CHAPTER 13

The Rise of Symmetry

How not to solve an equation

Around 1850 mathematics underwent one of the most significant changes in its entire history, although this was not apparent at the time. Before 1800, the main objects of mathematical study were relatively concrete: numbers, triangles, spheres. Algebra used formulas to represent manipulations with numbers, but the formulas themselves were viewed as symbolic representations of *processes*, not as things in their own right. But by 1900 formulas and transformations were viewed as *things*, not processes, and the objects of algebra were much more abstract and far more general. In fact, pretty much anything went as far as algebra was concerned. Even the basic laws, such as the commutative law of multiplication, $ab = ba$, had been dispensed with in some important areas.

Group Theory

These changes came about largely because mathematicians discovered group theory, a branch of algebra that emerged from unsuccessful attempts to solve algebraic equations, especially the quintic, or fifth degree, equation. But within 50 years of its discovery, group theory had been recognized as the correct framework for studying the

concept of *symmetry*. As the new methods sank into the collective consciousness, it became clear that symmetry is a deep and central idea, with innumerable applications to the physical sciences, indeed to the biological ones as well. Today, group theory has become an indispensable tool in every area of mathematics and science, and its connections with symmetry are emphasized in most introductory texts. But this point of view took several decades to develop. Around 1900 Henri Poincaré said that group theory was effectively the whole of mathematics reduced to its essentials, which was a bit of an exaggeration, but a defensible one.

The turning-point in the evolution of group theory was the work of a young Frenchman, Évariste Galois. There was a long and complicated pre-history – Galois's ideas did not arise from a vacuum. And there was an equally complicated and often somewhat muddled post-history, as mathematicians experimented with the new concept, and tried to work out what was important and what was not. But it was Galois, more than anyone else, who understood clearly the need for groups, worked out some of their fundamental features, and demonstrated their value in core mathematics. Not entirely surprisingly, his work went almost unnoticed during his lifetime. It was a little too original, perhaps, but it has to be said that Galois's personality, and his fierce involvement in revolutionary politics, did not help. He was a tragic figure living in a time of many personal tragedies, and his life was one of the more dramatic, and perhaps romantic, among those of major mathematicians.

Solving equations

The story of group theory goes right back to the ancient Babylonian work on quadratic equations. As far as the Babylonians were concerned, their method was intended for practical use; it was a computational technique, and they seem not to have asked deeper questions about it. If you knew how to find square roots, and had mastered basic arithmetic, you could solve quadratics.

The symmetries of a quadratic

Consider a quadratic equation, in the slightly simplified form

$$x^2+px+q = 0$$

Suppose that the two solutions are $x = a$ and $x = b$

$$x^2+px+q = (x-a)\ (x-b)$$

Then this tells us that

$$a + b = -p \qquad ab = q$$

So although we don't yet know the solutions, we do know their sum and their product – without doing any serious work.

Why is this? The sum $a + b$ is the same as $b + a$ – it does not change when the solutions are permuted. The same goes for $ab = ba$. It turns out that *every* symmetric function of the solutions can be expressed in terms of the coefficients p and q. Conversely, any expression in p and q is always a symmetric function of a and b. Taking a broad view, the connection between the solutions and the coefficients is determined by a symmetry property.

Asymmetric functions do not behave like this. A good example is the difference $a - b$. When we swap a and b, this becomes $b - a$, which is different. However – the crucial observation – it is not *very* different. It is what we get from $a - b$ by changing its sign. So the square $(a - b)^2$ is fully symmetric. But any fully symmetric function of the solutions must be some expression in the coefficients. Take the square root, and we have expressed $a - b$ in terms of the coefficients, using nothing more esoteric than a square root. We already know $a + b$ – it is equal to $-p$. Since we also know $a - b$, the sum of these two numbers is $2a$ and the difference is $2b$. Dividing by 2, we obtain formulas for a and for b.

