partition function of string theory with the boundary condition that the field ϕ has the value ϕ_0 on the boundary AdS. It is also clear that ϕ_0 is a function of the four variables that parameterize the boundary of AdS_5.

The expression (230) is general for any field ϕ. In other words every field propagating on AdS_5 is in a one to one correspondence with an operator in the field theory. Clearly there should be a relation between the mass of the field ϕ and the scaling dimension of the operator in the conformal field theory. To see this more explicitly let us consider Euclidean AdS_5 or $H = \{(z_0, \vec{z}), z_0 > 0, \vec{z} \in \mathbf{R}^4\}$ with Poincare metric

$$ds^2 = \frac{R^2}{z^2}(dz_0^2 + d\vec{z}^2) \ . \tag{231}$$

with boundary $\partial H = \mathbf{R}^4$. It is convenient to represent this space graphically as a disc whose boundary is a circle. The metric diverges at the boundary $z_0 = 0$ where the overall scale factor blows up. This scale factor may be removed by a Weyl rescaling of the metric but such a rescaling is not unique. A unique *well-defined limit to the boundary of* AdS_5 *can only exist if the boundary theory is scale invariant.* For finite values of $z_0 > 0$ the geometry the theory still have 4−dimensional Poincare invariance but it does not have to be invariant.

We know that superconformal $\mathcal{N} = 4$ Yang-Mills theory is scale invariant and so that we can say that it can live at the boundary of ∂H. Further, the dynamical observables of $\mathcal{N} = 4$ SYM are the local invariant polynomial operators that we introduced in previous sections. These operators naturally live on the boundary ∂H and are characterized by their dimension, Lorentz group $SO(1,3)$ and $SU(4)_R$ quantum numbers.

Following discussion given above we decompose all 10−dimensional fields on Kaluza-Klein towers on S^5 so that effective all fields $\phi_\triangle(z)$ propagate on AdS_5. We label these fields by their dimension $\triangle$ where other quantum numbers are implicit. We also presume that these fields are asymptotically free where of course there is a possibility that they interact in the bulk of AdS. The free field than satisfy the equations

$$(\frac{1}{\sqrt{g_{AdS}}}\partial_\mu[\sqrt{g_{AdS}}g^{\mu\nu}_{AdS}\partial_\nu\phi^0_\triangle] - m^2_\triangle\phi^0_\triangle = 0 \tag{232}$$

Let us now concentrate on the $z-$dependence so that the equation above takes the form

$$\frac{1}{\sqrt{g_{AdS}}}\frac{d}{dz}\left[\sqrt{g_{AdS}}g^{zz}\frac{d\phi^0_\triangle}{dz}\right] - m^2\phi^0_\triangle = \tag{233}$$

where we also presume an asymptotic behavior $\phi_0^\triangle \sim z^\Gamma$ for $\Gamma = \text{const}$. Then the equation above takes the form

$$\frac{1}{R^2}\Gamma(\Gamma-4)z^\Gamma - m^2\phi^0_\triangle = (\Gamma(\Gamma-4) - m^2R^2)\phi^0_\triangle = 0 \tag{234}$$

so that we have two roots of the equation above

$$\Gamma_+ = 2 + 2\sqrt{4+m^2R^2}\ , \Gamma_- = 2 - 2\sqrt{4+m^2R^2}\ . \tag{235}$$

We see that these two solutions have different behavior when we approach $z_0 \to 0$

$$\begin{aligned}\phi_0^+ &\sim z_0^{\Gamma_+} = z_0^{2+\sqrt{4+m^2R^2}} \text{....normalizable}\ ,\\ \phi_0^- &\sim z_0^{\Gamma_-} = z_0^{2-\sqrt{4+m^2R^2}} \text{...non}-\text{normalizable}\ .\end{aligned} \tag{236}$$

There is a relation between the mass of the field ϕ and the scaling dimension of the operator in the conformal field theory. Let us be more general and consider the AdS_{d+1} space where the metric has the form

$$ds^2 = \frac{R^2}{z^2}(dz^2 + (dx^1)^2 + \dots (dx^d)^2) \tag{237}$$

In this space the equation of motion takes the form

$$\begin{aligned} &\frac{1}{\sqrt{g_{AdS}}}\frac{d}{dz}\left[\sqrt{g_{AdS}}g^{zz}\frac{d\phi}{dz}\right] - m^2\phi = \\ &= \frac{z^{d+1}}{R^{d+1}}\frac{d}{dz}[\frac{R^{d+1}}{z^{d+1}}\frac{z^2}{R^2}\frac{d\phi}{dz}] - m^2\phi = 0 \end{aligned} \tag{238}$$

that for asymptotic $\phi \sim z^\Gamma$ implies

$$\Gamma(\Gamma - d) - m^2R^2 = 0 \Rightarrow \Gamma_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2R^2} \ . \tag{239}$$

We see that the solution with Γ_- is blowing up for $z \to 0$ while field with behaves as z_+^Γ vanishes for $z \to 0$. In other words the second mode corresponds to the normalizable solution while the first one appears as the source in the CFT action. In other words we find that the field that should be source in CFT action has to have following asymptotic behavior

$$\phi(\vec{x}, t) = \epsilon^{\Gamma_-}\phi_0(\vec{x}) = \epsilon^{d-\triangle}\phi_0(\vec{x}) \ , \tag{240}$$

and eventually we take the limit $\epsilon \to 0$. Note that we wrote Γ_- as

$$\Gamma_- = \frac{d}{2} - \sqrt{\frac{d^2}{4} + R^2m^2} = d - (\frac{d}{2} + \sqrt{\frac{d^2}{4} + R^2m^2}) \equiv d - \triangle \ . \tag{241}$$

We presume that ϕ is dimensionless so that ϕ_0 has dimension $[\text{length}]^{-\Gamma_-}$ that implies, through the left side of the equation (230) that the dimension of the operator $\mathcal{O}$ is

$$-d + [\mathcal{O}] + d - \triangle = 0 \Rightarrow [\mathcal{O}] = \triangle \ . \tag{242}$$

It turns out that in the case of the interacting fields the solutions will have the same asymptotic behaviors as in the free case. It was argued that the normalizable modes determine the vacuum expectation values of the operators of associated dimensions and quantum numbers. On the other hand the non-normalizable solutions do not correspond to the bulk excitations due to the fact that they are not square normalizable. In fact, they represent the coupling of external sources to the superagravity or string theory.

5 Explicit Witten's presentation

When we calculate the correlation functions it is conventient to use the Euclidian signature of the metric which can be described at several different ways. Consider a Euclidean space $\mathbf{R}^{d+1}$ with coordinates $y_0, \dots, y_d$ and let $\mathbf{B}_{d+1}$ is the open unit ball given by the condition

$$\sum_{i=0}^{d} y_i^2 < R^2$$

. Then AdS can be identified with the unit ball with the metric

$$ds^2 = 4R^4 \frac{\sum_{i=0}^{d} dy_i^2}{(R^2 - |y^2|)^2} \ . \tag{243}$$

With this parameterization is easy to compactify given AdS_{d+1} when get the closed unit ball $\bar{B}_{d+1}$ defined by the equation

$$\sum_{i=0}^{d} y_i^2 \leq R^2 \ . \tag{244}$$

Its boundary is the sphere $\mathbf{S}^d$ that is defined by the condition

$$\sum_{i=0}^{d} y_i^2 = R^2 \ . \tag{245}$$

$\mathbf{S}^d$ is the Euclidean version of the conformal compactification of Minkowski space. Since $\mathbf{S}^d$ is the boundary of $\bar{B}_{d+1}$ we find that this statement is the Euclidean version of the statement that Minkowski space is the boundary of AdS_{d+1}. We also see that the metric (243) that is defined on B_{d+1} cannot be extended on $\bar{B}_{d+1}$ since the metric is singular at $\sum_{i=0}^{d} y_i^2 = R^2$. In order to find a metric that extends on $\bar{B}_{d+1}$ we can take a function f on $\bar{B}_{d+1}$ that is positive on B_{d+1} and has a first order zero on the boundary, as for example $f = R^2 - |y|^2$. Then we consider the line element as

$$d\tilde{s}^2 = f^2 ds^2 \ , \tag{246}$$

then we can restrict the metric $d\tilde{s}^2$ on $\mathbf{S}^d$. Since there is not some prefered choice of f we see that this metric is only well defined up to conformal transformation. For example, if we consider the function f with another one

$$f \to f e^w \tag{247}$$

where w is real function on $\bar{B}_{d+1}$ we find that this replacement will induce the conformal transformation

$$d\tilde{s}^2 \to e^{2w} d\tilde{s}^2 \tag{248}$$

in the metric of $\mathbf{S}^d$.

In order to have contact with previous definitions let us write the metric ds^2 as

$$ds^2 = \frac{1}{(1 - \frac{y^2}{R^2})^2}(dy^2 + y^2 d\Omega) \ , \tag{249}$$

where $d\Omega^2$ is the metric on the unit sphere. Then we introduce variable r as

$$y = R\tanh\frac{\rho}{2} \ , dy = \frac{R}{2\cosh^2\frac{\rho}{2R}} d\rho \tag{250}$$

so that the line element has the form

$$\begin{aligned} ds^2 &= 4\cosh^4\frac{\rho}{2}(R^2\frac{1}{4\cosh^4\frac{\rho}{2}}d\rho^2 + + \frac{\sinh^2\frac{\rho}{2}}{\cosh^2\frac{\rho}{2}}d\Omega) = \\ &= R^2(d\rho^2 + \sinh^2\rho d\Omega) \ . \end{aligned} \tag{251}$$

5.1 Massless Field Equations

As a simple example we consider the massless equations that allow to reach the most nice result. In this case the equation of motion takes the form

$$\frac{1}{\sqrt{g_{AdS}}}\partial_\mu[\sqrt{g_{AdS}} g^{\mu\nu}_{AdS} \partial_\nu \phi] = 0 \ . \tag{252}$$

The important property about AdS_{d+1} is that if we have function $\phi(\Omega)$ on the boundary $\mathbf{S}^d$ then there is a unique extension of ϕ to a function on $\bar{B}_{d+1}$ that has given boundary values and obeys the field equation, where uniqueness means that there is non non-zero square-integrable solution of the equation above which, if it existed, could be added to any given solution above and hence destroys the uniqueness.

Let us now presume that the boundary value ϕ_0 is the source for the operator $\mathcal{O}$ and we have to calculate the classical action for ϕ that obeys the equation of motion with prescripted boundary conditions. To do this we try to find Green function which we mean a solution K of the Laplace equation on B_{d+1} whose boundary value is a delta function at a point P on the boundary. It is convenient to use the representation of B_{d+1} as the upper half space with metric

$$ds^2 = \frac{R^2}{x_0^2} \sum_{i=0}^{d} (dx^i)^2 \ , \tag{253}$$

where the boundary is a copy of $\mathbf{R}^d$ at $x^0 = 0$ together with a single point P at $x^0 = \infty$, where the point $x^0 = \infty$ consists of a single point since the metric in x^i direction vanishes as $x_0 \to \infty$. Note that without the point P the boundary would consist the space $\mathbf{R}^d$. On the other hand we know that the conformal compactification of $\mathbf{R}^d$ is obtained by adding in a point P at infinity which in the end gives the sphere $\mathbf{S}^d$. Since we argued before the the boundary of the Euclidean AdS is sphere it is natural to include the point P.

Returning to our example we see that the boundary conditions and metric are invariant under translations of the x_i so that K should obey this symmetry and to be a function of x_0 only. Then the equation of motion has the form

$$\frac{d}{dx_0} x_0^{-d+1} \frac{d}{dx_0} K(x_0) = 0 \tag{254}$$

with the solution

$$K(x_0) = c x_0^d \ . \tag{255}$$

where c is a constant. This function grows at infinity so that there is some sort of singularity at the boundary point P. Let us now consider $SO(1, d+1)$ transformation that maps P to a finite point. Explicitly we consider the transformation

$$x_i \to \frac{x_i}{x_0^2 + \sum_{j=1}^{d} x_j^2} \ , i = 0, \dots, d \tag{256}$$

that transforms the point $x_0 = \infty$ to the origin $x_i = 0, i = 0, \ldots, d$. Under these transformations the function K has the form

$$K(x) = c\frac{x_0^d}{(x_0^2 + \sum_{j=1}^d x_j^2)^d} \ . \tag{257}$$

It is easy to see that for $x_0 \to 0$ K vanishes except at $x_1 = x_2 \cdots = x_d = 0$. Further, K is positive. In other words for x_0 K is a delta function with support at $x_i = 0$. Using this Green function the solution of the Laplace equation on the upper half space with the boundary values ϕ_0 has the form

$$\phi(x_0, x_i) = c\int d\mathbf{x}'\frac{x_0^d}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^d}\phi_0(x') \ . \tag{258}$$

where $d\mathbf{x}' = dx_1' dx_2' \ldots dx_d'$ and $|\mathbf{x} - \mathbf{x}'|^2 = \sum_{j=1}^d (x_i - x_x')^2$. Then we find

$$\begin{aligned}\frac{\partial \phi}{\partial x_0} = dc\int d\mathbf{x}'[\frac{x_0^{d-1}}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^d}\phi_0(\mathbf{x}') - \\ -\frac{x_0^{d+1}}{(x_0 + |\mathbf{x} - \mathbf{x}'|^2)^{d+1}} = \\ = dcx_0^{d-1}\int d\mathbf{x}\frac{|\mathbf{x} - \mathbf{x}'|^2}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^{d+1}}\phi_0(\mathbf{x}') \ .\end{aligned} \tag{259}$$

In the limit $x_0 \to 0$ we can neglect x_0^2 in denominator and we obtain

$$\lim_{x_0 \to 0}\frac{\partial \phi}{\partial x_0} = dcx_0^{d-1}\int d\mathbf{x}\frac{1}{|\mathbf{x} - \mathbf{x}'|^{2d}}\phi_0(\mathbf{x}') \ . \tag{260}$$

Let us now determine the on-shell value of the action for the scalar field

$$\begin{aligned}S(\phi) = \frac{1}{2}\int d^{d+1}x\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \\ = \int d^{d+1}x\partial_\mu[\phi\sqrt{g}g^{\mu\nu}\partial_\nu\phi] - \int d^{d+1}x\phi\partial_\mu[\sqrt{g}^{\mu\nu}\partial_\nu\phi] = \\ = \lim_{\epsilon \to 0}\int_{T_\epsilon} d\mathbf{x}\sqrt{h}\phi(\vec{n}\cdot\vec{\nabla}\phi) \ ,\end{aligned} \tag{261}$$

where in the first step we used integration by parts and in the second step we used the fact that the second expression on the second line vanishes since the field ϕ obeys the equation of motion. Finally the surface T_ϵ is the surface $x_0 = \epsilon, h$ is its induced metric and n is unit normal vector to T_ϵ. From the definition of the metric we have that $h_{ij} = \frac{1}{x_0^2}\mathrm{diag}(1,\ldots,1)$ and hence $\sqrt{h} = x_0^{-d}$ so that $\sqrt{h}|_{T_\epsilon} = \epsilon^{-d}$. Further, since $\vec{n}$ has to obey the equation

$$n^\mu n^\nu g_{\mu\nu} = 1 \Rightarrow n^0 g_{00} n^0 = (n^0)^2 \frac{1}{x_0^2} = 1 \tag{262}$$

we find that $n^0 = x_0$ that on the boundary has the value $n_{T_\epsilon} = \epsilon$. Then we obtain

$$n^\mu \partial_\mu \phi = n^0 \partial_{x_0} \phi = x_0 \frac{\partial \phi}{\partial x_0} \ . \tag{263}$$

Using the value of $\partial_{x_0}\phi$ determined above and the fact that $\phi \to \phi_0$ for $x_0 \to 0$ by definition we obtain that the on-shell value of the action has the form

$$\begin{aligned} S(\phi)^{on\ shell} &= \lim_{\epsilon\to 0} \int_{T_\epsilon} d\mathbf{x}\sqrt{h}\phi n^0 \frac{\partial \phi}{\partial x^0}) = \\ = \lim_{\epsilon\to 0} \int_{T_\epsilon} \epsilon^{-d}\epsilon & \left(\epsilon_0^{d-1} \int d\mathbf{x}' \frac{1}{|\mathbf{x}-\mathbf{x}'|^{2d}} \phi_0(\mathbf{x}') \right) = \\ &= cd \int d\mathbf{x} d\mathbf{x}' \frac{\phi_0(\mathbf{x})\phi_0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{2d}} \end{aligned} \tag{264}$$

so that the two point function of the operator $\mathcal{O}$ is a multiple of $|\mathbf{x}-\mathbf{x}'|^{-2d}$ as expected for a field $\mathcal{O}$ of a conformal dimension d.

