CHAPTER 16

The Fourth Dimension

Geometry out of this world

In his science fiction novel *The Time Machine*, Herbert George Wells described the underlying nature of space and time in a way that we now find familiar, but which must have raised some eyebrows among his Victorian readers: 'There are really four dimensions, three which we call the three planes of Space, and a fourth, Time'. To set up the background for his story, he added: 'There is, however, a tendency to draw an unreal distinction between the former three dimensions and the latter, because it happens that our consciousness moves intermittently in one direction along the latter from the beginning to the end of our lives. But some philosophical people have been asking why three dimensions particularly – why not another direction at right angles to the three? – and have even tried to construct a four-dimensional geometry'. His protagonist then goes one better, overcomes the alleged limitations of human consciousness and travels along the fourth dimension of time, as if it were a normal dimension of space.

The fourth dimension

The art of the science fiction writer is suspension of disbelief, and Wells achieved this by informing his readers that 'Professor Simon Newcomb was expounding this to the New York Mathematical Society only a month or so ago'. Here Wells was probably referring to a real event; we know that Newcomb, a prominent astronomer, gave a lecture on four-dimensional space at roughly the right time. His lecture reflected a major change in mathematical and scientific thinking, freeing these subjects from the traditional assumption that space must always have three dimensions. This does not imply that time travel is possible, but it gave Wells an excuse to make penetrating observations about present-day human nature by displacing his time traveller into a disturbing future.

The Time Machine, published in 1895, resonated with a Victorian obsession with the fourth dimension, in which an additional, unseen dimension of space was invoked as a place for ghosts, spirits, or even God to reside. The fourth dimension was championed by charlatans, exploited by novelists, speculated upon by scientists and formalized by mathematicians. Within a few decades, not only was four-dimensional space standard in mathematics: so were spaces with any number of dimensions – five, ten, a billion, even infinity. The techniques and thought-patterns of multidimensional geometry were being used routinely in every branch of science – even biology and economics.

Higher-dimensional spaces remain almost unknown outside the scientific community, but very few areas of human thought could now function effectively without these techniques, remote though they may seem from ordinary human affairs. Scientists trying to unify the two great theories of the physical universe, relativity and quantum mechanics, are speculating that space may actually have nine dimensions, or ten, rather than the three that we normally perceive. In a rerun of the fuss about non-Euclidean geometry, space of three dimensions is increasingly being viewed as just one possibility out of many, rather than the only kind of space that is possible.

These changes have come about because terms like space and dimension are now interpreted in a more general manner, which agrees with the usual dictionary meanings in the familiar contexts of a TV screen or our normal surroundings, but opens up new possibilities. To mathematicians, a *space* is a collection of objects together with some notion of the distance between any two of those objects. Taking a hint from Descartes's idea of coordinates, we can define the dimension of such a space to be *how many numbers* are required to specify an object. With points as the objects, and the usual notion of distance in the plane or space, we find that the plane has two dimensions and space has three. However, other collections of objects may have four dimensions, or more, depending on what the objects are.

For example, suppose that the objects are spheres in three-dimensional space. It takes four numbers (x, y, z, r) to specify a sphere: three coordinates (x, y, z) for its centre, plus the radius r. So the space of all spheres in ordinary space has *four* dimensions. Examples like this show that natural mathematical questions can easily lead to higher-dimensional spaces.

Indeed, modern mathematics goes further. Abstractly, space of four dimensions is *defined* as the set of all quadruples (x_1, x_2, x_3, x_4) of numbers. More generally, space of n dimensions – for any whole number n – is defined as the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of numbers. In a sense, that is the whole story; the intriguing and baffling notion of many dimensions collapses to a triviality: long lists of numbers.

That viewpoint is now clear, but historically, it took a long time to become established. Mathematicians argued, often very forcibly, about the meaning and reality of higher-dimensional spaces. It took about a century for the ideas to become widely accepted. But the applications of such spaces, and the geometric imagery that went

with them, proved so useful that the underlying mathematical issues ceased to be controversial.