What we've done is to prove that there must *exist* a formula for the solutions a and b involving nothing more esoteric than a square root, based on general features of the symmetries of algebraic expressions. This is impressive: we have proved that the problem possesses a solution, without going to the bother of working out all the messy details that tell us what it is. In a sense, we have tracked down *why* the Babylonians were able to find a method. This little story puts the word 'understand' in a new light. You can understand *how* the Babylonian method produces a solution, by working through the steps and checking the logic. But now we have understood why there had to be some such method – not by exhibiting a solution, but by examining *general* properties of the presumed solutions. Here, the key property turned out to be symmetry.

With a bit more work, leading to an explicit expression for $(a - b)^2$, this method yields a formula for the solutions. It is equivalent to the formula that we learn in school, and to the method used by the Babylonians.

There are a few hints, in surviving clay tablets, that the Babylonians also thought about cubic equations, even some quartic equations. The Greeks, and after them the Arabs, discovered geometric methods for solving cubic equations based on conic sections. (We now know that the traditional Euclidean lines and circles cannot solve such problems exactly. Something more sophisticated was necessary; as it happens, conics do the job.) One of the prominent figures here was the Persian, Omar Khayyam. Omar solved all possible types of cubic by systematic geometric methods. But, as we have seen, an algebraic solution of cubic and quartic equations had to wait until the Renaissance, with the work of Del Ferro, Tartaglia, Fior, Cardano and his pupil Ferrari.

The pattern that seemed to be emerging from all this work was straightforward, even though the details were messy. You can solve any cubic using arithmetical operations, plus square roots, plus cube roots. You can solve any quartic using arithmetical operations, plus square roots, plus cube roots, plus fourth roots – though the latter can be reduced to two square roots taken in succession. It seemed plausible that this pattern would continue, so that you could solve any quintic using arithmetical operations plus square roots, cube roots, fourth roots and fifth roots. And so on, for equations of any degree. No doubt the formulas would be very complicated, and finding them would be even more complicated, but few seem to have doubted that they existed.

As the centuries went by, with no sign of any such formulas being found, a few of the great mathematicians decided to take a closer look at the whole area, to work out what was really going on behind the scenes, unify the known methods and simplify them so that it became obvious why they worked. Then, they thought, it would just be a matter of applying the same general principles, and the quintic would give up its secret.

The most successful and most systematic work along these lines was carried out by Lagrange. He reinterpreted the classical formulas in terms of the solutions that were being sought. What mattered, he said, was how certain special algebraic expressions in those solutions behaved when the solutions themselves were permuted – rearranged. He knew that any fully symmetric expression – one that remained exactly the same no matter how the solutions were shuffled – could be expressed in terms of the coefficients of the equation, making it a known quantity. More interesting were expressions that only took on a few different values when the solutions were permuted. These seemed to hold the key to the whole issue of solving the equation.

Lagrange's well-developed sense of mathematical form and beauty told him that this was a major idea. If something similar

could be developed for cubic and quartic equations, then he might discover how to solve the quintic.

Using the same basic idea, he found that partly symmetric functions of the solutions allowed him to reduce a cubic equation to a quadratic. The quadratic introduced a square root, and the reduction process could be sorted out using a cube root. Similarly, any quartic equation could be reduced to a cubic, which he called the resolvent cubic. So you could solve a quartic using square and cube roots to deal with the resolvent cubic and fourth roots to relate the answer to the solutions you wanted. In both cases, the answers were identical to the classical Renaissance formulas. They had to be, really – those were the answers. But now Lagrange knew *why* those were the answers, and better still, he knew why answers existed to be found.

He must have got quite excited at this stage of his research. Moving on to the quintic, and applying the same techniques, you expect to get a resolvent quartic – job done. But, presumably to his disappointment, he didn't get a resolvent quartic. He got a resolvent sextic – an equation of the sixth degree. Instead of making things simpler, his method made the quintic more complicated.

Was this a flaw in the method? Might something even cleverer solve the quintic? Lagrange seems to have thought so. He wrote that he hoped his new viewpoint would be useful to anyone trying to develop a way to solve quintics. It does not seem to have occurred to him that there might not *be* any such method; that his approach failed because in general, quintics do not *have* solutions in 'radicals' – which are expressions involving arithmetical operations and various roots, such as fifth roots. To confuse things, *some* quintics do have such solutions, for instance, $x^5 - 2 = 0$ has the solution $x = \sqrt[5]{2}$. But that is a rather simple case, and not really typical.