In what follows we will be more precise with the definition of the correspondence. As we argued above the correspondence has the general form

$$\exp\left\{-\Gamma[\bar{\phi}_\triangle]\right\} = \left\langle \exp\left\{ \int_{\partial H} \bar{\phi}_\triangle \mathcal{O}_\triangle \right\} \right\rangle \tag{265}$$

where these expressions are underestood to hold order by order in a perturbative expansion in the number of fields $\bar{\phi}_D$. On the AdS side we assume that we have an action $S[\phi_\triangle]$ that summarizes the dynamics of Type IIB string theory on $AdS_5 \times S^5$. In the supergravity approximation, $S[\phi_\triangle]$ is just

Type IIB supergravity action on $AdS_5 \times S^5$. Beyond the supergravity approximation, $S[\phi_\triangle]$ will include α' corrections due to the massive string effects. In this case we have

$$\Gamma[\bar{\phi}_\triangle] = \text{extr} S[\phi_\triangle] \ , \tag{266}$$

where the extremum on the right side is taken over all fields $\phi_\triangle$ that satisfy the assymptotic behavior for the boundary fields $\bar{\phi}_\triangle$. These sources correspond to the non normalizable modes and correspond to the sources of the SYM operators $\mathcal{O}_\triangle$.

5.2 Quantum Expansion in $1/N-$Witten Diagrams

The actions that we study have an overall coupling constant factor. For example, in case of the part of Type IIB supergravity for the dilaton Φ and the acion C in the presence of a metric $G_{\mu\nu}$ in the Einstein fame is given by

$$S = \frac{1}{2\kappa_5^2} \int_H \sqrt{G}[-R_G + \Lambda + \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} e^{2\Phi} \partial_\mu C \partial^\mu C] \tag{267}$$

Now the dimensional analysis gives that

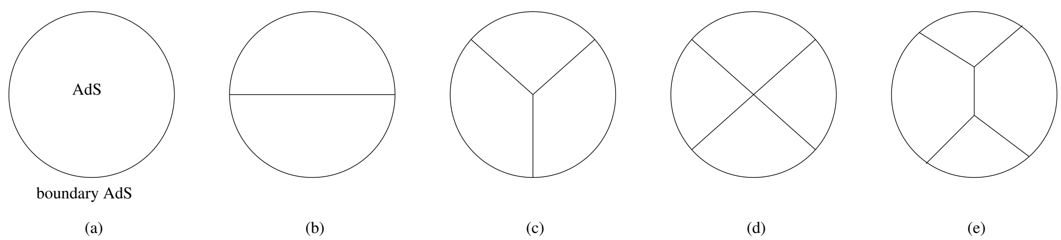
$$[\kappa_5^2] = M^{-3} \ , [\Phi] = [C] = M^0 \ . \tag{268}$$

We show later that κ_5^2 is equal to

$$\kappa_5^2 = \frac{\pi R^3}{2N^2} \tag{269}$$

and hence it implies that for large N κ_5 is small and then we can perform small κ_5, or semi-classical expansion of the correlators generated by this action. As a result we obtain set of rules that has similar form as Feynmann diagrams which are known as *Witten diagrams*. The Witten diagram is represented by a disc whose interior corresponds to the interior of AdS while the boundary circle corresponds to the boundary of AdS. The graphical rules are as follows:

- Each external source to $\bar{\phi}_\triangle(\vec{x}_I)$ is located at the boundary circle of the Witten diagram at a point $\vec{x}_I$.
- We include a propagator from the external source at $\vec{x}_I$ that points either to another boundary point or to an interior interaction point. This second propagator is known as *boundary to bulk propagator*.



- The structure of the interior interaction points is determined from the integration terms in the bulk action in the same way as in case of Feynman diagrams.
- Two interior interaction points may be connected by *bulk-to bulk propagators* in the same way as in case of ordinary Feynman diagrams.

In the diagram below we see $2-, 3-$ and $4-$ point function contributions.

5.3 AdS propagators

In the previous section we determined the propagator of the massless scalar field. In given section we determine more general form of these propagators. We work in Euclidean AdS_{d+1} space that we denote by H with the Poincare metric

$$ds^2 = \frac{R^2}{z_0^2}(dz_0^2 + d\vec{z}^2) \ . \tag{270}$$

Due to the fact that the space H has an isometry $SO(1,d+1)$ the Green functions depend upon the $SO(1,d+1)$ invariant distance between two points in H. The geodetic distance is given by

$$d(z,w) = \int_w^z ds \ln\left(\frac{1+\sqrt{1-\zeta^2}}{\zeta}\right) \ , \zeta = \frac{2z_0w_0}{z_0^2 + w_0^2 + (\vec{z}-\vec{w})^2} \ . \tag{271}$$

To see this it is instructive to solve the geodesic equation in this space. We do it when we study motion of point particle in AdS that is governed by the action

$$S = -m\int d\tau\sqrt{-g_{MN}\partial_\tau X^M\partial_\tau X^N} \tag{272}$$

Varying given action we derive the equation of motion in the form

$$\partial_\tau\left[\frac{g_{MN}\partial_\tau X^N}{\sqrt{-g_{PR}\partial_\tau X^P\partial_\tau X^R}}\right] - \frac{\delta g_{PR}}{\delta X^M}\frac{\partial_\tau X^P\partial_\tau X^R}{2\sqrt{-g_{PR}\partial_\tau X^P\partial_\tau X^R}} = 0 \tag{273}$$

Note that the Minkowski form of AdS metric has the form

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dx^i dx_i + dz^2) \tag{274}$$

so that we presume that there are no motion in the x_i directions and the only time dependent variable is $z_0 = z_0(t)$. First of all the equation of motion for $X^0 = t$ has the form

$$\partial_\tau \left[\frac{g_{tt}\partial_\tau X^0}{\sqrt{-g_{PR}\partial_\tau X^P \partial_\tau X^R}} \right] = 0 \tag{275}$$

We take the ansatz $t = \tau$ and hence then this equation implies the conserved quantity

$$\frac{g_{tt}}{\sqrt{-g_{tt} - g_{zz}(\partial_\tau Z)^2 - g_{ij}\partial_\tau X^i \partial_\tau X^j}} = C \tag{276}$$

on the other hand the equation of motion for X^i has the form

$$\partial_\tau \left[\frac{g_{ij}\partial_\tau X^j}{\sqrt{-g_{PR}\partial_\tau X^P \partial_\tau X^R}} \right] = 0 \tag{277}$$

that can be solved by the ansatz $X^i = v^i t + X^i_0$ due to the equation (275). Then instead of solving the equation of motion for z directly we use the equation (275) to find the differential equation for Z

$$\begin{gathered} \partial_t Z = \frac{\sqrt{(1-v^2)Z^2 - 1}}{CZ} \Rightarrow \\ Z^2 = (1-v^2)(Ct + C_0)^2 + \frac{1}{1-v^2} \ . \end{gathered} \tag{278}$$

In Euclidean theory we have similar equation

$$\begin{gathered} \frac{g_{zz}}{\sqrt{g_{zz} + g_{z_0 z_0}(\partial_z z^0)^2 + g_{ij}v^i v^j}} = C \Rightarrow \\ \frac{1}{C^2 Z_0^2} = (1 - v^2 - (\frac{dZ_0}{dz})^2) \end{gathered} \tag{279}$$

that can be easily solved with the result

$$\frac{1}{C^2(1+v^2)} = (z+K)^2 + (1+v^2)(Z_0)^2 \ . \tag{280}$$

that is an equation for ellipse. Note that for $v = 0$ we find the circle with radius $\frac{1}{C^2}$ and we see that the geodesic starts at $z = -\frac{1}{C^2} - K$, reaches the turning point at $z = -K$ and then returns back to the boundary that reaches at $z = \frac{1}{C^2} - K$. It is also clear that we should generalize this equation to the any point on the boundary performing $S0(4)$ rotation. so that we have the equation

$$\frac{1}{C^2} = (\vec{Z} - \vec{K})^2 + (Z_0)^2 \ . \tag{281}$$

We could be also interested in the distance between two points in the bulk of AdS. So that we should choose the initial condition that for given vector $\vec{Z}_i$ the point we are at the point Z_i^0 and for given $\vec{Z}_f$ we are at the point Z_f^0 so that we find two equations

$$\begin{aligned} \frac{1}{C^2} &= |\vec{Z}_i - \vec{K}|^2 + (Z_i^0)^2 \ , \\ \frac{1}{C^2} &= |\vec{Z}_f - \vec{K}|^2 + (Z_f^0)^2 \ , \end{aligned} \tag{282}$$

that gives

$$\vec{K} = \frac{1}{2}(\vec{Z}_f - \vec{Z}_i)(1 + \frac{(Z_f^0)^2 - (Z_i^0)^2}{|\vec{Z}_f - \vec{Z}_i|^2}) \ . \tag{283}$$

And then C can be found from the expression above using $\vec{K}$.

Generally, using this dependence we can find the geodetic distance between two points $(z_0, \vec{z})$ and $(w_0, \vec{w})$. However it is much more convenient to proceed following way. We know that Euclidean Anti-De Sitter space is described by the equation

$$-X_0^2 + X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -R^2 \tag{284}$$

in the space time $\mathbf{R}^{d,1}$ with the metric

$$ds^2 = -dX_0^2 + dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 = 0 \ , \tag{285}$$

so that the metric is $\eta_{MN} = (-1, 1, \ldots, 1)$. Let us introduce the action that determines the evolution of the particle on given space

$$S = \int d\tau (\dot{X}^M \eta_{MN} \dot{X}^N + \Gamma (X^M \eta_{MN} X^N + R^2)) \tag{286}$$

where Γ is Lagrange multiplicator that imposes the constraint that the particle moves on anti-de Sitter space. Then the equation of motion has the form

$$-\ddot{X}^N + \Gamma X^N = 0 \ , \tag{287}$$

while variation with respect to Γ we obtain the constraint $X^M \eta_{MN} X^N = -R^2$. Lagrange multiplicator Γ we derive from the first equation when we multiply it $\eta_{MN} X^N$ we obtain

$$\begin{aligned} -\ddot{X}^M X_M + \Gamma X^M X_M = 0 \Rightarrow \dot{X}^M \dot{X}_M - \Gamma R^2 = 0 \Rightarrow \\ \Rightarrow \Gamma = \frac{1}{R^2} \dot{X}^M \dot{X}_M \end{aligned} \tag{288}$$

where we used the fact that

$$\begin{aligned} \frac{d}{d\tau}(-R^2) = \frac{d}{d\tau}(X^M \eta_{MN} X^N) = 0 \Rightarrow \dot{X}^M \eta_{MN} X^N = 0 \Rightarrow \\ \ddot{X}^M \eta_{MN} X^N + \dot{X}^M \dot{X}_N = 0 \end{aligned} \tag{289}$$

Then the equation of motion for X has the form

$$\ddot{X}^M - \frac{1}{R^2} (\dot{X}^N \dot{X}_N) X^M = 0 \tag{290}$$

Further, using the equation of motion for X^M and multiply it with $\dot{X}_M$ we obtain

$$-\ddot{X}^M \dot{X}_M + \Gamma X^M \dot{X}_M = 0 \Rightarrow \frac{d}{d\tau}(\dot{X}^M \dot{X}_M) = 0$$

that implies

$$\dot{X}^M \dot{X}_M = \text{const} \equiv K^2 \tag{291}$$

Then the equation of motion for X^M has simple form

$$\ddot{X}^M - \frac{K^2}{R^2} X^M = 0 \tag{292}$$

Let us presume an ansatz in the form

$$X^M = m^M \exp\left(\frac{K}{R}\lambda\right) + n^M \exp\left(-\frac{K}{R}\lambda\right) \ , \tag{293}$$

Then

$$X^M X_M = m^M m_M \exp\left(2\frac{K}{R}\lambda\right) + n^M n_M \exp\left(-2\frac{K}{R}\lambda\right) + 2n^N m_M = -R^2 \tag{294}$$

from which we derive the conditions on the vectors n and m

$$n^N n_N = m^M m_M = 0 \ , n^M m_N = -\frac{R^2}{2} \ . \tag{295}$$

Now it is easy to determine the geodesic distance between two points $X(\lambda_1)$ and $X(\lambda_2)$ that is defined as

$$d = \int_1^2 ds = \int_1^2 d\lambda \sqrt{\dot{X}^M \dot{X}_M} = K(\lambda_2 - \lambda_1) \ . \tag{296}$$

To proceed further we use following expression

$$X^M(\lambda_2) X_M(\lambda_1) = m^M n_M [\exp\left(\frac{K}{R}(\lambda_2 - \lambda_1)\right) + \exp\left(-\frac{K}{R}(\lambda_2 - \lambda_1)\right)] \tag{297}$$

from which we obtain

$$e^{\frac{K}{R}(\lambda_2 - \lambda_1)} = \frac{1}{R^2}(-X(\lambda_2)^M X_M(\lambda_1) + \sqrt{(X^M(\lambda_2) X_M(\lambda_1))^2 - R^4}) \tag{298}$$

from which we obtain

$$K(\lambda_2 - \lambda_1) = R \ln \frac{1}{R^2}(-X(\lambda_2)^M X_M(\lambda_1) + \sqrt{(X^M(\lambda_2) X_M(\lambda_1))^4 - R^4}) \tag{299}$$