Three- or four-dimensional space

Ironically, today's conception of higher-dimensional spaces emerged from algebra, not geometry, as a consequence of a failed attempt to develop a three-dimensional number system, analogous to the two-dimensional system of complex numbers. The distinction between two and three dimensions goes back to Euclid's *Elements*. The first part of the book is about the geometry of the plane, a space of two dimensions. The second part is about solid geometry – the geometry of three-dimensional space. Until the 19th century, the word dimension was limited to these familiar contexts.

Greek geometry was a formalization of the human senses of sight and touch, which allow our brains to build internal models of positional relationships of the outside world. It was constrained by the limitations of our own senses, and of the world in which we live. The Greeks thought that geometry described the *real* space in which we live, and they assumed that physical space has to be Euclidean. The mathematical question 'can four-dimensional space exist in some conceptual sense?' became confused with the physical question 'can a *real* space with four dimensions exist?' And that question was further confused with 'can there be four dimensions *within our own familiar space?*' to which the answer is 'no'. So it was generally believed that four-dimensional space is impossible.

Geometry began to free itself from this restricted viewpoint when the algebraists of Renaissance Italy unwittingly stumbled upon a profound extension of the number concept, by accepting the existence of a square root of minus one. Wallis, Wessel, Argand and Gauss worked out how to interpret the resulting complex numbers as points in a plane, freeing numbers from the one-dimensional shackles of the real number line. In 1837, The Irish mathematician William Rowan Hamilton reduced the whole topic to algebra, by

William Rowan Hamilton
1805–1865

Hamilton was so precocious mathematically that he was made Professor of Astronomy at Trinity College Dublin while still an undergraduate, at the age of 21. This appointment made him Royal Astronomer of Ireland.

He made numerous contributions to mathematics, but the one that he himself believed to be most significant was the invention of quaternions. He tells us that 'Quaternions ... started into life, fully grown, on the 16th of October, 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge. That is to say, I then and there felt the galvanic circuit of thought closed, and the sparks which fell from it were the fundamental equations between i, j, k; *exactly such* as I have used them ever since. I pulled out, on the spot, a pocketbook, which still exists, and made an entry, on which, *at the very moment*, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come. I felt a *problem* to have been at that moment *solved*, an intellectual *want relieved*, which had haunted me for at least *fifteen years* before.'

Hamilton immediately carved the equation

$$i^2 = j^2 = k^2 = ijk = -1$$

in the stone of the bridge.

defining a complex number $x + iy$ to be a *pair* of real numbers (x, y). He further defined addition and multiplication of pairs by the rules

$$(x, y) + (u, v) = (x + u, y + v)$$

$$(x, y)(u, v) = (xu - yv, xv + yu)$$

In this approach, a pair of the form $(x,0)$ behaves just like the real

number x, and the special pair $(0,1)$ behaves like i. The idea is simple, but appreciating it requires a sophisticated concept of mathematical existence.

Hamilton then set his sights on something more ambitious. It was well known that complex numbers make it possible to solve many problems about the mathematical physics of systems in the plane, using simple and elegant methods. A similar trick for three-dimensional space would be invaluable. So he tried to invent a *three-dimensional number system*, in the hope that the associated calculus would solve important problems of mathematical physics in three-dimensional space. He tacitly assumed that this system would satisfy all the usual laws of algebra. But despite heroic efforts, he could not find such a system.

Eventually, he discovered why. It's impossible.

Among the usual laws of algebra is the *commutative law of multiplication*, which states that $ab = ba$. Hamilton had been struggling for years to devise an effective algebra for three dimensions. Eventually he found one, a number system that he called *quaternions*. But it was an algebra of four dimensions, not three, and its multiplication was not commutative.