All quintic equations *have* solutions, by the way; in general these are complex numbers, and they can be found numerically to any accuracy. The problem was about algebraic formulas for the solutions.

Search for a solution

As Lagrange's ideas started to sink in, there was a growing feeling that perhaps the problem could not be solved. Perhaps the general quintic equation cannot be solved by radicals. Gauss seems to have thought so, privately, but expressed the view that this was not a problem he thought was worth tackling. It is perhaps one of the few instances where his intuition about what is important let him down; another was Fermat's Last Theorem, but here the necessary methods were beyond even Gauss, and took a couple of centuries to emerge. But, ironically, Gauss had already initiated some of the necessary algebra to prove the insolubility of the quintic. He had introduced it in his work on the construction of regular polygons by ruler and compass. And he had also, in this work, set a precedent, by proving (to his own satisfaction, at any rate) that some polygons could not be constructed in that manner. The regular 9-gon was an example. Gauss knew this, but never wrote down a proof; one was supplied a little later by Pierre Wantzel. So Gauss had established a precedent for the proposition that some problems might not be soluble by particular methods.

The first person to attempt a proof of the impossibility was Paolo Ruffini, who became a mathematics professor at the University of Modena in 1789. By pursuing Lagrange's ideas about symmetric functions, Ruffini became convinced that there is no formula, involving nothing more esoteric than *n*th roots, to solve the quintic. In his *General Theory of Equations* of 1799 he claimed a proof that 'The algebraic solution of general equations of degree greater than four is always impossible'. But the proof was so long – 500 pages – that no one was willing to check it, especially since there were rumours of mistakes. In 1803 Ruffini published a new, simplified proof, but it fared no better. During his lifetime, Ruffini never managed to secure the credit for proving that the quintic is insoluble.

Ruffini's most important contribution was the realization that permutations can be combined with each other. Until then, a

permutation was a rearrangement of some collection of symbols. For instance, if we number the roots of a quintic as 12345, then these symbols can be rearranged as 54321, or 42153, or 23154 or whatever. There are120 possible arrangements. Ruffini realized that such a rearrangement could be viewed in another way: as a recipe for rearranging *any other set* of five symbols. The trick was to compare the standard order 12345 with the rearranged order. As a simple example, suppose the rearranged order was 54321. Then the rule for getting from the initial standard order to the new order was simple: reverse the order. But you can reverse the order of *any* sequence of five symbols. If the symbols are *abcde*, the reverse is *edcba*. If the symbols start out as 23451, then their reverse is 15432. This new way of viewing a permutation meant that you could perform two permutations in turn – a kind of multiplication of permutations. The algebra of permutations, multiplied in this way, held the key to the secrets of the quintic.

Abel

We now know that there was a technical error in Ruffini's proof, but the main ideas are sound and the gap can be filled. He did achieve one thing: his book led to a vague but widespread feeling that the quintic is not soluble by radicals. Hardly anyone thought that Ruffini had *proved* this, but mathematicians began to doubt that a solution could exist. Unfortunately the main effect of this belief was to dissuade anyone from working on the problem.

An exception was Abel, a young Norwegian with a precocious talent for mathematics, who thought that he had solved the quintic while still at school. He eventually discovered a mistake, but remained intrigued by the problem, and kept working on it intermittently. In 1823 he found a proof of the impossibility of solving the quintic, and this proof was completely correct. Abel used a similar strategy to Ruffini's but his tactics were better. At first he was unaware of Ruffini's research; later he clearly knew of it, but he stated that it was incomplete. However, he did not refer to any specific problem with Ruffini's proof. Ironically, one step in Abel's proof is exactly what is needed to fill the gap in Ruffini's.

We can get a general idea of Abel's methods without going into too many technicalities. He set up the problem by distinguishing two kinds of algebraic operation. Suppose we start with various quantities – they may be specific numbers or algebraic expressions in various unknowns. From them we can build many other quantities. The easy way to do this is to combine the existing quantities by adding them, subtracting them, multiplying them or dividing them. So from a simple unknown, x, we can create expressions like x^2, $3x + 4$ or $\frac{x+7}{2x-3}$. Algebraically, all of these expressions are on much the same footing as x itself.

