Physics goes mathematical

Newton's *Principia* was impressive, with its revelation of deep mathematical laws underlying natural phenomena. But what happened next was even more impressive. Mathematicians tackled the entire panoply of physics – sound, light, heat, fluid flow, gravitation, electricity, magnetism. In every case, they came up with differential equations that described the physics, often very accurately.

The long-term implications have been remarkable. Many of the most important technological advances, such as radio, television and commercial jet aircraft depend, in numerous ways, on the mathematics of differential equations. The topic is still the subject of intense research activity, with new applications emerging almost daily. It is fair to say that Newton's invention of differential equations, fleshed out by his successors in the 18th and 19th centuries, is in many ways responsible for the society in which we now live. This only goes to show what is lurking just behind the scenes, if you care to look.

CHAPTER 10

Impossible Quantities

Can negative numbers have square roots?

Mathematicians distinguish several different kinds of number, with different properties. What really matters is not the individual numbers, but the system to which they belong – the company they keep.

Four of these number systems are familiar: the *natural numbers*, 1, 2, 3, ...; the *integers*, which also include zero and negative whole numbers; the *rational numbers*, composed of fractions p/q where p and q are integers and q is not zero; and the *real numbers*, generally introduced as decimals that can go on forever – whatever that means – and represent both the rational numbers, as repeating decimals, and irrational numbers like $\sqrt{2}$, e and π whose decimal expansions do not ever repeat the same block of digits.

Integers

The name integer just means whole; the other names give the impression that the systems concerned are sensible, reasonable things – natural, rational and of course real. These names reflect, and encourage, a long-prevailing view that numbers are features of the world around us.

Many people think that the only way you can do mathematical research is to invent new numbers. This view is almost always wrong; a lot of mathematics is not about numbers at all, and in any case the usual aim is to invent new theorems, not new numbers. Occasionally, however, 'new numbers' do arise. And one such invention, a so-called 'impossible' or 'imaginary' number, completely changed the face of mathematics, adding immeasurably to its power. That number was the square root of minus one. To early mathematicians, such a description seemed ridiculous, because the square of any number is always positive. So, negative numbers cannot have square roots.

But just suppose they *did*. What would happen?

It took mathematicians a long time to appreciate that numbers are artificial inventions made by human beings; very effective inventions for capturing many aspects of nature, to be sure, but no more a *part* of nature than one of Euclid's triangles or a formula in calculus. Historically, we first see mathematicians starting to struggle with this philosophical question when they began to realize that imaginary numbers were inevitable, useful and somehow on a par with the more familiar real ones.

Problems with cubics

Revolutionary mathematical ideas are seldom discovered in the simplest and (with hindsight) most obvious context. They almost always emerge from something far more complicated. So it was with the square root of minus one. Nowadays, we normally introduce this number in terms of the quadratic equation $x^2 + 1 = 0$, the solution of which is the square root of minus one – whatever that means. Among the first mathematicians to wonder whether it had a sensible meaning were the Renaissance algebraists, who ran into square roots of negative numbers in a surprisingly indirect way: the solution of cubic equations.

Recall that del Ferro and Tartaglia discovered algebraic solutions to cubic equations, later written up by Cardano in his *Ars Magna*. In modern symbols, the solution of a cubic equation $x^3 + ax = b$ is

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}}$$

The Renaissance mathematicians expressed this solution in words and abbreviations, but the procedure was the same.

Sometimes this formula worked beautifully, but sometimes it ran into trouble. Cardano noticed that when the formula is applied to the equation $x^3 = 15x + 4$, with the obvious solution $x = 4$, the result is expressed as

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

This expression seemed to have no sensible meaning, however, because -121 has no square root. A puzzled Cardano wrote to Tartaglia, asking for clarification, but Tartaglia missed the point and his reply was distinctly unhelpful.

An answer of sorts was provided by Rafael Bombelli in his three-volume book *L'Algebra*, printed in Venice in 1572 and Bologna in 1579. Bombelli was worried that Cardano's *Ars Magna* was rather obscure, and he set out to write something clearer. He operated on the troublesome square root as if it were an ordinary number, noticing that

$$(2 + \sqrt{-1})^3 = 2 + \sqrt{-121}$$

and deducing the curious formula

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}$$

Similarly, Bombelli obtained the formula

$$\sqrt[3]{2 + \sqrt{-121}} = 2 - \sqrt{-1}$$

Now he could rewrite the sum of the two cube roots as

$$(2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4$$

So this strange method yielded the right answer, a perfectly normal integer, but it got there by manipulating 'impossible' quantities.