Quaternions resemble complex numbers, but instead of one new number, i, there are three: i, j, k. A quaternion is a combination of these, for example $7 + 8i - 2j + 4k$. Just as the complex numbers are two-dimensional, built from two independent quantities 1 and i, so the quaternions are *four-dimensional*, built from four *independent* quantities 1, i, j and k. They can be formalized algebraically as quadruples of real numbers, with particular rules for addition and multiplication.

Higher-dimensional space

When Hamilton made his breakthrough, mathematicians were already aware that spaces of high dimension arise entirely naturally, and have sensible physical interpretations, when the basic elements of space are something other than points. In 1846 Julius Plücker pointed out that it takes *four* numbers to specify a *line* in space. Two of those numbers determine where the line hits some fixed plane; two more determine its direction relative to that plane. So, considered as a collection of *lines*, our familiar space already has four dimensions, not three. However, there was a vague feeling that this construction was rather artificial, and that spaces made from four dimensions worth of points were unnatural. Hamilton's quaternions had a natural interpretation as rotations, and their algebra was compelling. They were as natural as complex numbers – so four-dimensional space was as natural as a plane.

The idea quickly went beyond just four dimensions. While Hamilton was promoting his beloved quaternions, a mathematics teacher named Hermann Günther Grassmann was discovering an extension of the number system to spaces with any number of dimensions. He published his idea in 1844 as *Lectures on Lineal Extension*. His presentation was mystical and rather abstract, so the work attracted little attention. In 1862, to combat the lack of interest, he issued a revised version, often translated as *The Calculus of Extension*, which was intended to be more comprehensible. Unfortunately, it wasn't.

Despite its cool reception, Grassmann's work was of fundamental importance. He realized that it was possible to replace the four units, 1, i, j and k, of quaternions by any number of units. He called combinations of these units *hypernumbers*. He understood that his approach had limitations. You had to be careful not to expect too much from the arithmetic of hypernumbers; slavishly following the traditional laws of algebra seldom led anywhere.

Meanwhile, physicists were developing their own notions of higher-dimensional spaces, motivated not by geometry, but by Maxwell's equations for electromagnetism. Here both the electric and magnetic fields are *vectors* – having a *direction* in three-dimensional space as well as a magnitude. Vectors are arrows, if you wish, aligned with the electric or magnetic field. The length of the arrow

shows how strong the field is, and its direction shows which way the field is pointing.

In the notation of the time, Maxwell's equations were eight in number, but they included two groups of three equations, one for each component of the electric or the magnetic field in each of the three directions of space. It would make life much easier to devise a formalism that collected each such triple into a single vector equation. Maxwell achieved this using quaternions, but his approach was clumsy. Independently, the physicist Josiah Willard Gibbs and the engineer Oliver Heaviside found a simpler way to represent vectors algebraically. In 1881 Gibbs printed a private pamphlet, *Elements of Vector Analysis*, to help his students. He explained that his ideas had been developed for convenient use rather than mathematical elegance. His notes were written up by Edwin Wilson, and they published a joint book *Vector Analysis* in 1901. Heaviside came up with the same general ideas in the first volume of his *Electromagnetic Theory* in 1893 (the other two volumes appeared in 1899 and 1912).

The different systems of Hamilton's quaternions, Grassmann's hypercomplex numbers and Gibbs's vectors rapidly converged to the same mathematical description of a vector: it is a triple (x, y, z) of numbers. After 250 years, the world's mathematicians and physicists had worked their way right back to Descartes – but now the coordinate notation was only part of the story. Triples did not just represent points: they represented directed magnitudes. It made a huge difference – not to the formalism, but to its interpretation, its physical *meaning*.

Mathematicians wondered just how many hypercomplex number systems there might be. To them, the question was not 'are they useful?' but 'are they interesting?' So, mathematicians mainly focused on the algebraic properties of systems of n-hypercomplex numbers, for any n. These were, in fact, n-dimensional spaces, *plus* algebraic operations, but to begin with everyone thought algebraically and the geometric aspects were played down.