Let us introduce coordinates u, t_E, x^i as

$$\begin{aligned} X_0 = \frac{1}{2u}(1 + u^2(R^2 + \vec{x}^2 + t_E^2)) \ , X_{d+1} = Rut_E \ , \\ X^i = Rux^i \ , i = 1, \dots, d-1 \ , \\ X_{d+1} = Rut_E \ , X^d = \frac{1}{2u}(1 - u^2(R^2 - \vec{x}^2 - t_E^2)) \ , \end{aligned} \tag{300}$$

that obeys the equation that defines an embedding of AdS

$$-X_0^2 + X_{d+1}^2 + \sum_{i=1}^{d-1} X_i^2 + X_d^2 = \tag{301}$$

Finally we introduce $z_0 = \frac{1}{u}$ so that

$$\begin{aligned} X_0 = \frac{z_0}{2}(1 + \frac{1}{z_0^2}(R^2 + \vec{x}^2 + t_E^2)) \ , X_{d+1} = R\frac{1}{z_0}t_E \ , \\ X^i = \frac{R}{z_0}x^i \ , i = 1, \ldots, d-1 \ , \\ X_{d+1} = \frac{R}{z_0}t_E \ , X^d = \frac{z_0}{2}(1 - \frac{1}{z_0^2}(R^2 - \vec{x}^2 - t_E^2)) \ , \end{aligned} \tag{302}$$

So that we finally obtain

$$X^M(1)X_M(2) = -\frac{1}{2}R^2\left(\frac{w_0^2 + z_0^2 + (\vec{z} - \vec{w})^2}{z_0 w_0}\right) \equiv -\frac{R^2}{\xi} \tag{303}$$

and hence the geodesic distance takes the form

$$d = R \ln\left(\frac{1 + \sqrt{1 - \xi^2}}{\xi}\right) \ . \tag{304}$$

Since we see that the geodesic distance depends on ξ it is more natural to work with ξ than with the geodetic distance. Sometimes is also used the notion *chordal distance* that is defined as

$$u = \frac{1 - \xi}{\xi} \ . \tag{305}$$

Then we can find an inverted relation

$$\xi = \frac{2e^{d/R}}{1 + e^{2d/R}} = \frac{1}{\cosh d/R} \Rightarrow u = \cosh\frac{d}{R} - 1 \ . \tag{306}$$

Now we are ready to consider bulk to bulk propagator. Consider scalar field with the mass $m^2 = \triangle(\triangle - d)$. According to the discussion given previously we know that

$$\frac{d}{2} + \sqrt{\frac{d^2}{4} + R^2 m^2} = \triangle \Rightarrow m^2 = \frac{1}{R^2}\triangle(\triangle - d) \ . \tag{307}$$

Now the action for the massive field has the form

$$S_{\phi_\triangle} = \int_H d^{d+1}z\sqrt{g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_\triangle\partial_\nu\phi_\triangle + \frac{1}{2}m^2\phi_\triangle^2 - \phi_\triangle J\right] \tag{308}$$

The equation of motion for $\phi_\triangle$ has the form

$$(\Box_g + m^2)\phi_\triangle = J \ , \tag{309}$$

where the scalar Laplacian has the form

$$\Box_g = -\frac{1}{\sqrt{g}}\partial_\mu[\sqrt{g}g^{\mu\nu}\partial_\nu] = -z_0^2\partial_0^2 + (d-1)z_0\partial_0 - z_0^2\sum_{i=1}^d \partial_i^2 \ . \tag{310}$$

Let us now presume that the field $\phi_\triangle$ is given in response to the source by the equation

$$\phi_\triangle(z) = \int_H d^{d+1}z'\sqrt{g}(z')G_\triangle(z,z')J(z') \ , \tag{311}$$

Inserting this equation to the equation of motion above and using the fact that

$$J(z) = \int d^{d+1}z'\delta(z-z')J(z') \tag{312}$$

which says that $G(z,z')$ obeys the equation

$$(\Box_g + m^2)G_\triangle(z,z') = \frac{1}{\sqrt{g}}\delta(z-z') \ . \tag{313}$$

Now the Green function $G_\triangle(z,w)$ has the form

$$G_\triangle(z,w) = G_\triangle(\xi) = \frac{2^{-\triangle}C_\triangle}{2(\triangle - d)}\xi^\triangle F\left(\frac{\triangle}{2}, \frac{\triangle}{2} + \frac{1}{2}; \triangle - \frac{d}{2} + 1; \xi^2\right) \ , \tag{314}$$

where F is hypergeometric function and the overall normalization constant is defined by

$$C_\triangle = \frac{\Gamma(\triangle)}{\pi^{d/2}\Gamma(\triangle - \frac{d}{2})} \ . \tag{315}$$

Since $0 \leq \xi \leq 1$ the hypergeometric function is defined by its Teylor series for all ξ except at the coincident point $\xi = 1, \xi = w$. **The massive scalar boundary to bulk propagator** This propagator corresponds to the limit when the source is on the boundary of H.

The action to linearized order is given by

$$S_{\phi_\triangle} = \int_H d^{d+1}z\sqrt{z}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_\triangle\partial_\nu\phi_\triangle + \frac{1}{2}m^2\phi_\triangle^2\right] - \int_{\partial H} d^{d+1}\bar{\phi}_\triangle(\vec{z})\bar{J}(\vec{z}) \ , \tag{316}$$

where the bulk field $\phi_\triangle$ is related to the boundary field $\bar{\phi}_\triangle$ by the limiting relation

$$\bar{\phi}_\triangle(\vec{z}) = \lim_{z_0 \to \infty} z_0^{\triangle - d}\phi_\triangle(z_0, \vec{z}) \ . \tag{317}$$

The corresponding *boundary-to-bulk* propagator is the Poisson kernel

$$K_\triangle(z, \vec{z}) = C_\triangle\left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2}\right)^\triangle \ . \tag{318}$$

Then the bulk field that is generated by the boundary source $\bar{J}$ is given by

$$\phi_\triangle(z) = \int_{\partial H} d^d\vec{x}K_\triangle(z, \vec{x})\bar{J}(\vec{x}) \ . \tag{319}$$

It is also possible to extract these results in slightly diffrent way. Let us consider the scalar field that again obeys the equation of motion $(\Box_g - m^2)\phi = 0$ that has the form

$$z_0^5\partial_{z_0}[\frac{1}{z_0^3}\partial\phi] + z_0^2\partial_i[\delta^{ij}\partial_j\phi] - m^2\phi = 0 \ . \tag{320}$$

Let us presume an ansatz $\phi = e^{i\vec{p}\cdot\vec{x}}Z(pz_0)$ so that $\frac{\partial}{\partial z_0} = \frac{\partial}{\partial m}p$, where $p^2 = \vec{p}\cdot\vec{p}, m = zp$. Then the equation above takes the form

$$m^2\frac{d^2Z}{d^2m} - 3m\frac{dZ}{dm} - m^2Z - m^2R^2Z = 0 \ . \tag{321}$$

There are two independent solutions of the equation above which are

$$Z(m) = u^2I_{\triangle-2}(m)', Z(m) = m^2K_{\triangle-2}(m) \ , \tag{322}$$

where I_ν, K_ν are Bessel functions and where

$$\triangle = 2 + \sqrt{4 + m^2R^2} \ . \tag{323}$$

When we demand the regularity at the interior when $z \to \infty$ we have to select the second solution since $I_{\triangle-2}(m)$ increases exponentially as $m \to \infty$. Then the field has the form

$$\phi(z_0, \vec{z}) = C(pz_0)^2 K_{\triangle-2}(pz_0) e^{i\vec{p}\cdot\vec{z}} \ , \tag{324}$$

where the constant C is determined by imposing the boundary conditions

$$\begin{aligned}\phi(\vec{z}, z_0) = \phi_0(\vec{z}) = e^{i\vec{p}\cdot\vec{x}} = (\epsilon p)^2 K_{\triangle-2}(p\epsilon) e^{i\vec{p}\cdot\vec{x}} C \Rightarrow \\ \Rightarrow C = \frac{1}{(\epsilon p)^2 K_{\triangle-2}(p\epsilon)}\end{aligned} \tag{325}$$

and hence if we define the bulk to boundary propagator as $K_{\vec{p}}(\vec{z}, z_0) = \frac{\phi_p(\vec{z}, z_0)}{\phi_0(\vec{z})}$ we find the form of the boulk-to boundary propagator in the form

$$K_p(\vec{z}, z_0) = \frac{(pz)^2 K_{\triangle-2}(pz)}{(p\epsilon)^2 K_{\triangle-2}(p\epsilon)} \ . \tag{326}$$

As we argued above the boundary valules of string fields act as sources for the gauge-invariant operators in the field theory. Genearlly we denote the bulk fileds as $\phi(\vec{z}, z_0)$ with values $\phi_0(\vec{z})$ at $z_0 = \epsilon$. The true boundary of AdS is at $z = 0$ and $\epsilon \neq 0$ serves as a cutoff that could be eventually removed. When we restrict to the supergravity approximation we choose ϕ_0 arbitrary as the boundary conditions at $z = \epsilon$, while the bulk fields have to obey the equations of motion. Then we evaluate the action on this classical solution with fixed boundary valules. In other words we have

$$W_{gauge}[\phi_0] = \ln \left\langle e^{\int d^4 z \phi_0(x) \mathcal{O}(z)} \right\rangle = extremum I_{SUGRA}[\phi]_{\phi|_{z_0=\epsilon}=\phi_0} \ . \tag{327}$$

In other words the generator of connected Green functions in the gauge theory in the large λ limit is the on-shell supergravity action.

It is important to stress that it is not clear whether the right side of the correspondence above is independent on ϵ. Instead we perform this calculation and then we eventually perform a wave-function renormalization on either $\mathcal{O}$ ir ϕ so that the final answer is independent of the cutoff. Now we explicitly the two point function. Note that the two point function can be calculated from the generator of connected Green functions when write

$$\begin{aligned} \int d^d x \phi_0(x) \mathcal{O}(x) = \\ \int d^d x \int d^d k \int d^d l e^{i\vec{k}\cdot\vec{x}} \phi_k e^{i\vec{l}\cdot\vec{x}} \mathcal{O}(\vec{l}) = \\ = \int d^d k \phi_k \mathcal{O}(-\vec{k}) \ . \end{aligned} \tag{328}$$

On the other hand on the right side of the correspondence we have

$$\begin{aligned} I_{SUGRA} = \frac{1}{2} \int_{z_0=\epsilon} d^d z \sqrt{h} \phi(\vec{z}, z_0) n^\mu \partial_\mu \phi(\vec{z}, z_0) = \\ = \epsilon^{-d} \frac{1}{2} \int_{z_0=\epsilon} d^d z \int d^d k \int d^l \phi_k e^{i\vec{k}\cdot\vec{l}} \partial_{z_0} K_p \phi_l e^{i\vec{l}\cdot\vec{x}} = \\ = \frac{1}{2} \int d^d k \int d^d l \phi_k \delta(p+l) \partial_{z_0} K_p \phi_l \ . \end{aligned} \tag{329}$$

To proceed further note that for small x the Bessel function has following asymptotic behavior

$$K_\alpha \sim \frac{1}{\Gamma(\alpha+1)} \left(\frac{x}{2}\right)^\alpha \tag{330}$$

and hence we can derive corresponding Green functions.

5.4 Wilson Loops

The Wilson loop is defined as

$$W[\mathcal{C}] = \mathrm{Tr}[P \exp(i \oint_{\mathcal{C}} A)] \tag{331}$$

depends on a loop $\mathcal{C}$ that is embedded in four dimensional space. This definition includes the path ordered integral of the gauge connection along the contour. The trace is taken over some representation of the gauge group but we consider the case when the gauge field is in the fundamental representation. From the exceptational value of the Wilson loop operator $< W(\mathcal{C}) >$

we can determine the quark-anti-guark potential. For that reason we consider rectangular loop with sides of length T and L in Euclidean space. If we interpret T as the time direction it is clear that for large T the expectation value behaes as e^{-TE} where E is the lowest possible energy of the quark-anti quark configuration.

For example, consider $U(1)$ gauge theory with the gauge transformation given by

$$\begin{aligned} \phi'(x) &= e^{i\alpha(x)}\phi(x) \ , \\ A'_\mu(x) &= A_\mu(x) + \partial_\mu\alpha(x) \ . \end{aligned} \tag{332}$$

Through these rules it is clear that the non-local operator $\phi(x)\phi^*(y)$ is not gauge invariant in general and hence cannot be considered as an observable. On the other hand it can be shown that following quantity is gauge invariant

$$\phi(x)e^{i\int_P dx^\mu A_\mu}\phi^*(y) \ , \tag{333}$$

where P is an arbitrary path from point x to y. To see this note that the object above transforms under gauge transformation as

$$\begin{aligned} & \phi(x)e^{i\int_P dx^\mu A_\mu}\phi^*(y) \rightarrow \phi(x)e^{i\alpha(x)}e^{i\int_P dx^\mu(A_\mu+\partial_\mu\alpha)}e^{-i\alpha(y)}\phi^*(y) = \\ = & \ \phi(x)e^{i\alpha(x)}e^{i(\alpha(y)-\alpha(x))}e^{i\int_P dx^\mu A_\mu}\phi^*(y) = \\ = & \ \phi(x)e^{i\int_P dx^\mu A_\mu}\phi^*(y) \end{aligned} \tag{334}$$

We see that there is a natural object in the theory that is either Wilson line

$$W_P(x,y) = e^{i\int_P dx^\mu A_\mu} \tag{335}$$

or Wilson loop

$$W_P(x,x) = e^{i\oint dx^\mu A_\mu} \tag{336}$$

The Wilson line represents the coupling of the gauge field to the test charge. Let us consider a charge particle with world-line $y(\lambda)$. Then the corresponding current has the form

$$J^\mu(x) = \int d\lambda \frac{dy^\mu}{d\lambda}\delta(x - y^\mu(\lambda)) \ . \tag{337}$$

For a given closed path, $\frac{dy}{d\lambda}$ can be both positive and negative so that we have both a positive charge and a negative charge when we accept the fact that the sign of the charge depends on the sign of $\frac{dy}{d\lambda}$, where $dy/d\lambda > 0$ for a positive charge. We can use J^μ as the coupling of the gauge field to the point particle as

$$\delta S = \int d^4x A_\mu J^\mu = \int d^4x A_\mu(x) \oint d\lambda \frac{dy^\mu}{d\lambda} \delta(x^\mu - y^\mu(\lambda)) = \oint d\lambda \frac{dy^\mu}{d\lambda} A_\mu(y(\lambda)) . \tag{338}$$