This was all very interesting, but *why did it work?*

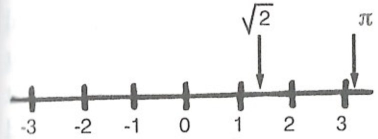
Imaginary numbers

To answer this question, mathematicians had to develop good ways to think about square roots of negative quantities, and do calculations with them. Early writers, among them Descartes and Newton, interpreted these imaginary numbers as a sign that a problem has no solutions. If you wanted to find a number whose square was minus one, the formal solution, square root of minus one, was imaginary, so no solution existed. But Bombelli's calculation implied that there was more to imaginaries than that. They could be used to *find* solutions; they could occur when solutions *did* exist.

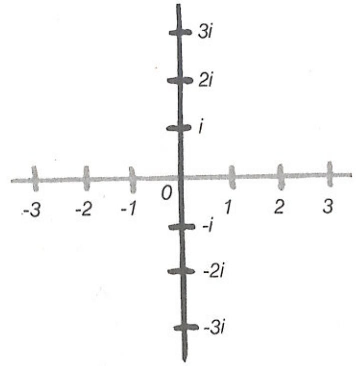
In 1673 John Wallis invented a simple way to represent imaginary numbers as points in a plane. He started from the familiar representation of real numbers as a line, with the positive numbers on the right and the negative ones on the left.

Then he introduced another line, at right angles to the first, and along this new line he placed the imaginaries.

This is like Descartes's algebraic approach to plane geometry, using coordinate axes. Real numbers form one axis in the picture, imaginaries another. Wallis did not state the idea in quite this form – his version was closer to Fermat's approach to coordinates than Descartes's. But the underlying point is the same. The remainder of the plane corresponds to complex numbers, which consist of two parts: one real, one imaginary. In Cartesian coordinates, we measure the real part along the real line and measure the imaginary part parallel to the imaginary line. So 3 + 2i lies 3 units to the right of the origin and 2 units up.

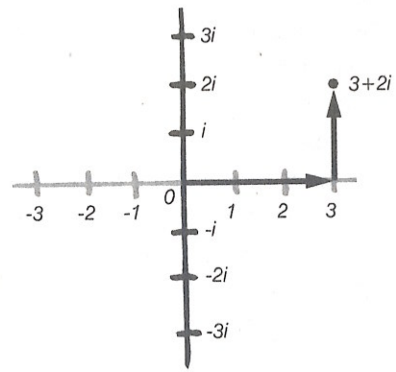


The real number line



Two copies of the real number line, placed at right angles

Wallis's idea solved the problem of giving meaning to imaginary numbers, but no one took the slightest notice. However, his idea slowly gained ground subconsciously. Most mathematicians stopped worrying that the square root of minus one could not occupy any position on the real line, and realized that it could live somewhere in the wider world of the complex plane. Some failed to appreciate the idea: in 1758 François Daviet de Foncenex, in a paper about



The complex plane according to Wessel

imaginary numbers, stated that it was pointless to think of imaginaries as forming a line at right angles to the real line. But others took it to heart and understood its importance.

The idea that a complex plane could extend the comfortable real line and give imaginaries a home was implicit in Wallis's work, but slightly obscured by the way he presented it. It was made explicit by the Norwegian Caspar Wessel in 1797. Wessel was a surveyor, and his main interest was to represent the geometry of the plane in terms of numbers. Working backwards, his ideas could be viewed as a method for representing complex numbers in terms of planar geometry. But he published in Danish, and his work went unnoticed until a century later, when it was translated into French. The French mathematician Jean-Robert Argand independently published the same representation of complex numbers in 1806, and Gauss discovered it independently of them both in 1811.

Complex analysis

If complex numbers had been good only for algebra, they might have remained an intellectual curiosity, of little interest outside pure mathematics. But as interest in calculus grew, and it took on a more rigorous form as analysis, people began to notice that a really interesting fusion of real analysis with complex numbers – *complex analysis* – was not only possible, but desirable. Indeed, for many problems, essential.