Differential geometry

Geometers responded to the algebraists' invasion of their territory by reinterpreting hypercomplex numbers geometrically. The key figure here was Riemann. He was working for his 'Habilitation' which gave him the right to charge lecture fees to students. Candidates for Habilitation must give a special lecture on their own research. Following the usual procedure, Gauss asked Riemann to propose a number of topics, from which Gauss would make the final choice. One of Riemann's proposals was On the Hypotheses Which Lie at the Foundation of Geometry, and Gauss, who had been thinking about the same question, chose that topic.

Riemann was terrified – he disliked public speaking and he hadn't fully worked out his ideas. But what he had in mind was explosive: a geometry of n dimensions, by which he meant a system of n coordinates $(x_1, x_2, \ldots, x_n)$, equipped with a notion of distance between nearby points. He called such a space a manifold. This proposal was radical enough, but there was another, even more radical feature: manifolds could be curved. Gauss had been studying the curvature of surfaces, and had obtained a beautiful formula which represented curvature intrinsically – that is, in terms of the surface alone, not of the space in which it was embedded.

Riemann had intended to develop a similar formula for the curvature of a manifold, generalizing Gauss's formula to n dimensions. This formula would also be intrinsic to the manifold – it would not make explicit use of any containing space. Riemann's efforts to develop the notion of curvature in a space of n dimensions led him to the brink of a nervous breakdown. What made matters worse was that at the same time, he was helping Gauss's colleague Weber, who was trying to understand electricity. Riemann battled on, and the interplay between electrical and magnetic forces led him to a new concept of force based on geometry. He had the same insight that led Einstein to general

relativity, decades later: forces can be replaced by the curvature of space.

In traditional mechanics, bodies travel along straight lines unless diverted by a force. In curved geometries, straight lines need not exist and paths are curved. If space is curved, what you experience when you are obliged to deviate from a straight line feels like a force. Now Riemann had the insight he needed to develop his lecture, which he gave in 1854. It was a major triumph. The ideas quickly spread, to growing excitement. Soon scientists were giving popular lectures on the new geometry. Among them was Hermann von Helmholtz, who gave talks about beings that lived on a sphere or some other curved surface.

The technical aspects of Riemann's geometry of manifolds, now called differential geometry, were further developed by Eugenio Beltrami, Elwin Bruno Christoffel and the Italian school under Gregorio Ricci and Tullio Levi-Civita. Later, their work turned out to be just what Einstein needed for general relativity.

Matrix algebra

Algebraists had also been busy, developing computational techniques for n-variable algebra – the formal symbolism of n-dimensional space. One of these techniques was the algebra of matrices, rectangular arrays of numbers, introduced by Cayley in 1855. This formalism arose naturally from the idea of a change of coordinates. It had become commonplace to simplify algebraic formulas by replacing variables such as x and y by their linear combinations, for example

$$u = ax + by$$

$$v = cx + dy$$

for constants a, b, c and d. Cayley represented the pair (x, y) as a column vector and the coefficients by a 2×2 table, or *matrix*. With a suitable definition of multiplication, he could rewrite the coordinate change as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The method extended easily to tables with any number of rows and columns, representing linear changes in any number of coordinates.