In other words the perturbation of the action corresponds to the exponent of the Wilson loop and hence we can write that the Wilson loop represents a partition function in the presence of a test charge

$$< W_p >= \frac{Z[J]}{Z[0]} . \tag{339}$$

In fact, let us write the Euclidean partition function as

$$Z = \left\langle f | e^{-H\tau} | i \right\rangle , \tag{340}$$

where $|i\rangle , |f\rangle$ are initial and the final states, respectively. Using the complete set of energy eigenstates

$$H |n\rangle = E_n |n\rangle \tag{341}$$

we can write

$$\begin{aligned} Z &= \sum_{n,m} \langle f|n\rangle \langle n| e^{-H\tau} |m\rangle \langle m|i\rangle = \sum_{n,m} \langle f|m\rangle e^{-E_n \tau} \langle m|n\rangle \langle n|i\rangle = \\ &= \sum_m \langle f|m\rangle e^{-E_m\tau} \langle m|i\rangle \end{aligned} \tag{342}$$

that in the limit $\tau \to \infty$ gives

$$Z \xrightarrow{\tau\to\infty} e^{-E_0\tau} . \tag{343}$$

In other words in the limit $\tau \to \infty$ the Euclidean partition function is dominated by the ground state and gives the ground state energy. In case when we can neglect the kinetic energy, it gives the quark-antiquark potential energy

$$\langle W_p \rangle \simeq e^{-V(R)\tau} , \tag{344}$$

where R is separation between quark-antiquark. When the quarks are confined as in case of QCD then the potential grows with the separation R so that $V \simeq \sigma R$, where σ is called the string tension. Then we obtain

$$\langle W_p \rangle \simeq e^{-\sigma R \tau} = e^{-\sigma A} \ , \tag{345}$$

where A is the area of the Wilson loop $A = R\tau$ when we consider rectangular Wilson loop, where the spatial dimension is R while the temporal size has length τ. This behavior is known as area low. In case of unconfined potential we have different situation, as for example in case of the Coulomb potential where the potential decays with the separation. Then in case when $\tau = R \gg 1$ we find

$$\langle W_p \rangle \simeq e^{-\mathcal{O}(R)} \ . \tag{346}$$

In this way the Wilson loop provides a criterion for the confinement. Now we are ready to proceed to the discussion of the Wilson loop in AdS/CFT context. The matter fields in the $\mathcal{N} = 4$ SYM are all in the adjoint representation. On the other hand quarks correspond to the matter fields in the fundamental representation. We know that the adjoin representation on the stack of N D3-branes has the origin in the open strings ending on different D-branes in the stack. In order to describe probe in the fundamental representation we consider an infinitely long string. In this case the string can end on N different branes which means that the string transforms in the fundamental representation of $SU(N)$ gauge theory. Such a long string represents a quark. Such a string has an extension and tension so the string has a large mass. In other words long string represents a heavy quark. This is the natural object for the discussion of the Wilson loop in AdS/CFT. These open strings couple to the Wilson loop operator. Then, following the case of the expectation values of the local operators in QFT and the partition functions for the supergravity modes on $AdS_5 \times S^5$ we see that it is natural to propose that the vacuum expectation value of the Wilson loop in the conformal field theory is given by the partition function of the fundamental string on $AdS_5 \times S^5$ with the condition that we have a string world-sheet that ends on the loop $\mathcal{C}$. In the supergravity regime, when $g_s N$ is large, the leading contribution to the partition function will be given by the area of the string world-sheet. This area is measured by AdS metric.

However there is a subtlety in the calculation of this are. It turns out that the area that is calculated is divergent which is a consequence of the fact that the string is going all the way to the boundary of AdS. Then if we evaluate the area up to some radial distance $U = r$ we find that for large r it diverges as $r|\mathcal{C}|$, where $|\mathcal{C}|$ is the length of the loop in the field theory. On the other hand the perturbative computation in the field theory shows that $< W >$ is finite. Then the divergence in the Wilson loop would imply that there is a mass renormalization of BPS particle which cannot be certainly true. This apparent conflict between the divergence of the area of the minimum surface in AdS and the finiteness of the field theory computation can be explained by the fact that the appropriate action for the string world-sheet is not the area itself but its Legendre transformation with respect to the radial coordinate u. The reason for this procedure is the fact that these string coordinates obey the Neumann boundary conditions rather than the Dirichlet ones that were appropriate for the original one. The Legendre transformation subtracts the divergent term $r|\mathcal{C}|$ leaving the resulting action finite.

Using the supergravity approximation we can compute the quark-antiquark potential in the supergravity approximation. We consider a quark at $x = -L/2$ and anti-quark at $x = L/2$. We consider Nambu-Gotto action for fundamental string

$$S = \frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-\mathbf{a}_{\alpha\beta}} \ , \mathbf{a}_{\alpha\beta} = G_{MN}\partial_\alpha X^M \partial_\beta X^N \ , \tag{347}$$

where g_{MN} is $AdS_5 \times S^5$ metric. The equation of motion that follows from this action has the form

$$\frac{1}{2}\partial_M g_{KL}\partial_\alpha X^K \partial_\beta X^L (\mathbf{a}^{-1})^{\beta\alpha}\sqrt{-\det \mathbf{a}} - \partial_\alpha[g_{MN}\partial_\beta x^N (\mathbf{a}^{-1})^{\beta\alpha}\sqrt{-\det \mathbf{a}}] \ , = 0 \tag{348}$$

where $\mathbf{a}_{\alpha\beta} = G_{MN}\partial_\alpha X^M \partial_\beta X^N$ and where we used the fact that

$$\sqrt{\det \mathbf{a}} = \exp \mathrm{Tr} \ln \mathbf{a} \ . \tag{349}$$

Let us now consider the metric in the form

$$ds^2_{AdS} = \frac{r^2}{R^2}(-dt^2 + d\mathbf{x}^2) + \frac{R^2}{r^2}dr^2 \tag{350}$$

Now we impose the static gauge in the form

$$\tau = t \ , \sigma = r \ , x = x(\sigma) \tag{351}$$

Of course, strictly speaking there is a subtlety with this gauge since the string has the turning point at $r = r_{min}$ which means that our gauge is not well defined. However this is not problem since we can consider only one half of the string due to the symmetry. Of course, we could also impose the gauge $t = \tau\ , x = \sigma, r = r(x)$, but this would lead to slightly more complicated result and also this gauge fixing cannot be used for the case of the string stretched from the boundary to the bulk of AdS. With the help of this gauge we obtain

$$\mathbf{a}_{\tau\tau} = -\frac{r^2}{R^2}\ , \mathbf{a}_{\sigma\sigma} = \frac{R^2}{r^2} + \frac{r^2}{R^2} x'^2\ , \mathbf{a}_{\tau\sigma} = 0\ , \tag{352}$$

where $x' \equiv \partial_\sigma x$. Note that with this ansatz the equation of motion for t are obeyed while the equation of motion for x gives

$$\begin{aligned} \partial_\alpha[g_{xx}\partial_\sigma X(\mathbf{a}^{-1})^{\sigma\alpha}\sqrt{-\det \mathbf{a}}] = \\ \partial_\sigma[g_{xx}x'(\mathbf{a}^{-1})^{\sigma\sigma}\sqrt{-\det \mathbf{a}}] = 0 \Rightarrow \\ \frac{r^4}{R^4}\frac{x'}{\sqrt{1+\frac{r^4}{R^4}x'^2}} = K \end{aligned} \tag{353}$$

The question is how to determine the constant K. Recall that the boundary is at $r = \infty$ and that the string has the turning point at $r = r_{min}$. At this turning point, by definition

$$\partial_\sigma x|_{\sigma = r_{min}} = \infty \tag{354}$$

so that we obtain

$$\lim_{r\rightarrow r_{min}} \frac{r^4}{R^4}\frac{x'}{\sqrt{1+\frac{r^4}{R^4}x'^2}} = \frac{r_{min}^2}{R^2} = K \tag{355}$$

Now we can solve the equation (357) for x and we obtain

$$x'^2 = \frac{R^4}{r^4}\frac{1}{\frac{r^4}{r_{min}^4} - 1}\ . \tag{356}$$

In order to find the string configuration we can solve the equation above so that we obtain (when we take the boundary condition $x = 0, \sigma = r_{min}$ so that the dependence of x on $\sigma = r$ is given by the integral

$$\int_0^x dx = \int_{r_{min}}^r \left(\frac{R}{r'}\right)^2 \frac{dr'}{\sqrt{\left(\frac{r'}{r_{min}}\right)^4 - 1}} \ . \tag{357}$$

For $r \to \infty$ we demand the quark to be at $x = L$ and hence we obtain

$$\begin{aligned} L &= \frac{R^2}{r_{min}} \int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} = \\ &= \frac{R^2}{r_{min}^2} \frac{\sqrt{2}\pi^{3/2}}{\Gamma(\frac{1}{4})^2} \ . \end{aligned} \tag{358}$$

Note that this equation gives

$$r_{min} \simeq \frac{R^2}{L} \ . \tag{359}$$

Further, for $r \gg r_{min}$ we find from (357)

$$\begin{aligned} x = \frac{R^2}{r_{min}} \int_1^{r/r_{min}} \frac{dy}{y^2\sqrt{y^4-1}} = \\ = \frac{R^2}{r_{min}} \int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} - \frac{R^2}{r_{min}} \int_{r/r_{min}}^\infty \frac{dy}{y^2\sqrt{y^4-1}} = \\ = L - \frac{R^2}{r_{min}} \int_{r/r_{min}}^\infty \frac{dy}{y^4} \Rightarrow \\ x - L \simeq \frac{R^2}{r_{min}} y^{-3}|_{r/r_{min}}^\infty \simeq -r^{-3} \end{aligned} \tag{360}$$

using

$$\int_1^\infty = \int_1^{r/r_{min}} + \int_{r/r_{min}}^\infty \tag{361}$$

and we used the fact that in the second integral $r/r_{min} \gg 1$ and hence $y \gg 1$ in the whole integration interval. We see that for large r the string quickly approaches the point $x = L$.

Now we evaluate the action for this string configuration in order to evaluate the quark potential. We obtain

$$S = -\frac{1}{2\pi\alpha'} \int d\tau \int_{r_{min}}^{\infty} \frac{dr}{r^2} \sqrt{1 + \frac{R^4}{r^4} x'^2} = \\ = -\frac{1}{2\pi\alpha'} \int d\tau \int_{r_{min}}^{\infty} \frac{\left(\frac{r}{r_{min}}\right)^2}{\sqrt{\left(\frac{r}{r_{min}}\right)^4 - 1}} . \tag{362}$$

Since

$$S = -\int d\tau V \tag{363}$$

and since we have two half of the string we find

$$V = \frac{2}{2\pi\alpha'} r_{min} \int_1^{\infty} \frac{dy y^2}{\sqrt{y^4 - 1}} \tag{364}$$

This potential diverges which has natural physical interpretation since in this case the quark is a probe with infinite heavy mass. On the other hand we would like to separate the finite contribution by substracting this large mass contribution. This string configuration is described by the ansatz $x' = 0$ so that the corresponding string action has the form

$$\begin{aligned} S_{str} &= -\frac{1}{2\pi\alpha'} \int d\tau \int d\sigma \sqrt{-g_{tt}(g_{rr} + g_{xx} x'^2)} = \\ &= -\frac{1}{2\pi\alpha'} \int d\tau \int_0^{\infty} dr = -\frac{1}{2\pi\alpha'} r_m \int d\tau [\int_0^1 dy + \int_1^{\infty} dy] = \\ &= -\frac{1}{2\pi\alpha'} r_m - \frac{1}{2\pi\alpha'} \int_1^{\infty} dy \equiv -\int d\tau V_0 \end{aligned} \tag{365}$$

and hence the finite potential has the form

$$V_{fin} = V - V_0 = \frac{2}{2\pi\alpha'} r_{min} [\int_1^{\infty} dy [\frac{y^2}{\sqrt{y^4 - 1}} - 1] - 1] . \tag{366}$$

We see that this potential is proportional to r_{min} but r_{min} but we know that $r_{min} \simeq \frac{R^2}{L}$ so that we know that $E \simeq \frac{1}{L}$ that has the form of the Coulomb potential. In fact, explicit calculation gives

$$V_{fin} = -\frac{4\pi^2}{\Gamma(\frac{1}{4})^4} \frac{\lambda^{1/2}}{L} \ . \tag{367}$$

Let us outline given procedure:

- It is very simple to add probe to the original system.
- The procedure how the Wilson loop is calculated using AdS/CFT correspondence can be generalized to the more general background as for example asymptotically AdS spacetimes.
- This probe analysis can be extended to the dynamical strings in given background.

It is also very interesting to see how the confining phase can be described using the dual geometry. Recall that confining phase is characterized by linear dependence of the potential. More explicitly, the AdS/CFT correspondence can be extended to the backgrounds that are dual to the field theories at finite temperature. At finite temperature the large N and strong coupling limit of $d = 4, \mathcal{N} = 4$ SYM is dual to the near-horizon geometry of near extremal D3-branes in Type IIB string theory which is given by a Schwarzschild-anti-de Sitter Type IIB supergravity compactification

$$ds^2 = \left[\frac{U^2}{R^2}(-Hdt^2 + d\mathbf{x}^2) + R^2 \frac{1}{H} \frac{dU^2}{U^2} + R^2 d\Omega_5^2)\right] \ , \tag{368}$$

where

$$H = 1 - \frac{U_0^4}{U^4} \ , \quad U_0^4 = \frac{2^7\pi^4}{3} g_{eff}^4 \frac{\mu}{N^2} \ , R^2 = \sqrt{4\pi g N} \ , \quad g_{eff} = g_{YM}^2 N \ . \tag{369}$$

where the parameter μ is interpreted as the free energy density on the near extremal D3-brane and hence

$$\mu \approx \frac{4\pi^2}{45} N^2 T^4 \ . \tag{370}$$

Note that in the field theory limit $\alpha' \to 0$ μ remains finite.

At finite critical temperature $T = T_c$ pure SU(N) theory exhibits a deconfinement phase transition. It turns out that the relevant order parameter is the Wilson-Polyakov loop

$$P(x) = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left(i\int_0^{\frac{1}{T}} A_0(x)dt\right) \tag{371}$$

Below the critical temperature $T < T_c$ we find that $< P >= 0$ which means that QCD confines. Above $T > T_c$ we have $< P >$ is non-zero and takes the value in $\mathbf{Z}_N$ which is the center of $SU(N)$.