This discovery stemmed from early attempts to think about complex *functions*. The simplest functions, such as the square or the cube, depended only on algebraic manipulations, so it was easy to define these functions for complex numbers. To square a complex number, you just multiply it by itself, the same process that you would apply to a real number. Square roots of complex numbers are marginally trickier, but there is a pleasant reward for making the effort: *every* complex number has a square root. Indeed, every non-zero complex number has precisely two square roots, one equal to minus the other. So not only did augmenting the real numbers with

a new number, i, provide −1 with a square root, it provided square roots for everything in the enlarged system of complex numbers.

What about sines, cosines, the exponential function and the logarithm? At this stage, things started to get very interesting, but also very puzzling, especially when it came to logarithms.

Like i itself, logarithms of complex numbers turned up in purely real problems. In 1702 Johann Bernoulli was investigating the process of integration, applied to reciprocals of quadratics. He knew a clever technique to carry out this task whenever the quadratic equation concerned has two real solutions, r and s. Then we can rewrite the expression to be integrated in terms of 'partial fractions'

$$\frac{1}{ax^2 + bx + c} = \frac{A}{x - r} + \frac{B}{x - s}$$

which leads to the integral

$$A \log (x - r) + B \log (x - s)$$

But what if the quadratic has no real roots? How can you integrate the reciprocal of $x^2 + 1$, for instance? Bernoulli realized that once you have defined complex algebra, the partial fraction trick still works, but now r and s are complex numbers. So, for example,

$$\frac{1}{x^2 + 1} = \frac{i/2}{x + i} + \frac{i/2}{x - i}$$

and the integral of this function takes the form

$${}^{i}\!/_{2} \log (x + i) + {}^{i}\!/_{2} \log (x - i)$$

This final step was not fully satisfactory, because it demanded a definition of the logarithm of a complex number. Was it possible to make sense of such a statement?

Bernoulli thought it was, and proceeded to use his new idea to excellent effect. Leibniz also exploited this kind of thinking. But the mathematical details were not straightforward. By 1712 the two of them were arguing about a very basic feature of this approach. Forget *complex* numbers – what was the logarithm of a *negative real* number? Bernoulli thought that the logarithm of a negative real number should be real; Leibniz insisted that it was complex. Bernoulli had a kind of proof for his claim: assuming the usual formalism of calculus, the equation

$$\frac{d(-x)}{-x} = \frac{dx}{x}$$

can be integrated to yield

$$\log(-x) = \log(x)$$

However, Leibniz was unconvinced, and believed that the integration was correct only for positive real x.

This particular controversy was sorted out by Euler in 1749, and Leibniz was right. Bernoulli, said Euler, had forgotten that any integration involves an arbitrary constant. What Bernoulli should have deduced was that

$$\log(-x) = \log(x) + c$$

for some constant c. What was this constant? If logarithms of negative (and complex) numbers are to behave like logarithms of positive real numbers, which is the point of the whole game, then it should be true that

$$\log(-x) = \log(-1 \times x) = \log(-1) + \log x$$

so that $c = \log(-1)$. Euler then embarked on a series of beautiful calculations that produced a more explicit form for c. First, he found a way to manipulate various formulas involving complex numbers, assuming they behaved much like real numbers, and deduced a relation between trigonometric functions and the exponential:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

a formula that had been anticipated in 1714 by Roger Cotes. Putting $\theta = \pi$, Euler obtained the delightful result that

$$e^{i\pi} = -1$$

relating the two fundamental mathematical constants e and π. It is remarkable that any such relation should exist, and even more remarkable that it is so simple. This formula regularly tops league tables for the 'most beautiful formula of all time'.

Taking the logarithm, we immediately deduce that

$$\log(-1) = i\pi$$

revealing the secret of that enigmatic constant c above: it is $i\pi$. As such, it is imaginary, so Leibniz was right and Bernoulli was wrong.

However, there is more, and it opens Pandora's box. If we put $\theta = 2\pi$, then

$$e^{2i\pi} = 1$$

So $\log(1) = 2i\pi$. Then the equation $x = x \times 1$ implies that

$$\log x = \log x + 2i\pi$$

from which we conclude that if n is any integer whatever,

$$\log x = \log x + 2ni\pi$$

At first sight, this makes no sense – it seems to imply that $2ni\pi = 0$ for all n. But there is a way to interpret it that does make sense. Over the complex numbers, the logarithmic function is many-valued. Indeed, unless the complex number z is zero, the function $\log z$ can take infinitely many distinct values. (When $z = 0$, the value $\log 0$ is not defined.)