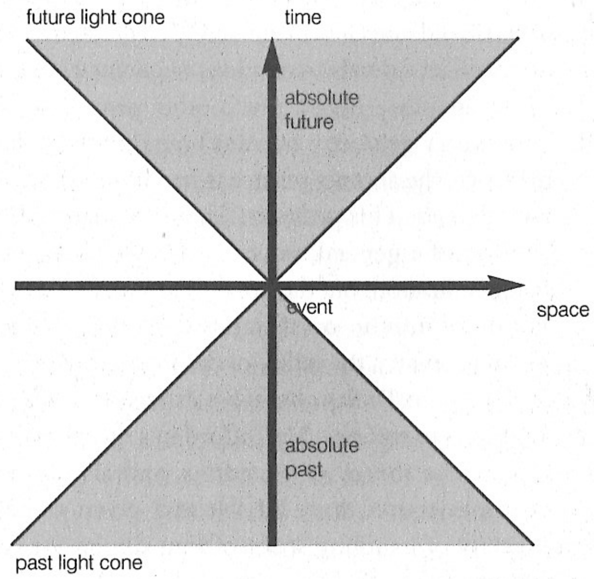
Matrix algebra made it possible to calculate in n-dimensional space. As the new ideas sank in, a geometric language for n-dimensional space came into being, supported by a formal algebraic computational system. Cayley thought that his idea was no more than a notational convenience, and predicted that it would never acquire any applications. Today, it is indispensable throughout science, especially in areas like statistics. Medical trials are a great consumer of matrices, which are used to work out which associations between cause and effect are statistically significant.

The geometric imagery made it easier to prove theorems. Critics countered that these newfangled geometries referred to spaces that didn't exist. The algebraists fought back by pointing out that the algebra of n variables most certainly did exist, and anything that helped advance many different areas of mathematics must surely be interesting. George Salmon wrote: 'I have already completely discussed this problem [solving a certain system of equations] when we are given three equations in three variables. The question now before us may be stated as the corresponding problem in space of p dimensions. But we consider it as a purely algebraic question, apart from any geometrical considerations. We shall however retain a little of geometrical *language*... because we can thus more readily see how to apply to a system of p equations, processes analogous to those which we have employed in a system of three.'

What high-dimensional geometry did for them

Around 1907 the German mathematician Hermann Minkowski formulated Einstein's theory of special relativity in terms of a four-dimensional *space-time*, combining one-dimensional time and three-dimensional space into a single mathematical object. This is known as *Minkowski space-time*.

The requirements of relativity imply that the natural metric on Minkowski space-time is not the one determined by Pythagoras's theorem, in which the square of the distance from a point (x, t) to the origin is $x^2 + t^2$. Instead, this expression should be replaced by the interval $x^2 - c^2t^2$, where c is the speed of light. The crucial change here is the minus sign, which implies that events in space-time are associated with two cones. One cone (here a triangle because space has been reduced to one dimension) represents the future of the event, the other its past. This geometric representation is employed almost universally by modern physicists.



Real space

Do higher dimensions exist? Of course, the answer depends on what we mean by 'exist', but people tend not to understand that kind of thing, especially when their emotions are aroused. The issue came to a head in 1869. In a famous address to the British Association, later reprinted as *A Plea for the Mathematician*, James Joseph Sylvester pointed out that generalization is an important way to advance mathematics. What matters, said Sylvester, is what is *conceivable*, not what corresponds directly to physical experience. He added that with a little practice it is perfectly possible to visualize four dimensions, so four-dimensional space is conceivable.

This so infuriated the Shakespearean scholar Clement Ingleby that he invoked the great philosopher Immanuel Kant to prove that three-dimensionality is an essential feature of space, completely missing Sylvester's point. The nature of real space is irrelevant to the mathematical issues. Nevertheless, for a time most British mathematicians agreed with Ingleby. But some continental mathematicians did not. Grassmann said: 'The theorems of the Calculus of Extension are not merely translations of geometrical results into an abstract language; they have a much more general significance, for while the ordinary geometry remains bound to three dimensions of [physical] space, the abstract science is free of this limitation'.

Sylvester defended his position: 'There are many who regard the alleged notion of a generalized space as only a disguised form of algebraic formulization; but the same might be said with equal truth of our notion of infinity, or of impossible lines, or lines making a zero angle in geometry, the utility of dealing with which no one will be found to dispute. Dr Salmon in his extension of Chasles's theory of characteristics to surfaces, Mr Clifford in a question of probability, and myself in the theory of partitions, and also in my paper on barycentric projection, have all felt and given evidence on the practical utility of handling space of four dimensions as if it were conceivable space'.