When we study the field theory at finite temperature we transform from the Minkowski metric to the Euclidean one. First of all let us consider the analytic continuation to Euclidean signature $t_E = it$ where the metric has the form

$$ds_E^2 = f(r)dt_E^2 + \frac{dr^2}{f(r)} + \dots\ . \tag{372}$$

Now we would like to analyse the region near the horizon $r \simeq r_0$, where by definition $f(r_0) = 0$. Introducing coordinate $\tilde{r}$ as $r = r_0 + \tilde{r}$ we can write

$$f(\tilde{r} + r_0) = f(r_0) + f'(r_0)\tilde{r} \tag{373}$$

so that the metric has the form

$$ds_E^2 = f'(r_0)\tilde{r}dt_E^2 + \frac{1}{f'(r_0)\tilde{r}}d\tilde{r}^2 + \dots \tag{374}$$

Finally we introduce the coordindate ρ defined as

$$\rho = 2\sqrt{\frac{\tilde{r}}{f'_(r_0)}}\ , d\rho = \frac{1}{\sqrt{f'(r_0)\tilde{r}}} \tag{375}$$

so that the relevant part of the line element has the form

$$ds^2 = d\rho^2 + \rho^2 d\left(\frac{f'(r_0)}{2}t_E\right)^2\ . \tag{376}$$

Now we see that the metric has the same form as a plane in polar coordinates if $f'(r_0)t_E/2$ has the period 2π. Then the period of t_E is equal to

$$f'(r_0)\beta/2 = 2\pi \Rightarrow \beta = \frac{4\pi}{|f'(r_0)|}\ . \tag{377}$$

On the other hand the compactified Euclidean time direction means that we have the system at finite temperature which is equal to

$$T = \frac{1}{\beta} = \frac{f'(r_0)}{4\pi} \ . \tag{378}$$

For example, in case of Schwarzschild black hole we have $f(r) = 1 - \frac{r_0}{r}$ so that $f' = \frac{r_0}{r^2} = \frac{1}{r_0}$ and hence

$$T = \frac{1}{4\pi r_0} \ . \tag{379}$$

In our case the relevant part of the metric is

$$ds^2 = \frac{U^2}{R^2}(Hdt_E^2 R^2 \frac{1}{H}\frac{dU^2}{U^2} = |U = \frac{R^2}{Y}| = \frac{R^2}{Y^2}[H(Y)dt_E^2 + \frac{1}{H(Y)}dY^2] \tag{380}$$

where the horizon is localized at $Y_0 = \frac{R^2}{U_0}$ and the function $H(Y)$ is

$$H(Y) = 1 - \frac{U_0^4}{R^8}Y^4 \ , H'(Y) = -4\frac{U_0^4}{R^8}U^3 \ , |H'(Y_0)| = 4\frac{U_0}{R^2} \tag{381}$$

and hence

$$T = \frac{H'(Y_0)}{4\pi} = \frac{U_0}{\pi R^2} \ . \tag{382}$$

Let us now consider string as a probe in given background

$$S = -\frac{1}{2\pi\alpha'}\int d\tau\sqrt{\det h} \ , h = g_{MN}\partial_\alpha X^M \partial_\beta X^N \ . \tag{383}$$

We again impose the static gauge

$$t_E = \tau \ , U = \sigma \ , x = x(U) \tag{384}$$

so that the equation of motion for x gives

$$\begin{aligned} \partial_u[\frac{\sqrt{g_{tt}}g_{xx}x'}{\sqrt{g_{uu} + g_{xx}x'^2}}] = 0 \Rightarrow \\ x'^2 = \frac{C^2 g_{uu}}{g_{tt}g_{uu} - C^2} = \\ x'^2 = \frac{R^4 C^2}{U^4 H(\frac{U^4}{R^4}H - C^2)} \end{aligned} \tag{385}$$

As in previous case we determine the integration constant C as the point where $x' = \infty$ that gives

$$\sqrt{g_{tt}g_{xx}}|(U_{min}) = C \Rightarrow C^2 = \frac{1}{R^4}(U_{min}^4 - U_0^4) \ . \tag{386}$$

Then we obtain

$$x' = R^2\sqrt{1-\frac{U_0^4}{U_{min}^4}}\frac{1}{\sqrt{(U^4-U_0^4)(\frac{U^4}{U_{min}^4}-1)}} \tag{387}$$

and hence we obtain

$$x = \frac{R^2}{U_{min}}\sqrt{\epsilon}\int_1^{U/U_{min}}\frac{dy}{\sqrt{(y^4-1)(y^4-1+\epsilon)}} \ , \tag{388}$$

where

$$\epsilon = 1 - \frac{U_0^4}{U_{min}^4} \ . \tag{389}$$

We can again find relation between U_{min} and L when as

$$L = \frac{R^2}{U_{min}}\sqrt{\epsilon}\int_1^{\infty}\frac{dy}{\sqrt{(y^4-1)(y^2-1+\epsilon)}} \ . \tag{390}$$

We calculate potential in the same way as in case of zero energy however we have to take into account that now t_E coordinate is compactified with period of β. Further, we have to substract the contributions of two heavy strings which however do not end at $U = 0$ but at the horizon $U = U_0$. In other words, this contribution is equal to

$$S = \frac{1}{2\pi\alpha'}\int d\tau \int_{U_0}^{\infty} dU \tag{391}$$

and hence the finite potential is equal to

$$\begin{aligned} V &= \frac{2}{2\pi\alpha'}\int_{U_{min}}^{\infty} dU\frac{\sqrt{U^4-U_0^4}}{\sqrt{U^4-U_{min}^4}} - \frac{2}{2\pi\alpha'}\int_{U_0}^{U_{min}} dU - \frac{2}{2\pi\alpha'}\int_{U_{min}}^{\infty} dU = \\ &= \frac{2}{2\pi\alpha'}\left\{U_{min}\int_1^{\infty}\left(\frac{\sqrt{y^4-1+\epsilon}}{\sqrt{y^4-1}}-1\right) - U_{min} + U_0\right\} \end{aligned} \tag{392}$$

This is the potential between quark and antiquark at finite temperature that is function of L and T. However in order to find this we should eliminate U_{min} that appears in given expression and that can be determined from the equation above. This can be done only numerically so that it is instructive to analyze the behavior if given potential at different limits. First of all the region $\epsilon \approx 1$ corresponds to $U_0 \ll 1$ which means that $T \ll 1$ and hence we are at the low temperature region. At the same way the case $\epsilon \approx 0$ corresponds to the high temperature region $TL \gg 1$. For small temperature we can determine the dependence of L on r_{min} as in the free zero temperature case $r_{min}^2 \sim \frac{R^2}{L}$. Then we find that the lowest correction to the potential is proportional to $(TL)^4$, where we have to take into account that the theory is in conformal phase so that the temperature has to appear in the combination TL. In principle we can calculate this energy from the result given above however it is instructive to proceed in slightly different way. Let us consider the pure AdS spacetime with the metric

$$ds^2 = \left(\frac{r}{R}\right)^2 (-dt^2 + dx^2 + \dots) \tag{393}$$

where the line element comes with the factor r^2. We also see that we measure the gauge theory time and distance using t and x which differs from the proper time and distance of the AdS space-time.

Let us denote the quark-antiquark separation as $\triangle x = 2L$. The quark-antiquark pair is represented by a string that connects the pair. As we know string has tension which implies that the string wants to minimalize its length so that naively we would say that the state with the lowest energy is the straight string at $r = \infty$, where we recall that the boundary is at $r = \infty$. However this is not true since we have to take into account the fact that the metric comes with the factor r^2. It turns out that state with less eneergy corresponds to the string that goes inside the AdS space-time ($r \neq \infty$) since the line element has the factor r^2 so that the string is shorter at the origin where the line element is $\triangle x r^2$. In fact, we can roughly divide the string that connects two quarks into two parts. The first one where the string extends vertically and the part where the string extend horizontally. In other words we can approximate the string by rectangular string where only the horizontal string contributes to the quark potential since this part of the string changes as we vary the separation of the quarks $2L$, while the vertical part of the string does not change very rapidly and it simply corresponds to the quark mass. As we also showed above the turning point occurs at

$$r_{max} \sim \frac{R^2}{L} \ . \tag{394}$$

Now from the line element we find that the length of the horizontal string is $\frac{r}{R}2L$ so that the string energy is approximately

$$E(r) \sim \left(\frac{r}{R}\right) L \ . \tag{395}$$

It is also important to stress that given energy does not correspond to the gauge theory energy which follows from the fact that the time-like direction comes with the prefactor r^2 too. In fact, the gauge theory time is the coordinate time t since the proper time τ_r defined as $-d\tau^2 = ds^2$ is related to the gauge theory time by

$$\tau_r = \frac{r}{R}t \tag{396}$$

so that we have an inverse relation between the proper energy and the gauge theory energy as

$$E_t = \left(\frac{r}{R}\right) E(r) \ . \tag{397}$$

Then the potential in the gauge theory is given as

$$E_t = \left(\frac{r_m}{R}\right) E(r) = \left(\frac{r_m}{R}\right)^2 L \sim \frac{R^2}{L} \ . \tag{398}$$

We see that in this picture we derived the Coulomb potential $E = \frac{1}{L}$ and not the confining potential $E \sim L$. We see that even if the quark-antiquark are connected by string it still corresponds to the unconfinning potential. We also see that this potential is proportional to R^2, which, according to AdS/CFT relations, means that $R^2 \sim \lambda^{1/2}$ which means that it represents non-perturbative effect. In fact, this potential can be also calculated non-perturbatively from the field theory point of view and it behaves as $\lambda^{1/2}$ at strong coupling.

The previous discussion can be generalized to the more general form of the metric where

$$ds^2 = g_{00}dt^2 + g_{xx}dx^2 + \dots \ . \tag{399}$$

Now the proper time is related to the coordinate time through the relation $\tau = \sqrt{-g_{00}}t$ and hence the gauge energy is related to the proper energy of the string as

$$E_t = \sqrt{-g_{00}}|_{r_{max}} E(r) = \frac{1}{2\pi\alpha'}\sqrt{-g_{00}g_{xx}}|_{r_{max}} L \tag{400}$$

where the metric is again evaluated at r_{min}.

Confining Phase

We would like to see whether confining phase can be described using AdS/CFT correspondence as well. As we know the pure AdS space-time corresponds to the $\mathcal{N} = 4$ SYM which is not QCD. Further, $\mathcal{N} = 4$ SYM is scale invariant and hence the confining phase does not exist even at zero temperature. Then in order to describe theory that has similar properties ad QCD we have to modify AdS geometry in some way.

In fact, there are number of models that deform the AdS space-time with corresponding modification of the dual theory. Let us however consider following model. The AdS spacetime extends from $r = \infty$ to $r = 0$ but let us presume that we cut off the AdS spacetime at $r = r_c$. Let us also presume that confinement occurs at some low energy phase Λ. As we argued above $r-$coordinate has the interpretation as the gauge theory energy scale so that the confinement means that the AdS space-time is modofied deep insider AdS. It is also clear that there is no modification of the potential when the string is far enough from the cutoff $r = r_c$. In this case we find the pure AdS space-time. On the other hand if the separation of the quarks is large enough we find new effect. To see this note that in the AdS space-time the turning point behaves as $r_{max} \sim 1/L$. But in the cutoff AdS space-time the string reaches $r = r_c$ for large enough L. Since this is the end of the space-time string cannot go further and hence the energy of the horizontal string is

$$E_t = \sqrt{-g_{00}}|_{r=r_c} L \sim r_c^2 L \tag{401}$$

that is really the energy of the confining potential.

The same arguments can be used in case of non-zero temperature AdS space-time which is known as the plasma phase. As we argued above the dual geometry is AdS black hole with the horizon at $r = r_0$. Again, if the string is far enough from the black hole the geometry is approximately the AdS space-time with corresponding Coulomb potential. However when the string reaches the horizon the new effect occurs. As we know the line element is

$$ds^2 = -\left(\frac{r}{L}\right)^2 (1 - \left(\frac{r_0}{r}\right)^4) dt^2 + \dots \tag{402}$$

so that $g_{00} = 0$ at horizon $r = r_0$. Then we however find that $E_t = 0$ which means that the horizontal string has no contribution to the energy. In other words there is no force where the quark separation is large enough. This is known as the Debye screening in AdS/CFT. In other words we have following picture. When the separation of the string is small we have standard Coulomb like behavior while for large separations of the quarks we find that these quarks become free due to the screening by the thermal bath.

Despite of this fact we would like to see how we can find simple model that can describe the confinement. Let us consider AdS black hole with line element in the form

$$ds^2 = \left(\frac{r}{R}\right)^2 (-hdt^2 + dx^2 + dy^2 + dz^2) + R^2 \frac{dr^2}{hr^2}, \tag{403}$$

where

$$h = 1 - \left(\frac{r_0}{r}\right)^4 \ . \tag{404}$$

Let us now compactify $z-$ direction so that $0 \leq z < l$ so that we have solution with asymptotic geometry $R^{1,2} \times S^1$. However there is another solution with this asymptotic geometry. To see this we perform double Wick rotation

$$z' = it \ , z = it' \tag{405}$$

of the black hole and we obtain the metric

$$ds_5^2 = \left(\frac{r}{R}\right)^2 (-dt'^2 + dx^2 + dy^2 + hdz'^2) + R^2 \frac{dr^2}{hr^2} \tag{406}$$

that has the same asymptotic structure $R^{1,2} \times S^1$. This geometry is known as AdS soliton. Clearly they are the same as Euclidean geometries but they have different Lorentzian interpretation. The AdS solition is not a black hole since due to the factor h in front of dz'^2 we should say that the space-time ends at $r = r_0$. Then from the analysis performed above we obtain the energy of the quark-antiquark potential in the form

$$E_t = \sqrt{-g_{t't'} g_{xx}} L = \left(\frac{r_0}{R}\right)^2 L \tag{407}$$

that is confining potential.

6 Thermal $\mathcal{N} = 4$ SYM and AdS/CFT Correspondence

In this section we discuss aspects of the AdS/CFT correspondence at finite temperature. As we show in previous lecture the thermal field theory could be dual to the background with thermal properties. In this lecture we will be more precise with this statement.

We start with the Schwarzschild-AdS black hole (SAdS). It is possible to consider either Schwarzschild black hole in AdS with spherical horizon or we can consider AdS black hole with planar horizon known also as a AdS black branes.

The $SAdS_5$ black hole is a solution of the Einstein equation with a negative cosmological spatime with the metric

$$
\begin{aligned}
ds_5^2 &= -\left(\frac{r}{L}\right)^2 h(r)dt^2 + \frac{dr^2}{\left(\frac{r}{L}\right)^2 h(r)} + \left(\frac{r}{L}\right)^2 (dx^2 + dy^2 + dz^2) \ , \\
h(r) &= 1 - \left(\frac{r}{r_0}\right)^4 \ .
\end{aligned}
\tag{408}
$$

The horizon is located at $r = r_0$. The coordinates (x, y, z) represent R^3 coordinates and we see that the horizon extends in $(x, y, z)-$directions. Note that this is different from the case of Schwarzschild black hole where this line element has the form $r^2 d\Omega^2$. Note that the presence of the black hole in the AdS space-time means that now $SAdS_5$ is not the space-time with constant curvature as opposite to the case of AdS_5 with curvature singularity at $r = 0$.

As we know AdS space-time is invariant under scaling

$$
x^\mu \rightarrow a x^\mu \ , r \rightarrow \frac{r}{a} \ . \tag{409}
$$

Now under this scaling the relevant part of the metric element of SAdS scale as

$$
ds_5^2 \rightarrow -\left(\frac{r}{L}\right)^2 \left\{1 - \left(\frac{ar_0}{r}\right)^4\right\} dt^2 \tag{410}
$$

which means that we derive the same SAdS when we scale the horizon as $r_0 \rightarrow \frac{1}{a} r_0$. In other words black holes with different horizon radii are all equivalent. Further, we show previously that the temperature of the black hole is $T \propto r_0$so that we see that we can change the temperature by scaling. In other words all temperatures are equivalent and the physics is the same, of course with exception of zero temperature. This also means that there is no characteristic temperature which could correspond to a phase translation. This is a consequence of the fact that $\mathcal{N} = 4$ SYM is scale invariant and there is no dimensionful quantity except the temperature. Of course, this is valid for the SYM in flat space-time but the situation could be different when we consider theory on S^3.