The second way to get new quantities from existing ones is to use radicals. Take one of the above mentioned harmless modifications of existing quantities, and extract some root. Call such a step adjoining a radical. If it is a square root, say that the degree of the radical is 2, if a cube root, then the degree is 3, and so on.

In these terms, Cardano's formula for the cubic can be summarized as the result of a two-step procedure. Start with the coefficients of the cubic (and any harmless combination of them). Adjoin a radical of degree 2. Then adjoin a further radical of degree 3. That's it. This description tells us what *kind* of formula arises, but not exactly what it is. Often the key to answering a mathematical riddle is not to focus on fine details, but to look at broad features. Less can mean more. When it works, this trick is spectacular, and here it worked beautifully. It allowed Abel to reduce any hypothetical formula for solving the quintic to its essential steps: extract some sequence of radicals, in some order, with various degrees. It is always possible to arrange for the degrees to be prime – for instance, a sixth root is the cube root of a square root.

Call such a sequence a *radical tower*. An equation is soluble by radicals if at least one of its solutions can be expressed by a radical

tower. But instead of trying to find a radical tower, Abel merely assumed that there was a radical tower, and asked what the original equation must look like.

Without realizing it, Abel now filled the gap in Ruffini's proof. He showed that whenever an equation can be solved by radicals, there must exist a radical tower leading to that solution, involving only the coefficients of the original equation. This is called the Theorem on Natural Irrationalities and it states that nothing can be gained by including a whole pile of new quantities, unrelated to the original coefficients. This ought to be obvious, but Abel realized that it is in many ways the crucial step in the proof.

The key to Abel's impossibility proof is a clever preliminary result. Suppose we take some expression in the solutions x_1, x_2, x_3, x_4, x_5 of the equation, and extract its pth root for some prime number p. Moreover, assume that the original expression is unchanged when we apply two special permutations

$$S: x_1, x_2, x_3, x_4, x_5 \rightarrow x_2, x_3, x_1, x_4, x_5$$

and

$$T: x_1, x_2, x_3, x_4, x_5 \rightarrow x_1, x_2, x_4, x_5, x_3.$$

Then, Abel showed, the pth root of that expression is *also* unchanged when we apply S and T. This preliminary result leads directly to the proof of the impossibility theorem, by 'climbing the tower' step by step. Assume that the quintic can be solved by radicals, so there is a radical tower that starts with the coefficients and climbs all the way to some solution.

The first floor of the tower – the harmless expressions in the coefficients – is unchanged when we apply the permutations S and T, because those permute the solutions, not the coefficients. Therefore, by Abel's preliminary result, the second floor of the tower is *also* unchanged when we apply S and T, because it is reached by adjoining a pth root of something in the ground floor, for some prime p. By the same reasoning, the third floor of the tower is unchanged when we apply S and T. So is the fourth floor, the fifth floor, ..., all the way to the top floor.

However, the top floor contains some solution of the equation. Could it be x_1? If so, then x_1 must be unchanged when we apply S. But S applied to x_1 gives x_2, not x_1, so that's no good. For similar reasons, sometimes using T, the solution defined by the tower cannot be x_2, x_3, x_4 or x_5 either. *All five solutions* are excluded from any such tower – so the hypothetical tower cannot, in fact, contain a solution.

There is no escape from this logical trap. The quintic is unsolvable because any solution (by radicals) must have self-contradictory properties, and therefore cannot exist.

Galois

The pursuit not just of the quintic, but of all algebraic equations, was now taken up by Évariste Galois, one of the most tragic figures in the history of mathematics. Galois set himself the task of determining which equations could be solved by radicals, and which could not. Like several of his predecessors, he realized that the key to the algebraic solution of equations was how the solutions behaved when permuted. The problem was about symmetry.

Ruffini and Abel had realized that an expression in the solutions did not have to be either symmetric or not. It could be partially symmetric: unchanged by some permutations but not by others. Galois noticed that the permutations that fix some expression in the roots do not form any old collection. They have a simple, characteristic feature. If you take any two permutations that fix the expression, and multiply them together, the result *also* fixes the expression. He called such a system of permutations a *group*. Once you've realized that this is true, it is very easy to prove it. The trick is to notice it, and to recognize its significance.