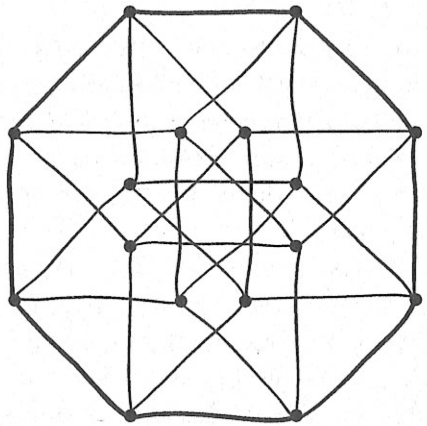
Multidimensional space

In the end, Sylvester won the debate. Nowadays mathematicians consider something to exist if it is not logically contradictory. It may contradict physical experience, but that is irrelevant to *mathematical* existence. In this sense, multidimensional spaces are just as real as the familiar space of three dimensions, because it is just as easy to provide a formal definition.

The mathematics of multidimensional spaces, as now conceived, is purely algebraic, and based on obvious generalizations from low-dimensional spaces. For example, every point in the plane (a two-dimensional space) can be specified by its two coordinates, and every point in three-dimensional space can be specified by its three coordinates. It is a short step to define a point in four-dimensional space as a set of four coordinates, and more generally to define a point in n-dimensional space as a list of n coordinates. Then n-dimensional space itself (or n-space for short) is just the set of all such points.

Similar algebraic machinations let you work out the distance between any two points in n-space, the angle between any two lines, and so on. From there on out, it's a matter of imagination: most sensible geometric shapes in two or three dimensions have straightforward analogues in n dimensions, and the way to find them is to describe the familiar shapes using the algebra of coordinates and then extend that description to n coordinates.

For example, a circle in the plane, or a sphere in 3-space, consists of all points that lie at a fixed distance (the radius) from a chosen point (the centre). The obvious analogue in n-space is to consider all points that lie at a fixed distance from a chosen point. Using the formula for distances, this becomes a purely algebraic condition, and the resulting object is known as an $(n-1)$-dimensional hypersphere, or $(n-1)$-sphere for short. The dimension drops from n to $n-1$ because, for example, a circle in 2-space is a curve, which is a one-dimensional object; similarly a sphere in space is a two-



A four-dimensional hypercube, projected on to the plane

dimensional surface. A *solid* hypersphere in n dimensions is called an n-ball. So the Earth is a 3-ball and its surface is a 2-sphere.

Nowadays, this point of view is called *linear algebra*. It is used throughout mathematics and science, especially in engineering and statistics. It is also a standard technique in economics. Cayley stated that his matrices were unlikely ever to have any practical application. He could not have been more wrong.

By 1900 Sylvester's predictions were coming true, with an explosion of mathematical and physical areas where the concept of multidimensional space was having a serious impact. One such area was Einstein's relativity, best considered as a special kind of four-dimensional space-time geometry. In 1908 Hermann Minkowski realized that the three coordinates of ordinary space, together with an extra one for time, form a four-dimensional *space-time*. Any point in space-time is called an *event*: it is like a point particle that winks into existence at just one moment in time, and then winks out again. Relativity is really about the physics of events. In traditional mechanics, a particle moving through space occupies coordinates

$(x(t), y(t), z(t))$ at time t, and this position changes as time passes. From Minkowski's space-time viewpoint, the collection of all such points is a curve in space-time, the *world line* of the particle, and it is a single object in its own right, existing for all time. In relativity, the fourth dimension has a single, fixed interpretation: *time*.