In other words we see that the black hole metric in AdS is invariant under the scaling

$$x^\mu \to a x^\mu , \quad , r \to \frac{1}{a} r \ , \quad r_0 \to \frac{1}{a} r_0 \ . \tag{411}$$

6.1 Thermodynamic quantities of AdS black hole

Here we determine thermodynamic quantities of the $SAdS_5$ black hole that, in the AdS/CFT correspondence they are interpreted as thermodynamic quantities of dual $\mathcal{N} = 4$ SYM at strong coupling. Recall that we have the relation between SYM and the bulk theory:

$$N^2 = \frac{\pi}{2} \frac{R^3}{G_5} \ , \lambda = \left(\frac{R}{l_s}\right)^4 \ . \tag{412}$$

As we know the temperature of the black hole is given by the relation

$$\begin{aligned} T = \frac{f'(r_0)}{4\pi} = \\ = \frac{2r}{R^2} \frac{1}{4\pi} (1 - (\frac{r_0}{r})^4 + \frac{1}{4\pi R^2} (\frac{r}{R})^2 \frac{r_0^4}{r^5} = \\ = \frac{r_0}{4\pi R^2} \ . \end{aligned} \tag{413}$$

Now since the black hole is infinite extended in all three spatial dimensions its entropy is divergent and hence it is more appropriate to use the entropy density s. Let us introduce infrared cutoff so that we impose upper limit on spatial coordinates $0 \leq x, y, z \leq L_x, L_y, L_z$, so that the gauge theory volume is $V_3 = L_x L_y L_z$. Now it is known from the physics of the black hole that the entropy of the black hole is proportional to the area of the horizon. Note that the spatial section of the black hole space-time in AdS is equal to (at constant r)

$$d\sigma^2 = \left(\frac{r}{R}\right)^2 (dx^2 + dy^2 + dz^2) \tag{414}$$

so that area of the horizon is (evaluated at $r = r_0$)

$$A = \int_{hor} d^3\sigma = \left(\frac{r_0}{R}\right)^2 L_x L_y L_z \tag{415}$$

when we introduced an infrared cutoff. Then the entropy of the black hole is equal to

$$S = \frac{A}{4G_5} = \frac{1}{4G_5}\left(\frac{r_0}{R}\right)^3 V_3 \tag{416}$$

and hence the entropy density is equal to

$$s = \frac{S}{V_3} = \frac{1}{4G_5}\left(\frac{r_0}{R}\right)^3 \tag{417}$$

If we now express r_0 using T and further using $R^3 = \frac{2G_5}{\pi} N^2$ we obtain

$$s = \frac{\pi^2}{2} N^2 T^3 \ . \tag{418}$$

We will discuss the physical interpretation of this result in the dual theory below. Finally the rest of the thermodynamical quantities can be determined using thermodynamic relations. From the first law

$$d\epsilon = T ds \tag{419}$$

we obtain

$$ds = \frac{3}{2}\pi^2 N^2 T^2 dT \ , d\epsilon = \frac{3}{2}\pi^2 N^2 T^3 dT \Rightarrow \epsilon = \frac{3}{8}\pi^2 N^2 T^4 \ . \tag{420}$$

Finally from the Euler relation

$$\epsilon = Ts - P \tag{421}$$

we obtain the pressure P as

$$P = \frac{1}{8}\pi^2 N^2 T^4 \tag{422}$$

that implies an important relation between the pressure and the energy density

$$P = \frac{1}{3}\epsilon \tag{423}$$

It is very interesting to discuss the dependence of the energy on temperature. The dependence $\epsilon \propto T^4$ is manifestation of the *Stefan-Boltzman law* which follows from the fact that $\mathcal{N} = 4$ SYM is scale invariant and there is no dimensionfull quantity except of temperature. Then since the energy density has a quantity of the mass dimension $[\epsilon] = M^{-4}$ it is clear that it has to be proportional to T^4. However clearly this dimensional analysis cannot determine the numerical factor so that using this black hole calculation we derive non-trivial result that should be compared with the calculation performed in case of the free gas.

We should also stress that the fact that the black hole obeys the Stefan-Boltzmann law is rather non-trivial. For example, in case of the five-dimensional Schwarzschild black hole we find $M \propto \frac{1}{T^2}$ and hence the Stefan-Boltzmann law is not reproduced.

Further, the Schwarzschild black hole has a negative heat capacity

$$C = \frac{dM}{dT} \propto -\frac{1}{T^3} \tag{424}$$

and hence there is no stable equilibrium. This is well known fact that Schwarzschild black hole in the flat assymptotic spacetime emits the Hawking radiation which means that the black hole loses its mass due to the radiation. This also leads to the growing of the temperature of the black hole and the black hole does not reach an equilibrium.

However in case of SAdS the situation is very different. Let us be more explicit and write

$$\begin{aligned} &(5-\text{dimensional})\ \text{Schwarzschild}: \quad M \simeq \frac{1}{G_5 T^2}\ , \\ &(5-\text{dimensional})\ \text{Schwarzschild}-\text{AdS}: \quad \epsilon = \frac{(\pi R)^3}{4G_5}T^4\ . \end{aligned} \tag{425}$$

We see that in case of Schwarzschild black hole we have two dimensionful quantities G_5 and T. On the other hand in case of AdS black hole there is another dimensionful quantity which is the radius of AdS R that we can combine with G_5 to give a dimensionless quantity N. Clearly this cannot be done in case of Schwarzschild black hole and hence Stefan-Boltzmann law cannot appear in this case.

N^2 **dependence** The entropy density is proportional to N^2. The fact that the entropy scales with N^2 implies that the theory is in unconfinement phase. More precisely, the entropy counts the number of degrees of freedom. $SU(N)$ theory is gauge theory with $N \times N$ hermitean matrices so that we have N^2 degrees of freedom approximately. In unconfined phase these degrees of freedom contribute to the entropy. In the confined phase only $SU(N)$ gauge singlets contribute to the entropy and consequently the entropy is not proportional to N^2.

Traceless stress energy tensor As we saw the energy density and the pressure satisfies the relation

$$\epsilon = 3P \ . \tag{426}$$

that implies that the energy momentum tensor is traceless. In fact, this is also consequence of the scale invariance of $\mathcal{N} = 4$ SYM.

Free gas result It is instructive to compare these results with the free gas calculation. Let us consider the partition function of free particles. Denote the energy level as ω_i and n_i as the number of particles that occupy the energy level ω_i. The total energy of the microstate is determined by numbers $n_1, n_2, \dots$ and is given by

$$E(n_1, n_2, \dots) = \sum_i n_i \omega_i \ . \tag{427}$$

The partition function is given by

$$Z = \sum e^{-\beta E(n_1, n_2, \dots)} = \sum_{n_1} \sum_{n_2} \dots e^{-\beta(n_1\omega_1 + n_2\omega_2 + \dots)} \ . \tag{428}$$

where $\beta = 1/T$. For bosons n_i takes the value from 0 to ∞ so that

$$Z_B = \left(\sum_{n_1=0}^{\infty} e^{-\beta n_1 \omega_1} \right) \times \dots = \prod_{i=1}^{\infty} \frac{1}{1 - e^{-\beta \omega_i}} \tag{429}$$

using

$$\sum_{n=1}^{\infty} e^{-\beta n \omega} = 1 + e^{-\beta\omega} + (e^{-\beta\omega})^2 + \dots = \frac{1}{1 - e^{-\beta\omega}} \ . \tag{430}$$

Then the thermodynamic quantities are evaluated as

$$
\begin{aligned}
\ln Z_B = -\sum_{i=1}^{\infty} \ln(1 - e^{-\beta\omega_i}) \rightarrow \\
\rightarrow -V \int \frac{d^3 q}{(2\pi)^3} \ln(1 - e^{-\beta\omega}) = \\
-\frac{V}{(2\pi)^3} 4\pi \int_0^{\infty} dq q^2 \ln(1 - e^{-\beta q}) = \\
-\frac{V}{2\pi^2} \frac{1}{\beta^3} \int_0^{\infty} dx x^2 \ln(1 - e^{-x}) = \\
= V \frac{\pi^2}{90\beta^3} \ .
\end{aligned}
\tag{431}
$$

In previous calculation we replaced the sum by integral and used the dispersion relation $\omega = |q|$ that is valid for massless particles. We also used the relation

$$
\int_0^{\infty} dx x^2 \ln(1 - e^{-x}) = -\frac{\pi^2}{45} \tag{432}
$$

Now, with the help of the partition function we determine corresponding thermodynamic quantities

$$
\begin{aligned}
\epsilon_B = -\frac{1}{V} \frac{\partial \ln Z_B}{\partial \beta} == \frac{\pi^2}{30\beta^4} \ , \\
s_B = \frac{1}{V} \ln Z_B + \beta \epsilon_B = \frac{2\pi^2}{45\beta^3} \ .
\end{aligned}
\tag{433}
$$

In case of fermions we hav that n_i takes the value either 0 or 1 so that

$$\begin{aligned}
Z_F = (1+e^{-\beta\omega_1})\times = \prod_{i=1}^{\infty}(1+e^{-\beta\omega_i}) \ , \\
\ln Z_F = \sum_{i=1}^{\infty}\ln(1+e^{-\beta\omega}) \rightarrow \\
\rightarrow V\int \frac{d^3q}{(2\pi)^3}\ln(1+e^{-\beta\omega}) = \\
= \frac{V}{(2\pi)^3}4\pi\int_0^{\infty} dqq^2\ln(1+e^{-\beta q}) = \\
= V\frac{7\pi^2}{720\beta^3} \ ,
\end{aligned} \tag{434}$$

where we used an integral

$$\int_0^{\infty} dxx^2\ln(1+e^{-x}) = \frac{7\pi^4}{360} \ . \tag{435}$$

Using this result we find

$$\ln Z_F = \frac{7}{8}\ln Z_B \tag{436}$$

Using these results we can determine the entropy of the SYM at the free gas approximation. Firstly, using the form of the entropy for the bosonic degree of freedom we find the photon contribution to the entropy density

$$s_{photon} = \frac{2\pi^2}{45}T^3\times 2 \ , \tag{437}$$

where the last factor 2 comes from the polarization of photons. However this is true for the photon gas only but in case of $\mathcal{N}=4$ SYM we have more degress of freedom. In fact, the number of the bosonic degrees of freedom is equal to

$$N_{bos} = (2+6)\times(N^2-1) \ , \tag{438}$$

where the factor 2 corresponds two states of the gauge field and the factor 6 corresponds to the number of the scalar fields. Finally the factor (N^2-1) is the number of degrees of freedom for the field in the adjoint representation of $SU(N)$. In case of fermions we find that their number is the same as the number of bosons due to the supersymmetry:

$$N_F = N_B \ . \tag{439}$$

On the other hand we know that fermions contribute 7/8 of the bosons entropy so that

$$N_{dof} = N_{bos} + \frac{7}{8}N_{fer} = (2+6) \times \frac{15}{8}(N^2-1) = 15(N^2-1) \ . \tag{440}$$

Consequently the entropy density in the free gas is equal to

$$s_{free} = \frac{2\pi^2}{45}N_{dof}T^3 = \frac{2\pi^2}{3}(N^2-1)T^3 \simeq \frac{2\pi^2}{3}N^2T^3 \ . \tag{441}$$

We see that this expression has the same functional form as in the case of the entropy calculated using the black hole but the coeficients are different. In fact, they are related by the formula

$$s_{BH} = \frac{3}{4}s_{free} \ . \tag{442}$$

Now we see that there is famous 3/4 discrepancy that has following origin. The black hole computation corresponds to the strong coupling result that is different from the free gas result. In other words AdS/CFT predicts that the entropy of $\mathcal{N} = 4$ SYM at strong coupling becomes 3/4 of the free gas result. It is very difficult to check this prediction but it was shown that similar behavior can be obtained in lattice simulations.

6.2 The AdS black hole with spherical horizon

The AdS black hole with spherical horizon is given by

$$ds_5^2 = -\left(\frac{r^2}{R^2} + 1 - \frac{r_0^4}{R^2r^2}\right)dt^2 + \frac{dr^2}{\frac{r^2}{R^2} + 1 - \frac{r_0^4}{R^2r^2}} + r^2d\Omega_3^2 \ , \tag{443}$$

and we see that for $r_0 = 0$ the metric reduces to the metric in the global coordinates. The horizon is located at $r = r_+$ where

$$\begin{aligned} \frac{r_+^2}{R^2} + 1 - \frac{r_0^4}{R^2r_+^2} = 0 \Rightarrow \\ r_+^4 + R^2r_+^2 - r_0^4 = 0 \Rightarrow \\ r_+^2 = \frac{1}{2}(\sqrt{R^4 + 4r_0^4} - R^2) \ . \end{aligned} \tag{444}$$

It turns out that spherical black holes with different horizon radii are not equivalent. This has very important consequence for the phase structure of the dual SYM theory where, however we have to presume that it is now defined on the compact space. Such a theory has very rich physics as phase transition.

7 Phase transition in AdS/CFT correspondence

It is well known that when we change parameters of thermodynamic system, as for example temperature, this system could perform a transition to a macroscopically different state that is more stable. This phenomena is known as phase transition. It turns out that in a phase transition a thermodynamic potential such as free energy become non-analytic. More precisely, the nth order phase transition is the transition when the analyticity is broken in the n-th derivative of the thermodynamic potential. Explicitly

- First order phase transitino when F is continuous and F' is disconinuous. Note that F' means the derivative with resepct to any independent variable of thermodynamic potential.

- Second-order phase transition F and F' are continuous but F'' is discontinuous.

Very nice example of such a system is a ferromagnet that has sponaneous magnetization M below te transition temperatur T_c and the magnetization vanishes at $T = T_c$. Macroscopic variable such as M that characterizes two phases is called the order parameter.

We introduce the free energy $F = F(T, M)$ for such a system. However it turns out that it is useful to use magnetic field as the control parameter and use the another thermodynamic potential which is Gibbs energy defined as Legendre transformation of the free energy

$$G(T, H) = F - MH. \tag{445}$$

so that

$$M = -\frac{\partial G}{\partial H} \ . \tag{446}$$

It turns out that in case of ferromagnet M is coninuous at the transition point that is defined as the point $H = 0$. This fact implies that given transition is the second-order phase transition.

After this brief introduction to the process of the phase transition we will study phase transitions in AdS/CFT case. We know that in $\mathcal{N} = 4$ there is no characteristic temperature and hence there is no phase transition in given theory. However it is interesting to consider theories and black holes that have properties of the phase transitions from following reasons:

- There are gauge theories with rich phase structure, as for example QCD. These phenomena can be nicely described using AdS/CFT correspondence.
- This is even more useful in case of the strongly coupled systems where the dual gravity description is very useful tool.