Évariste Galois
1811-1832

Évariste Galois was the son of Nicholas Gabriel Galois and Adelaide Marie Demante. He grew up in revolutionary France, developing distinctly left-wing political views. His great contribution to mathematics went unrecognized until 14 years after his death.

The French revolution had begun with the storming of the Bastille in 1789 and the execution of Louis XVI in 1793. By 1804 Napoleon Bonaparte had proclaimed himself Emperor, but after a series of military defeats he was forced to abdicate, and the monarchy was restored in 1814 under Louis XVIII. By 1824, Louis had died and the king was now Charles X.

In 1827 Galois began to display an unusual talent for—and an obsession with—mathematics. He tried to gain entrance to the prestigious École Polytechnique, but failed the examination. In 1829 his father, then town Mayor, hanged himself when his political enemies invented a phoney scandal. Shortly after, Galois tried once more to enter the École Polytechnique, and failed again. Instead, he went to the École Normale.

In 1830 Galois submitted his researches on the solution of algebraic equations for a prize offered by the Academy of Sciences. The referee, Fourier, promptly died, and the paper was lost. The prize went to Abel (who was by then dead of tuberculosis) and to Carl Jacobi. In the same year Charles X was deposed and fled for his life. The director of the École Normale locked his students in to prevent them joining in. Galois, furious, wrote a sarcastic letter attacking the director for cowardice, and was promptly expelled.

As a compromise, Louis-Phillipe was made king. Galois joined a republican militia, the Artillery of the National Guard, but the new king abolished it. Nineteen of the Guard's officers were arrested and tried for sedition, but the jury threw the charges out, and the Guard held a dinner to celebrate. Galois proposed an ironic toast to the king, holding a knife in his hand. He was arrested, but acquitted because (so he claimed) the toast had been 'To Louis-Phillipe, if he betrays', and not a threat to the king's life. But on Bastille Day Galois was arrested again, for wearing the now illegal uniform of the Guard.

In prison, he heard what had happened to his paper. Poisson had rejected it for being insufficiently clear. Galois tried to kill himself, but the other prisoners stopped him. His hatred of officialdom now became extreme, and he displayed signs of paranoia. But when a cholera epidemic began, the prisoners were released.

At that point Galois fell in love with a woman whose name was for many years a mystery; she turned out to have been Stephanie du Motel, the daughter of the doctor in Galois's lodgings. The affair did not prosper, and Stephanie ended it. One of Galois's revolutionary comrades then challenged him to a duel, apparently over Stephanie. A plausible theory, advanced by Tony Rothman, is that the opponent was Ernest Duchâtelet, who had been imprisoned along with Galois. The duel seems to have been a form of Russian roulette, involving a random choice from two pistols, only one being loaded, and fired at point blank range. Galois chose the wrong pistol, was shot in the stomach, and died the next day.

The night before the duel he wrote a long summary of his mathematical ideas, including a description of his proof that all equations of degree 5 or higher cannot be solved by radicals. In this work he developed the concept of a group of permutations, and took the first important steps towards group theory. His manuscript was nearly lost, but it made its way to Joseph Liouville, a member of the Academy. In 1843 Liouville addressed the Academy, saying that in Galois's papers he had found a solution 'as correct as it is deep of this lovely problem: given an irreducible equation of prime degree, decide whether or not it is soluble by radicals'. Liouville published Galois's papers in 1846, finally making them accessible to the mathematical community.

The upshot of Galois's ideas is that the quintic cannot be solved by radicals because it has *the wrong kind of symmetries*. The group of a general quintic equation consists of all permutations of the five solutions. The algebraic structure of this group is inconsistent with a solution by radicals.

Galois worked in several other areas of mathematics, making equally profound discoveries. In particular he generalized modular arithmetic to classify what we now call *Galois fields*. These are finite systems in which the arithmetical operations of addition, subtraction, multiplication and division can be defined, and all the usual laws apply. The size of a Galois field is always a power of a prime, and there is exactly one such field for each prime power.