The subsequent incorporation of gravity, achieved in general relativity, made heavy use of Riemann's revolutionary geometries, but modified to suit Minkowski's representation of the geometry of flat space-time – that is, what space and time do when no mass is present to cause gravitational distortions, which Einstein modelled as curvature.

Mathematicians preferred a more flexible notion of dimensionality and space and as the late 19th century flowed into the early 20th, mathematics itself seemed, ever more, to demand acceptance of multidimensional geometry. The theory of functions of two complex variables, a natural extension of complex analysis, required thinking about space of two complex dimensions – but each complex dimension boils down to two real ones, so like it or not you are looking at a four-dimensional space. Riemann's manifolds and the algebra of many variables provided further motivation.

Generalized coordinates

Yet another stimulus towards multidimensional geometry was Hamilton's 1835 reformulation of mechanics in terms of generalized coordinates, a development initiated by Lagrange in his *Analytical Mechanics* of 1788. A mechanical system has as many of these coordinates as it has degrees of freedom – that is, ways to change its state. In fact the number of degrees of freedom is just dimension in disguise.

For example, it takes six generalized coordinates to specify the configuration of a rudimentary bicycle: one for the angle at which the handlebars sit relative to the frame, one each for the angular

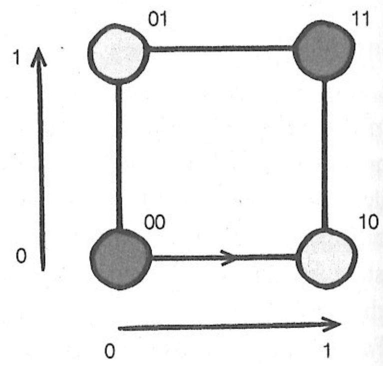
What high-dimensional geometry does for us

Your mobile phone makes essential use of multidimensional spaces. So do your Internet connection, your satellite or cable TV and virtually any other piece of technology that sends or receives messages. Modern communications are digital. All messages, even telephone voice messages, are converted into patterns of 0s and 1s – binary numbers.

Communications are not much use unless they are reliable – the message that is received should be exactly the same as the one that was sent. Electronic hardware cannot guarantee this kind of accuracy, because interference, or even a passing cosmic ray, can cause mistakes. So, electronic engineers use mathematical techniques to put signals into code, in such a way that errors can be detected, and even corrected. The basis of these codes is the mathematics of multidimensional spaces.

Such spaces turn up because a string of, say, ten binary digits, or *bits*, such as 1001011100, can profitably be viewed as a point in a ten-dimensional space with coordinates restricted either to 0 or to 1. Many important questions about error-detecting and error-correcting codes are best tackled in terms of the geometry of this space.

For example, we can detect (but not correct) a single error if we code every message by replacing every 0 by 00 and every 1 by 11. Then, a message such as 110100 codes as 111100110000. If this is received as 111000110000, with an error in the fourth bit, we know something has

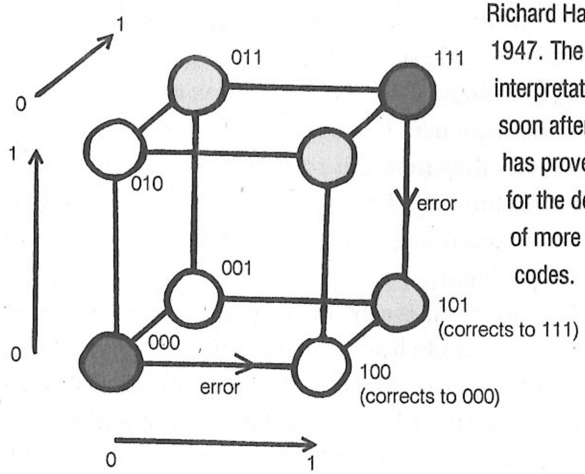


Geometry of pairs of binary digits

gone wrong because the boldface pair 10 should not occur. But we don't know whether it should have been 00 or 11. This can be neatly illustrated in a two-dimensional figure (corresponding to the length 2 of the code words 00 and 11). By thinking of the bits in the code words as coordinates relative to two axes (corresponding to the first and second digits of the code word, respectively) we can draw a picture in which the valid code words 00 and 11 are diagonally opposite corners of a square.