It is clear that it is very useful to study the phase space phenomena in the AdS space and try to find their interpretations in the dual theory. We start with the calculation of the black hole partition function that, in the saddle-point approximation is given as the evaluation of the action on the classical solution. Then we can interpret the exponential of given function as the partition function of the dual gauge theory

$$Z_{gauge} \simeq e^{-S^E_{onshell}} , \quad (447)$$

where $S^E_{onshell}$ is on-shell action that we derive when we insert the solution of the equation of motion into it. However it is important to stress that it is possible to have two solutions corresponding to given boundary conditions. Then we have to take these solutions so that

$$Z_{gauge} \simeq e^{-S^E_{onshell_1}} + e^{-S^E_{onshell_2}} \quad (448)$$

where $S_{onshell_{1,2}}$ are actions evaluated on the solutions $1, 2$ respectively. Clearly dominant contributions give the solution with lower free energy.

Let us consider compactified $\mathcal{N} = 4$ SYM compactified on S^1 with periodicity l. The $\mathcal{N} = 4$ SYM on R^4 is scale invariant and there is no dimensionful parameter except of temperature. Further, as we know, we can always change the temperature by scaling and all temperatures are equivalent. However introducing the scale l has crucial consequence for given theory. It is not possible to change the temperature by a scaling and it is paremeterized by a dimensionless parameter Tl. It turns out that given theory possesses the first order transition at $Tl = 1$.

As we showed previously there are two possible dual geometries that approach $R^{1,2} \times S^1$ asymptotically. The first one is the $SAdS_5$ black hole

$$ds_5^2 = \left(\frac{r}{R}\right)^2 (-hdt^2 + dx^2 + dy^2 + dz^2) + R^2 \frac{dr^2}{hr^2} \ , \\ h = 1 - \left(\frac{r_0}{r}\right)^4 \ , 0 \leq z < l \ . \tag{449}$$

The second solution is AdS soliton

$$ds_5^2 = \left(\frac{r}{R}\right)^2 (-dt'^2 + dx^2 + dy^2 + hdz'^2) + R^2 \frac{dr^2}{hr^2} \ , \\ l = \frac{\pi R^2}{r_0} \ . \tag{450}$$

that, as we know, is derived from the black hole by the double Wick rotation $z' = it \ , z = it'$.

The SAdS black hole describes the plasma phase wheras AdS soliton describes the confining phase which we also saw when we analyzed Wilson lines in these backgrounds. Explicitly, we found that Wilson loop describes linear potential in the background of AdS soliton. Further, AdS soliton does not have entropy due to the fact that it is not black hole.

At high temperature the AdS soliton undergoes a first-order phase transition to the SAdS black hole. This phase transition then describes a confinement/deconfinement transition in the dual gauge theory, where however we should stress that this transition does not correspond to the usual confinement in QCD from following reason. We still work with the background that is dual $\mathcal{N} = 4$ SYM which however is defined on compact space.

To see how the phase transition comes from let us evaluate the free energy of these space-times. We start with the partition function of AdS black hole. The gravitational action has following form

$$S_E = S_{bulk} + S_{GH} + S_{CT} \ , \tag{451}$$

where S_{bulk}, S_{GH} and S_{CT} are bulk action, the Gibbons-Hawking action and the counterterm action respectively. The bulk action is the five dimensional action that has the form

$$S = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}(R-2\Lambda) \ ,$$
$$2\Lambda = -\frac{12}{L^2} \ . \tag{452}$$

The Gibbons-Hawking action is added to the EH action in order to have a well-defined variational problem

$$S = -\frac{2}{16\pi G_5}\int d^4x\sqrt{\gamma}K \ , \tag{453}$$

which is a surface term evaluated at $u=0$. Further,γ is $4-$dimensional metric at the surface surface and K is the trace of the extrinsic curvature.

The surface term does not affect the equation of motion but its presence determines the value of the on-shell action. In fact, in some cases there is no bulk contribution to the cases so that the on-shell action is entirely determined by Gibbons-Hawking action.

Now we discuss these two actions evaluated on the classical solution corresponding o the SAdS. Then we will discuss the meaning of the counterterm action. Let us consider following background

$$ds_5^2 = \left(\frac{r}{L}\right)^2(hdt_E^2+dx^2+dy^2+dz^2)+L^2\frac{dr^2}{hr^2} \ , h = 1-\left(\frac{r_0}{r}\right)^4 \ , \\ = \tag{454}$$

Let us introduce coordinate

$$u = \frac{r_0}{r} \ , h = 1-u^4 \ , d^r = -\frac{r_0}{u^2}du \tag{455}$$

so that the line element takes the form

$$ds_5^2 = \left(\frac{r_0}{L}\right)^2\frac{1}{u^2}(hdt_E^2+dx^2+dy^2+dz^2)+R^2\frac{du^2}{hu^2} \ , h = 1-u^4 \ . \tag{456}$$

In the coordinate r the black hole horizon is located at $r=r_0$ and AdS boundary is located at $r=\infty$. In the coordinate u the horizon is located at $u=1$ and the AdS boundary is located at $u=0$.

Let us now return to the bulk action. The variation of the action with respect to g_{MN} implies following equation of motion

$$R_{MN} - \frac{1}{2}g)_{MN}R + \Lambda g_{MN} = 0 \ . \tag{457}$$

Taking the trace of given equation gives

$$R - \frac{5}{2}R + 5\Lambda = 0 \Rightarrow R = \frac{10}{3}\Lambda \tag{458}$$

that for given Λ gives

$$R = -\frac{20}{L^2} \ . \tag{459}$$

Then on-shell action is

$$S_{bulk} = -\frac{1}{16\pi G_5}\int d^5x\sqrt{g}\frac{4}{3}\Lambda = \frac{1}{2\pi G_5 L^2}\int d^5x\sqrt{g} \ . \tag{460}$$

For the black hole spacetime we have

$$g = \frac{r^6}{L^6} = \frac{r_0^8}{L^6u^8} \tag{461}$$

and hence we find

$$\begin{aligned} S_{bulk} &= \frac{1}{2\pi G_5 L^2}\int_0^\beta dt\int d^3x\int_0^1 du\frac{r_0^4}{L^3u^4} = \\ &= \lim_{u\to 0}\frac{\beta V_3 r_0^4}{8\pi G L^5}\left(\frac{1}{u^4}-1\right)|_{u=0} \equiv \frac{\beta V_3}{16\pi G_5}\frac{r_0^4}{L^4}\hat{S}_{bulk} \ , \end{aligned} \tag{462}$$

where V_3 is the volume of the spatial section. The bulk action diverges as $u \to 0$ since it is proportional to the space-time volume.

Let us now perform the calculation in case of Gibbons-Hawking action. Let us write the line element for the SAdS black hole in the form

$$ds_5^2 = g_{uu}du^2 + \gamma_{\mu\nu}dx^\mu dx^\nu \ . \tag{463}$$

We see that the 4−dimensional metric defined on the surface $u = \text{const}$ is given by $\gamma_{\mu\nu}$. Clearly the normal vector to this surface has the only one non-zero component n^u. If we require that its norm is one we find

$$n^u n^u g_{uu} = 1 \Rightarrow n^\mu = -\frac{1}{\sqrt{g_{uu}}} \ , \tag{464}$$

where we chosen the sign $-$ in front of the square root since the normal n^μ points in the direction od decreasing u. For diagonal metric the extrinsic curvature is given by

$$K_{\mu\nu} = \frac{1}{2} n^u \partial_u \gamma_{\mu\nu} \ . \tag{465}$$

Its trace is defined as

$$K = \gamma^{\mu\nu} K_{\mu\nu} = \frac{1}{2} n^u \partial_u \gamma_{\mu\nu} \gamma^{\mu\nu} \ . \tag{466}$$

Since $\det \gamma = \exp \operatorname{Tr} \ln \gamma$ we obtain

$$\partial_u \sqrt{g} = \frac{1}{2} \partial_u \gamma_{\mu\nu} \gamma^{\nu\mu} \sqrt{g} \tag{467}$$

Then comparing there two contributions we obtain

$$K = n^\mu \frac{1}{\sqrt{\gamma}} \partial_u \sqrt{\gamma} \ . \tag{468}$$

For AdS black hole we have

$$g_{uu} = \frac{L}{\sqrt{h} u} \ , \gamma = \left(\frac{r_0}{L}\right)^4 \frac{1}{u^4} \sqrt{h} \tag{469}$$

and hence we obtain

$$\lim_{u \to 0} \sqrt{\gamma} K = \frac{8}{u^4} - 4 \tag{470}$$

and hence we find that S_{GH} is equal to

$$S_{GH} = \frac{1}{16\pi G} \frac{\beta r_0^4 V_3}{L^4} [-\frac{8}{u^4} + 4]|_{u=0} \ . \tag{471}$$

We see that the bulk action together with GH action diverges in the limit $u \to 0$. For that reason we add another surface term to the action known as counterterm action.

In order to explain this notation note that in AdS/CFT correspondence this divergence is interpreted as the ultraviolet divergence of the dual field theory. The standard procedure how to deal with divergences in QFT is to add a finite number of local counterterms to we add these terms to the gravitational action in order to have finite partition function. This procedure is known as the holographic renormalization. Explicitly, we choose the counterterm action in the following way

- It is written in terms of boundary metric $\gamma_{\mu\nu}$ and the quantities calculated from it, as for example Ricci scalar $\mathcal{R}$.
- It consists only a finite number of terms.
- The coeficients of these terms are chosen n order to cancel divergences.

In case of $5-$ dimensional space-time the counterterm action is given b

$$S_{CT} = \frac{1}{16\pi G_5}\int d^4x\sqrt{\gamma}\left(\frac{6}{L}+\frac{L}{2}\mathcal{R}\right) \tag{472}$$

so that for the black hole solution we find

$$S_{CT} = \frac{1}{16\pi G_5}\frac{\beta V_3}{L^4}\frac{6\sqrt{h}}{u^4}|_{u=0} \rightarrow \frac{1}{16\pi G_5}\frac{\beta V_3}{L^4}\left(\frac{6}{u^4}-3\right) \ . \tag{473}$$

In summary, we have following result

$$\begin{aligned} S_{bulk} &= \frac{\beta V_3}{16\pi G_5}\frac{r_0^4}{L^4}\left(\frac{2}{u^4}-2\right)|_{u=0} \ , \\ S_{GH} &= \frac{\beta V_3}{16\pi G_5}\frac{r_0^4}{L^4}\left(-\frac{8}{u^4}+3\right)|_{u=0} \ , \\ S_{CF} &= \frac{\beta V_3}{16\pi G_5}\frac{r_0^4}{L^4}\left(\frac{6}{u^4}-3\right)|_{u=0} \ . \end{aligned} \tag{474}$$

In summary we obtain

$$S_E(on-shell) = S_{bulk}+S_{GH}+S_{CT} = -\frac{\beta V_3}{16\pi G_5}\frac{r_0^4}{L^5} \ . \tag{475}$$

The on-shell partition function is related to the partition function Z and the free energy F as

$$Z = e^{-S_E(on-shell)} \ , F = -\frac{1}{\beta}\ln Z == \frac{1}{\beta}S_E(on-shell) \ . \tag{476}$$

so that the free energy of the black hole has the form

$$F_{BH} = -\frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5} = -\frac{(\pi LT)^4}{16\pi G_5 L}V_3 = -\frac{1}{8}\pi^2 N_c^2 T^4 V_3 \ , \tag{477}$$

using $T = \frac{r_0}{\pi L^2}$. In case of the AdS soliton we can proceed in the same way. In fact, the AdS soliton has the same Euclidean geometry so the free energy for the AdS soliton takes the same form with the important difference that now r_0 is not related to the temperature T but it is related to the S^1 periodicity l

$$F_{soliton} = -\frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5} = -\frac{V_3 L^3}{16\pi G_5}\frac{\pi^4}{l^4} \ . \tag{478}$$

using the relation between periodicity l and r_0

$$l = \frac{\pi L^2}{r_0} \ . \tag{479}$$

Then the free energy difference is equal to

$$\triangle F = F_{BH} - F_{soliton} = -\frac{V_3 L^3}{16\pi G_5}\pi^4(T^4 - \frac{1}{l^4}) \tag{480}$$

and hence we have

- At low temperature $T < 1/l$ where $F_{soliton} < F_{BH}$ the stable solution is AdS soliton that describes confining phase.
- At high temperature $T > 1/l$ we have $F_{BH} < F_{soliton}$ and the stable solution is the black hole that describes unconfining phase.

We also see one important point. Black hole has entropy while the AdS soliton doesn't we see that entropy is discontinuous at the point $Tl = 1$. Since $S = -\partial_T F$ is discontinous we find that this is first-order phaseIntransition.

The phase transition that was described above is generally called the *Hawking-Page transition.* In fact, original Hawking-Page transition uses SAdS with spherical horizon where the line element has the form

$$ds_5^2 = -\left(\frac{r^2}{L^2} + 1 - \frac{r_0^4}{L^2 r^2}\right)dt^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1 - \frac{r_0^4}{L^2 r^2}} + r^2 d\Omega_3^2 \ . \tag{481}$$

In the limit $r \to \infty$ we find that the line element has the form

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + L^2 d\Omega_3^2) \tag{482}$$

so that the dual gauge theory is $\mathcal{N} = 4$ SYM on S^3 with the radius L.

Since there is a characteristic scale L we see that the theory is parameterized by a dimensionless parameter TL and the theory possesses the first order transition. The horizon is located at $r = r_+$ that is solution of the equation

$$\frac{r_+^2}{L^2} + 1 - \frac{r_0^4}{L^2 r_+^2} = 0 \rightarrow r_0^4 = r_+^4 + L^2 r_+^2 \ . \tag{483}$$

and the temperature is given by

$$T = \frac{2r_+^2 + L^2}{2\pi r_+ L^2} \ . \tag{484}$$

For given value of T we have two values of r_+.

8 Advanced Topics: Gravity And Entaglement

Let us outline basic idea of AdS/CFT correspondence. The basic idea is that certain non-gravitational system on fixed space-time is equivalent to quantum gravitational theories whose states correspond to different spacetimes with given asymptotic behavior. In other words, each sat in the non-gravitational theory corresponds to the state in the dual gravitational theory and each observable in non-gravitational theory corresponds to some observable in the gravitatonal theory. Of course, the interpretation of these states and observables can be completely different on both sides of the correspondence. However there is one quantity that has the same interpretation wich is the energy: Energy of a CFT state corresponds to the total energy of the spacetime which is measured by the ADM mass. As we know from the previous parts of this lecture such a classical example of the non-gravitational system is a conformal field theory defined on some fixed space-time $\mathcal{B}$ where $\mathcal{B}$ is either Minkowski space $R^{d-1,1}$ or a sphere with a time direction $S^{d-1} \times R$.