Mathematicians were used to functions that could take several distinct values, the square root being the most obvious example: here, even a real number possessed two distinct square roots, one

What complex numbers did for them

The real and imaginary parts of a complex function satisfy the Cauchy–Riemann equations, which are closely related to the PDEs for gravitation, electricity, magnetism and some types of fluid flow in the plane. This connection made it possible to solve many equations of mathematical physics – but only for two-dimensional systems.

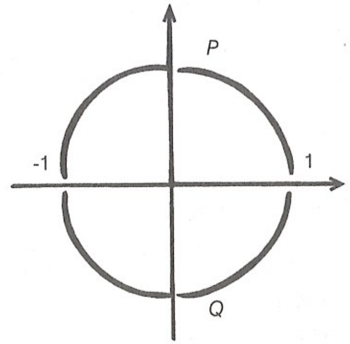
positive and the other negative. But *infinitely many* values? This was very strange.

Cauchy's theorem

What really put the cat among the pigeons was the discovery that you could do calculus – analysis – with complex functions, and that the resulting theory was elegant and *useful*. So useful, in fact, that the logical basis of the idea ceased to be an important issue. When something *works*, and you feel that you need it, you generally stop asking whether it makes sense.

The introduction of complex analysis seems to have been a conscious decision by the mathematical community – a generalization so obvious and compelling that any mathematician with any kind of sensitivity would want to see what transpired. In 1811 Gauss wrote a letter to a friend, the astronomer Friedrich Bessel, revealing his representation of complex numbers as points in a plane; he also mentioned some deeper results. Among them is a basic theorem upon which the whole of complex analysis hangs. Today we call it Cauchy's Theorem, because it was published by Cauchy, but Gauss had the idea much earlier in his unpublished writings.

This theorem concerns definite integrals of complex functions: that is, expressions



Two distinct paths *P* and *Q* from –1 to 1 in the complex plane

$$\int_a^b f(z)\,dz$$

where a and b are complex numbers. In real analysis this expression can be evaluated by finding an antiderivative $F(z)$ of $f(z)$, that is, a function $F(z)$ such that its derivative $dF(z)/dz = f(z)$. Then the definite integral is equal to $F(b) - F(a)$. In particular, its value depends only on the end points a and b, not on how you move from one to the other.

Complex analysis, said Gauss, is different. Now the value of the integral may depend on the *path* that the variable z takes as it moves from a to b. Because the complex numbers form a plane, their geometry is richer than that of the real line, and this is where that extra richness matters.

For example, suppose you integrate $f(z) = 1/z$ from $a = -1$ to $b = 1$. If the path concerned is a semicircle P lying above the real axis, then the integral turns out to be $-\pi i$. But if the path is a semicircle Q lying below the real axis, then the integral turns out to be πi. The two values are different, and the difference is $2\pi i$.

This difference, said Gauss, occurs because the function $1/z$ is badly behaved. It becomes infinite inside the region enclosed by the

Augustin-Louis Cauchy
1789–1857

Augustin-Louis Cauchy was born in Paris during a time of political turmoil. Laplace and Lagrange were family friends, so Cauchy was exposed to higher mathematics at an early age. He went to the École Polytechnique, graduating in 1807. In 1810 he carried out engineering work in Cherbourg, preparing for Napoleon's planned invasion of England, but he continued thinking about mathematics, reading Laplace's *Mécanique Céleste* (*Celestial Mechanics*) and Lagrange's *Théorie des Fonctions* (*Theory of Functions*).

He continually sought academic positions, without success, but kept working on mathematics. His famous paper on complex integrals, which effectively founded complex analysis, appeared in 1814, and he finally achieved his goal of an academic post, becoming assistant professor of analysis at the École Polytechnique a year later. His mathematics now flourished, and a paper on waves won him the 1816 prize of the Academy of Sciences. He continued to develop complex analysis, and in his 1829 *Leçons sur le Calcul Différentiel* he gave the first explicit definition of a complex function.