Jordan

The concept of a group first emerged in a clear form in the work of Galois, though with earlier hints in Ruffini's epic writings and the elegant researches of Lagrange. Within a decade of Galois's ideas becoming widely available, thanks to Liouville, mathematics was in possession of a well-developed *theory* of groups. The main architect of this theory was Camille Jordan, whose 667-page work *Traité de Substitutions et des Équations Algébriques* was published in 1870. Jordan developed the entire subject in a systematic and comprehensive way.

Jordan's involvement with group theory began in 1867, when he exhibited the deep link with geometry in a very explicit manner, by classifying the basic types of motion of a rigid body in Euclidean space. More importantly, he made a very good attempt to classify how these motions could be combined into groups. His main motivation was the crystallographic research of Auguste Bravais, who initiated the mathematical study of crystal symmetries, especially the underlying atomic lattice. Jordan's papers generalized the work of Bravais. He announced his classification in 1867, and published details in 1868–9.

What group theory did for them

One of the first serious applications of group theory to science was the classification of all possible crystal structures. The atoms in a crystal form a regular three-dimensional lattice, and the main mathematical point is to list all possible symmetry groups of such lattices, because these effectively form the symmetries of the crystal.

In 1891 Evgraf Fedorov and Arthur Schönflies proved that there are exactly 230 distinct crystallographic space groups. William Barlow obtained a similar but incomplete list.

Modern techniques for finding the structure of biological molecules, such as proteins, rely on passing X-rays through a crystal formed by that molecule and observing the resulting diffraction patterns. The symmetries of the crystal are important in deducing the shape of the molecule concerned. So is Fourier analysis.

Technically, Jordan dealt only with *closed* groups, in which the limit of any sequence of motions in the group is also a motion in the same group. These include all finite groups, for trivial reasons, and also groups like all rotations of a circle about its centre. A typical example of a non-closed group, not considered by Jordan, might be all rotations of a circle about its centre through rational multiples of 360°. This group exists, but does not satisfy the limit property (because, for example, it fails to include rotation by $360 \times \sqrt{2}$ degrees, since $\sqrt{2}$ is not rational). The non-closed groups of motions are enormously varied and almost certainly beyond any sensible classification. The closed ones are tractable, but difficult.

The main rigid motions in the plane are translations, rotations, reflections and glide reflections. In three-dimensional space, we also encounter *screw* motions, like the movement of a corkscrew: the

object translates along a fixed axis and simultaneously rotates about the same axis.

Jordan began with groups of translations, and listed ten types, all mixtures of continuous translations (by any distance) in some directions and discrete translations (by integer multiples of a fixed distance) in other directions. He also listed the main finite groups of rotations and reflections: cyclic, dihedral, tetrahedral, octahedral and icosahedral. He distinguished the group O(2) of all rotations and reflections that leave a line in space, the *axis*, fixed, and the group O(3) of all rotations and reflections that leave a point in space, the *centre*, fixed.

Later it became clear that his list was incomplete. For instance, he had missed out some of the subtler crystallographic groups in three-dimensional space. But his work was a major step towards the understanding of Euclidean rigid motions, which are important in mechanics, as well as in the main body of pure mathematics.

Jordan's book is truly vast in scope. It begins with modular arithmetic and Galois fields, which as well as providing examples of groups also constitute essential background for everything else in the book. The middle third deals with groups of permutations, which Jordan calls substitutions. He sets up the basic ideas of normal subgroups which are what Galois used to show that the symmetry group of the quintic is inconsistent with a solution by radicals, and proves that these subgroups can be used to break a general group into simpler pieces. He proves that the sizes of these pieces do not depend on how the original group is broken up. In 1889 Otto Hölder improved this result, interpreting the pieces as groups in their own right, and proved that their group structure, not just their size, is independent of how the group is broken up. Today this result is called the Jordan–Hölder Theorem.

A group is *simple* if it does not break up in this way. The Jordan–Hölder Theorem effectively tells us that the simple groups relate to general groups in the same way that atoms relate to molecules in chemistry. Simple groups are the atomic constituents of all groups. Jordan proved that the alternating group A_n, comprising all permutations of n symbols that switch an even number of pairs of symbols, is simple whenever $n \geq 5$. This is the main group-theoretic reason why the quintic is insoluble by radicals.