Any single error changes them to code words at the other two corners – which are not valid code words. However, because these corners are adjacent to both of the valid code words, different errors can lead to the same result. To get an error-correcting code, we can use code words of length three and encode 0 as 000 and 1 as 111. Now the code words live at the corners of a cube in three-dimensional space. Any single error results in an adjacent code word; moreover, each such invalid code word is adjacent to only one of the valid code words 000 or 111.

This approach to coding of digital messages was pioneered by Richard Hamming in 1947. The geometric interpretation came soon after, and it has proved crucial for the development of more efficient codes.



Error-correcting code using strings of length three positions of the two wheels, another for the axle of the pedals, two more for the rotational positions of the pedals themselves. A bicycle is, of course, a three-dimensional *object* – but the space of possible *configurations* of the bicycle is six-dimensional, which is one of the reasons why learning to ride a bicycle is hard until you get the knack. Your brain has to construct an internal representation of how those six variables interact – you have to learn to navigate in the six-dimensional geometry of bicycle-space. For a moving bicycle, there are six corresponding velocities to worry about too: the dynamics is, in essence, *twelve*-dimensional.

By 1920 this concurrence of physics, mathematics and mechanics had triumphed, and the use of geometric language for many-variable problems – multidimensional geometry – had ceased to raise eyebrows, except perhaps among philosophers. By 1950, the process had gone so far that mathematicians' natural tendency was to formulate everything in n dimensions from the beginning. Limiting theories to two or three dimensions seemed old-fashioned and ridiculously confining.

The language of higher-dimensional space rapidly spread into every area of science, and even invaded subjects like economics and genetics. Today's virologists, for instance, think of viruses as points in a space of DNA sequences that could easily have several hundred dimensions. By this they mean, at root, that the genomes of these viruses are several hundred DNA bases long – but the geometric image goes beyond mere metaphor: it provides an effective way to think about the problem.

None of this, however, means that the spirit world exists, that ghosts now have a credible home, or that one day we might (as in Edwin Abbott's *Flatland*) receive a visit from the Hypersphere, a creature from the Fourth Dimension, who would manifest himself to us as a sphere whose size kept mysteriously changing, able to shrink to a point and vanish from our universe. However, physicists working in the theory of superstrings currently think that our

universe may actually have *ten* dimensions, not four. Right now they think that we've never noticed the extra six dimensions because they are curled up too tightly for us to detect them.

Multidimensional geometry is one of the most dramatic areas in which mathematics appears to lose all touch with reality. Since physical space is three-dimensional, how can spaces of four or more dimensions exist? And even if they can be defined mathematically, how can they possibly be useful?

The mistake here is to expect mathematics to be an obvious, literal translation of reality, observed in the most direct manner. We are in fact surrounded by objects that can best be described by a large number of variables, the 'degrees of freedom' of those objects. To state the position of a human skeleton requires at least 100 variables, for example. Mathematically, the natural description of such objects is in terms of high-dimensional spaces, with one dimension for each variable.

It took mathematicians a long time to formalize such descriptions, and even longer to convince anyone else that they are useful. Today, they have become so deeply embedded in scientific thinking that their use has become a reflex action. They are standard in economics, biology, physics, engineering, astronomy... the list is endless.

The advantage of high-dimensional geometry is that it brings human visual abilities to bear on problems that are not initially visual at all. Because our brains are adept at visual thinking, this formulation can often lead to unexpected insights, not easily obtainable by other methods. Mathematical concepts that have no direct connection with the real world often have deeper, indirect connections. It is those hidden links that make mathematics so useful.