If we claim that CFT is holographic theory then there is a correspondence between various quantum states of the CFT and states in the dual theory, where these various states can describe different spacetime geometries. But what is important is that the assymptotic behavior of these geometries is the same. For example, for CFT on Minkowski space $R^{d-1,1}$ the vacuum state of the theory corresponds to $(d+1)-$dimensional AdS with a Minkowsi space boundary that can be described by the metric

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_\mu dx^\mu) \ , \tag{485}$$

where the boundary is localized at $z = 0$. Then more general excited states of the CFT are dual to different geometries that approach this geometry as $z \to 0$. This requirement implies that these geometries can be described as

$$ds^2 = \frac{{L^2}^2}{z} \ (dx^2 + \Gamma_{\mu\nu}(x,z) dx^\mu dx^\nu) \ , \tag{486}$$

where for small z we have

$$\Gamma_{\mu\nu}(x,z) = \eta_{\mu\nu} + \mathcal{O}(z^d) \ , \tag{487}$$

that implies that $\lim_{z\to 0} \Gamma_{\mu\nu}(x,z) = \eta_{\mu\nu}$ and really the space-time asymptotically approaches AdS space-time. For CFT defined on a more general space-time $\mathcal{B}$ the definition is the same with the exception that $\eta_{\mu\nu}$ in (487) is replaced with metric on $\mathcal{B}$.

Let us now discuss various states of CFT and their relation to the dual geometries. For small perturbations to the vacuum state of CFT the corresponding geometries are represented by small perturbations to AdS and this is well known from the previous parts of this lecture. On the other hand high-energy excited states have corresponding dual space-times with different geometries and even topologies. One such a famous example is the case of the thermal state of the CFT. As we know for the Minkowski-space CFT the geometry dual to a thermal state is the planar AdS black hole. The case when the CFT is defined on the sphere is more interesting and was analyzed in previous part of the lecture.

One can ask the question whether all CFT are holographic. It turns out that we do not have set of necessary and sufficient conditions which tell us whether any particular CFT is holographic. We believe that in order to have gravity that looks like Einstein gravity coupled to matter requires that we have a CFT with a large number of degrees of freedom and strong coupling.

8.1 Entropy and Geometry

Now we return to the problem of the black holes in AdS space-time. As we known AdS/CFT correspondence a Schwarzschild black hole in AdS space-time can be identified with a high-energy therma state of the corresponding

CFT on a sphere. Such a theory has a discrete spectrum of energy eingenvalues $|E_i\rangle$ and the thermal state corresponds to the corresponding canonical ensemble. However we can say more since we know that *any* CFT state is dual to asymptotically AdS space-time with or without a black hole. Further, for *any subsystem* of the CFT, entropy of the subsystem corresponds to a area of a particular surface in the corresponding space-time. In order to understand this relation better it is useful to review description of quantum subsystems.

8.2 Quantum Subsystems and Entaglement

Let us have a quantum system and imagine that it has its subsystem A. Let us denote the complement of this subsystem as $\bar{A}$. Then we can decompose the Hilbert space as

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}} \tag{488}$$

In other words the Hilbert space is given as a tensor product of Hilbert spaces of corresponding subsystems.

Let us now consider a state $|\Psi\rangle \in \mathcal{H}$ of the full system. Then we can ask the question, what is the state of the subsystem A? We could presume that it is possible to find some state $\left|\psi^A\right\rangle \in \mathcal{H}_A$ that carries all information about the subsystem. We could demand that for every operator $\mathcal{O}_A$ that acts on $\mathcal{H}_A$ alone

$$\left\langle\psi^A\right| \mathcal{O}_A \left|\psi^A\right\rangle = \langle\Psi| \mathcal{O}_A \otimes \mathbf{I}_B |\Psi\rangle \ , \tag{489}$$

where $\mathbf{I}_B$ is identity element acting on the subspace $\mathcal{H}_B$. However it turns out that for general $|\Psi\rangle$ there does not exist such a state $\left|\psi^A\right\rangle$. In other words, the state of the subsystem is not described by any single state in the Hilbert space $\mathcal{H}_A$. In fact, previous formula is valid when we can express $|\Psi\rangle$ as $|\Psi\rangle = \left|\psi^A\right\rangle \left|\psi^B\right\rangle$ since then

$$\begin{aligned} &\langle\Psi| \mathcal{O}_A \otimes I |\Psi\rangle = \left\langle\psi^B\right| \left\langle\psi^A\right| (\mathcal{O}_A \otimes \mathbf{I}_B) \left|\psi^A\right\rangle \left|\psi^B\right\rangle = \\ &= \langle\psi|_A \mathcal{O}_A |\psi\rangle_A \langle\psi_B| \mathcal{O}_B |\psi\rangle_B = bra\psi_A \mathcal{O}_A |\psi\rangle_A \ . \end{aligned} \tag{490}$$

8.2.1 Ensembles of quantum states

The right way how to describe given subsystem is to use an idea of an *Ensemble* of states that is known as a *Mixed state* which is opposite to the *pure* *state* which is single Hilbert space vector. Explicitly, we consider a collection $\{(\psi_i, p_i)\}$ of orthogonal states and corresponding probabilities. We define expectation value of an operator $\mathcal{O}$ in the ensemble to be the average of the expectation values for the individual states weighted by the probabilities

$$\langle \mathcal{O} \rangle_{ensemble} = \sum_i p_i \langle \psi_i | \mathcal{O} | \psi_i \rangle \,. \tag{491}$$

For system consisting from more parts with state$|\Psi\rangle$ or more generally an ensamble of states for the full system we can always find ensamble of states for subsystem A such that all expectation values of $\mathcal{O}_A$ are reproduced

$$\langle \Psi | (\mathcal{O}_A \otimes \mathbf{I}_B) | \Psi \rangle = \sum_i p_i \left\langle \psi_i^A \right| \mathcal{O}_A \left| \psi_i^A \right\rangle \tag{492}$$

Now p_i represent classical uncertainty about system in the state $|\psi_i\rangle$. We prove this prescription below.

Given such an ensemble we can define an operator

$$\rho_A \equiv \sum_i p_i \left| \psi_i^A \right\rangle \left\langle \psi_i^A \right| \tag{493}$$

that is known as *Density matrix* for subsystem. The density matrix is an Hermitian operator with unit trace and non-negative eigenvalues p_i. In fact, it is easy to see that $\left|\psi_i^A\right\rangle$ is eigenvector of this operator since

$$\rho_A \left| \psi_i^A \right\rangle = \sum_j p_j \left| \psi_j^A \right\rangle \left\langle \psi_j^A | \psi_i^A \right\rangle = p_i \left| \psi_i^A \right\rangle \tag{494}$$

using the fact that $\left|\psi_i^A\right\rangle$ are orthogonal. In other words density matrix provides a mathematical description of ensamble of states.

In order to compute exceptation value of operator using density of matrix wc calculate the trace this operator

$$\begin{gathered}\langle \mathcal{O}_A \rangle = \sum_j \left\langle \psi_j^A \right| \mathcal{O}_A \rho_A \left| \psi_j^A \right\rangle = \sum_j \left\langle \psi_j^A \right| \mathcal{O}_A \sum_i p_k \left| \psi_i^A \right\rangle \left\langle \psi_i^A | \psi_j^A \right\rangle = \\ \sum_j \left\langle \psi_j^A \right| \mathcal{O}_A \sum_i p_k \left| \psi_i^A \right\rangle \delta_j^i = \sum_i p_i \left\langle \psi_i^A \right| \mathcal{O}_A \left| \psi_i^A \right\rangle\end{gathered} \tag{495}$$

Now we are ready to calculate density matrix for subsystem. Let Ψ is the state of the full system that, when $\{|\psi_n\rangle\}$ is basis of subsystem A and $\{|\psi_N\rangle\}$ is basis for the complement of A that we denote as B. Then the state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \sum_{n,N} c_{N,n} |\psi_n\rangle \otimes |\psi_N\rangle \tag{496}$$

Since $|\Psi\rangle$ is normalized $\langle\Psi|\Psi\rangle = 1$ we obtain

$$\begin{aligned}\langle\Psi|\Psi\rangle = \sum_{N,n}\sum_{M,n} c^*_{N,n}c_{M,m} \langle\psi_n| \otimes \langle\psi_N| |\psi_m\rangle \otimes |\psi_M\rangle = \\ = \sum_{N,n}\sum_{M,n} c^*_{N,n}c_{M,m}\delta^n_m\delta^M_N = \sum_{n,N}|c_{n,N}|^2 = 1\end{aligned} \tag{497}$$

Then the operator

$$\rho \equiv |\Psi\rangle\langle\Psi| \tag{498}$$

represents density matrix for the whole system. Then the expectation value of operator $\mathcal{O}_A \otimes \mathbf{I}_B$ on subsystem we have

$$\begin{aligned}&\mathrm{Tr}((\mathcal{O}_A \otimes \mathbf{I}_B)\rho) = \sum_{k,K} \langle\psi_k| \otimes \langle\psi_K| ((\mathcal{O}_A \otimes \mathbf{I}_B)\rho) |\psi_k\rangle \otimes |\psi_K\rangle = \\ &= \sum_{k,K} \langle\psi_k| \otimes \langle\psi_K| (\mathcal{O}_A \otimes \mathbf{I}_B) \sum_{n,N}\sum_{m,M} c^*_{n,N}c_{m,M} |\psi_n\rangle \otimes |\psi_N\rangle \langle\psi_m| \otimes \langle\psi_M| |\psi_k\rangle \otimes |\psi_K\rangle = \\ &= \sum_{m,M} \langle\psi_m| \otimes \langle\psi_M| (\mathcal{O}_A \otimes \mathbf{I}_B) \sum_{n,N} c^*_{n,N}c_{m,M} |\psi_n\rangle \otimes |\psi_N\rangle = \\ &= \sum_{n,m}\sum_{N} c^*_{n,N}c_{m,N} \langle\psi_n| \mathcal{O}_A |\psi\rangle_m = \sum_k \langle\psi_k| \sum_{n,m}\sum_N \mathcal{O}_A |\psi_n\rangle\langle\psi_m| |\psi_k\rangle = \\ &= \mathrm{Tr}(\mathcal{O}_A\rho_A)\ .\end{aligned} \tag{499}$$

where the reduced density operator is defined as

$$\rho_A = \sum_{n,m}\sum_N c^*_{n,N}c_{m,N} |\psi_m\rangle\langle\psi_n| \tag{500}$$

Note that it can be defined as *partial trace* over the subspace B

$$\mathrm{Tr}_B(\rho) = \sum_K \langle\psi_K| \sum_{n,N}\sum_{m,M} c^*_{n,N} c_{m,M} |\psi_n\rangle \otimes |\psi_N\rangle \langle\psi_m| \otimes \langle\psi_M| |\psi_K\rangle =$$
$$= \sum_{n,N}\sum_{m,N} c^*_{n,N} c_{m,N} |\psi_n\rangle \langle\psi_n| \tag{501}$$

The previous result shows that quantum subsystem can be represented as ensamble. We will say about ensamble below.

8.3 Thermodynamics ensambles and entropy

We firstly define *microcanonical ensamble*. This ensamble is defined as ensamble of states with energy in the small region $[E, E+dE]$. More explicitly, this ensamble is collection of states

$$\left\{ |E_i\rangle \,, i = 1, \ldots, n, p_i = \frac{1}{n} \right\} \tag{502}$$

This ensamble is used when we calculate expected values of quntities when only the overall energy for some closed state is known. Then the ignorance about microstate can be expressed by entropy which is given as

$$S = \ln n \ , (k_B \equiv 1) \tag{503}$$

where n is number of microstates with the total energy E. We can think about each state to give individual contribution to entropy

$$S_{state} = \frac{1}{n} \ln n = -p \ln p \ , p = \frac{1}{n} \ . \tag{504}$$

If we now presume that the individual contribution of the state is the same for more general ensambles with different probability p_i we can define the total entropy as extensive quantity given as a sum of particular entropies

$$S = -\sum_i p_i \ln p_i = -\mathrm{Tr}\rho_A \ln \rho_A \ , \tag{505}$$

which is valid for any ensamble. We can take this formula as general definition of entropy for arbitrary ensamble that describes any quantum subsystem.

As the second ensamble familiar from standard statistical physics is thermal or canonical ensamble. This represents a system A which weakly interacts with the thermal bath which we denote as B. Note that A with B consists closed system. In terms of the energy eigenstates

$$|E_i\rangle \ \ , H_A |E\rangle_i = E_i |E\rangle_i \tag{506}$$

this ensamble is given as collections of states and corresponding probabilities

$$\left\{|E_i\rangle , p_i = e^{-\beta E_i}/Z\right\} \tag{507}$$

where

$$Z = \sum_i e^{-\beta E_i} \ . \tag{508}$$

8.4 Entanglement

There are several possibilities how to define entanglement. One such a possibility is to say that the state of system, which consists from subsystem and complement cannot be write as product state in the sense that

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \tag{509}$$

Another possibility is to say that subsystem A is entangled with the rest of the system if the ensamble that describes this subsystem has probabilities $p_i \neq 1$.

It is also important to say that entanglement is not the same for all systems, in other words, some systems are more entangled than the others. For example, let us consider two spins A, B where each spin is in the $\langle\uparrow|$ or $\langle\downarrow|$. The Hilbert space of this system is

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{510}$$

where $\mathcal{H}_A, \mathcal{H}_B$ are Hilbert spaces of particle A and B, respectively. An orthonormal basis of this systems consists four elements

$$|\uparrow\rangle_A \otimes |\uparrow\rangle_B \ \ , |\uparrow\rangle_B \otimes |\downarrow\rangle_B \ \ , \quad |\downarrow\rangle_A \otimes |\uparrow\rangle_B \ \ , \quad |\downarrow\rangle_A \otimes |\downarrow\rangle_B \ \ . \tag{511}$$

Then the entangled state of two spins has the form

$$A |\uparrow\rangle \otimes \tag{512}$$

Reference

[1] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *"Large N field theories, string theory and gravity,"* Phys. Rept. **323** (2000) 183 [hep-th/9905111].

[2] G. 't Hooft, *"A Planar Diagram Theory for Strong Interactions,"* Nucl. Phys. B **72** (1974) 461.

[3] J. Polchinski, *"Dirichlet Branes and Ramond-Ramond charges,"* Phys. Rev. Lett. **75** (1995) 4724 [hep-th/9510017].

[4] M. J. Duff, R. R. Khuri and J. X. Lu, *"String solitons,"* Phys. Rept. **259** (1995) 213 [hep-th/9412184].

[5] A. M. Polyakov, *"String theory and quark confinement,"* Nucl. Phys. Proc. Suppl. **68** (1998) 1 [hep-th/9711002].

[6] A. M. Polyakov, *"The Wall of the cave,"* Int. J. Mod. Phys. A **14** (1999) 645 [hep-th/9809057].

[7] W. Taylor, *"Lectures on D-branes, gauge theory and M(atrices),"* In *Trieste 1997, High energy physics and cosmology* 192-271 [hep-th/9801182].

[8] E. Witten, *"Anti-de Sitter space and holography,"* Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].