A major new development was Jordan's theory of linear substitutions. Here the transformations that make up the group are not permutations of a finite set, but linear changes to a finite list of variables. For example, three variables x, y, z might transform into new variables X, Y, Z by way of linear equations

$$X = a_1x + a_2y + a_3z$$

$$Y = b_1x + b_2y + b_3z$$

$$Z = c_1x + c_2y + c_3z$$

where the a's, b's and c's are constants. To make the group finite, Jordan usually took these constants to be elements of the integers modulo some prime, or more generally a Galois field.

Also in 1869, Jordan developed his own version of Galois theory and included it in the *Traité*. He proved that an equation is soluble if and only if its group is soluble, which means that the simple components all have prime order. He applied Galois's theory to geometric problems.

Symmetry

The 4000-year-old quest to solve quintic algebraic equations was brought to an abrupt halt when Ruffini, Abel and Galois proved that no solution by radicals is possible. Although this was a negative result, it had a huge influence on the subsequent development of both mathematics and science. This happened because the method introduced to prove the impossibility turned out to be central to the mathematical understanding of symmetry, and symmetry turned out to be vital in both mathematics and science.

What group theory does for us

Group theory is now indispensable throughout mathematics, and its use in science is widespread. In particular, it turns up in theories of pattern formation in many different scientific contexts.

One example is the theory of reaction–diffusion equations, introduced by Alan Turing in 1952 as a possible explanation of symmetric patterns in the markings of animals. In these equations, a system of chemicals can diffuse across a region of space, and the chemicals can also react to produce new chemicals. Turing suggested that some such process might set up a pre-pattern in a developing animal embryo, which later on could be turned into pigments, revealing the pattern in the adult.

Suppose for simplicity that the region is a plane. Then the equations are symmetric under all rigid motions. The only solution of the equations that is symmetric under all rigid motions is a uniform state, the same everywhere. This would translate into an animal without any specific markings, the same colour all over. However, the uniform state may be unstable, in which case the actual solution observed will be symmetric under some rigid motions but not others. This process is called *symmetry-breaking*.

A typical symmetry-breaking pattern in the plane consists of parallel stripes. Another is a regular array of spots. More complicated patterns are also possible. Interestingly, spots and stripes are among the commonest patterns in animal markings, and many of the more complicated mathematical patterns are also found in animals. The actual biological process, involving genetic effects, must be more complicated than Turing assumed, but the underlying mechanism of symmetry-breaking must be mathematically very similar.

The effects were profound. Group theory led to a more abstract view of algebra, and with it a more abstract view of mathematics. Although many practical scientists initially opposed the move towards abstraction, it eventually became clear that abstract methods are often more powerful than concrete ones, and most opposition has disappeared. Group theory also made it clear that negative results may still be important, and that an insistence on proof can sometimes lead to major discoveries. Suppose that mathematicians had simply assumed without proof that quintics cannot be solved, on the plausible grounds that no one could find a solution. Then no one would have invented group theory to explain *why* they cannot be solved. If mathematicians had taken the easy route, and assumed the solution to be impossible, mathematics and science would have been a pale shadow of what they are today.

That is why mathematicians insist on proofs.

CHAPTER 14

Algebra Comes of Age

Numbers give way to structures

By 1860 the theory of permutation groups was well developed. The theory of invariants – algebraic expressions that do not change when certain changes of variable are performed – had drawn attention to various infinite sets of transformations, such as the *projective group* of all projections of space. In 1868 Camille Jordan had studied groups of motions in three-dimensional space, and the two strands began to merge.

Sophisticated concepts

A new kind of algebra began to appear, in which the objects of study were not unknown numbers, but more sophisticated concepts: permutations, transformations, matrices. Last year's processes had become this year's things. The long-standing rules of algebra often had to be modified to fit the needs of these new structures. Alongside groups, mathematicians started to study structures called rings and fields, and a variety of algebras.

One stimulus to this changing vision of algebras came from partial differential equations, mechanics and geometry: the development of Lie groups and Lie algebras. Another source of inspiration was number theory: here algebraic numbers could be