After the revolution of 1830 Cauchy briefly went to Switzerland, and in 1831 he became professor of theoretical physics in Turin. His courses were reported as being highly disorganized. By 1833 he was in Prague tutoring the grandson of Charles X, but the prince disliked both mathematics and physics, and Cauchy often lost his temper. He returned to Paris in 1838, regaining his post at the Academy but did not regain his teaching positions until Louis Philippe was deposed in 1848. Altogether, he published an astonishing 789 research articles on mathematics.

two paths. Namely, at $z = 0$, which here is the centre of the circle formed by the two paths. 'But if this does not happen … I affirm,'

Gauss wrote to Bessel, 'that the integral has only one value even if taken over different paths provided [the function] does not become infinite in the space enclosed by the two paths. This is a very beautiful theorem, whose proof I shall give on a convenient occasion'. But he never did.

Instead, the theorem was rediscovered, and published by Augustin-Louis Cauchy, the true founder of complex analysis. Gauss may have had the ideas, but ideas are useless if no one gets to see them. Cauchy published his work. In fact, Cauchy seldom stopped publishing. It is said that the rule, still in force today, that the journal *Comptes Rendus de l'Academie Française* accepts papers no more than four pages long, was introduced explicitly to stop Cauchy filling it with his huge output. But when the rule was introduced, Cauchy just wrote lots of short papers. From his prolific pen the main outlines of complex analysis quickly emerged. And it is a simpler, more elegant and in many ways more complete theory than real analysis, where the whole idea started.

For instance, in real analysis a function may be differentiable, but its derivative may not be. It may be differentiable 23 times, but not 24. It may be differentiable as many times as you wish, but not possess a power series representation. None of these nasty things can happen in complex analysis. If a function is differentiable, then it can be differentiated as many times as you wish; moreover, it has a power series representation. The reason – closely related to Cauchy's Theorem and probably a fact used by Gauss in his unknown proof – is that in order to be differentiable, a complex function must satisfy some very stringent conditions, known as the *Cauchy–Riemann equations*. These equations lead directly to Gauss's result that the integral between two points may depend on the path chosen. Equivalently, as Cauchy noted, the integral round a *closed* path need not be zero. It is zero provided the function concerned is differentiable (so in particular is not infinite) at all points inside the path.

There was even a theorem – the residue theorem – that told you the value of an integral round a closed path, and it depended only on the locations of the points at which the function became infinite, and its behaviour near those points. In short, the entire structure of a complex function is determined by its singularities – the points at which it is badly behaved. And the most important singularities are its poles, the places where it becomes infinite.

The square root of minus one puzzled mathematicians for centuries. Although there seemed to be no such number, it kept turning up in calculations. And there were hints that the concept must make some kind of sense, because it could be used to obtain perfectly valid results which did not themselves involve taking the square root of a negative number.

As the successful uses of this impossible quantity continued to grow, mathematicians began to accept it as a useful device. Its status remained uncertain until it was realized that there is a logically consistent extension of the traditional system of real numbers, in which the square root of minus one is a new kind of quantity – but one that obeys all of the standard laws of arithmetic.

Geometrically, the real numbers form a line and the complex numbers form a plane; the real line is one of the two axes of this plane. Algebraically, complex numbers are just pairs of real numbers with particular formulas for adding the pairs or multiplying them.

Now accepted as sensible quantities, complex numbers quickly spread throughout mathematics because they simplified calculations by avoiding the need to consider positive and negative numbers separately. In this respect they can be considered as analogous to the earlier invention of negative numbers, which avoided the need to consider addition and subtraction separately. Today, complex numbers, and the calculus of complex functions, are routinely used as an indispensable technique in virtually all branches of science, engineering and mathematics.

What complex numbers do for us

Today, complex numbers are widely used in physics and engineering. A simple example occurs in the study of oscillations: motions that repeat periodically. Examples include the shaking of a building in an earthquake, vibrations in cars and the transmission of alternating electrical current.

The simplest and most fundamental type of oscillation takes the form $a \cos \omega t$, where t is time, a is the amplitude of the oscillation and ω is its frequency. It turns out to be convenient to rewrite this formula as the real part of the complex function $e^{i\omega t}$. The use of complex numbers simplifies calculations because the exponential function is simpler than the cosine. So engineers studying oscillations prefer to work with complex exponentials, and revert to the real part only at the end of the calculation.

Complex numbers also determine the stabilities of steady states of dynamical systems, and are widely used in control theory. This subject is about methods for stabilizing systems that would otherwise be unstable. An example is the use of computer-controlled moving control surfaces to stabilize the space shuttle in flight. Without this application of complex analysis, the space shuttle would fly like a brick.