CHAPTER 4

Lure of the Unknown

X marks the spot

The use of symbols in mathematics goes well beyond their appearance in notation for numbers, as a casual glance at any mathematics text will make clear. The first important step towards symbolic reasoning – as opposed to mere symbolic representation – occurred in the context of problem-solving. Numerous ancient texts, right back to the Old Babylonian period, present their readers with information about some unknown quantity, and then ask for its value. A standard formula (in the literary sense) in Babylonian tablets goes 'I found a stone but did not weigh it'. After some additional information – 'when I had added a second stone of half the weight, the total weight was 15 gin' – the student is required to calculate the weight of the original stone.

Algebra

Problems of this kind eventually gave rise to what we now call algebra, in which numbers are represented by letters. The unknown quantity is traditionally denoted by the letter x, the conditions that apply to x are stated as various mathematical formulas, and the

student is taught standard methods for extracting the value of x from those formulas. For instance, the Babylonian problem above would be written as $x + {}^{1}/_{2}\, x = 15$, and we would learn how to deduce that $x = 10$.

At school level, algebra is a branch of mathematics in which unknown numbers are represented by letters, the operations of arithmetic are represented by symbols and the main task is to deduce the values of unknown quantities from equations. A typical problem in school algebra is to find an unknown number x, given the equation $x^2 + 2x = 120$. This quadratic equation has one positive solution, $x = 10$. Here $x^2 + 2x = 10^2 + 2 \times 10 = 100 + 20 = 120$. It also has one negative solution, $x = -12$. Now $x^2 + 2x = (-12)^2 + 2\times(-12) = 144 - 24 = 120$. The ancients would have accepted the positive solution, but not the negative one. Today we admit both, because in many problems negative numbers have a sensible meaning and correspond to physically feasible answers, and because the mathematics actually becomes simpler if negative numbers are permitted.

In advanced mathematics, the use of letters to represent numbers is only one tiny aspect of the subject, the context in which it got started. Algebra is about the properties of symbolic expressions in their own right; it is about structure and form, not just number. This more general view of algebra developed when mathematicians started asking general questions about school-level algebra. Instead of trying to solve specific equations, they looked at the deeper structure of the solution process itself.

How did algebra arise? What came first were the problems and methods for solving them. Only later was the symbolic notation – which we now consider to be the essence of the topic – invented. There were many notational systems, but eventually one eliminated all of its competitors. The name 'algebra' appeared in the middle of this process, and it is of Arabic origin. (The initial 'al', Arabic for 'the', indicates its origin.)

Equations

What we now call the solution of equations, in which an unknown quantity must be found from suitable information, is almost as old as arithmetic. There is indirect evidence that the Babylonians were solving quite complicated equations as early as 2000 BC, and direct evidence for solutions of simpler problems, in the form of cuneiform tablets, dating from around 1700 BC.

The surviving portion of Tablet YBC 4652, from the Old Babylonian period (1800–1600 BC), contains eleven problems for solution; the text on the tablet indicates that originally there were 22 of them. A typical question is:

'I found a stone, but did not weigh it. After I weighed out six times its weight, added 2 *gin* and added one third of one seventh [of this new weight] multiplied by 24, I weighed it. The result was 1 *ma-na*. What was the original weight of the stone?'

A weight of 1 *ma-na* is 60 *gin*.

In modern notation, we would let x be the required weight in *gin*. Then the question tells us that

$$(6x + 2) + \frac{1}{3} \times \frac{1}{7} \times 24(6x + 2) = 60$$

and standard algebraic methods lead to the answer $x = 4^1/_3$ *gin*. The tablet states this answer but gives no clear indication of how it is obtained. We can be confident that it would not have been found using symbolic methods like the ones we now use, because later tablets prescribe solution methods in terms of typical examples – 'halve *this* number, add the product of these two, take the square root . . .' and so on.

This problem, along with the others on YBC 4652, is what we now call a *linear* equation, which indicates that the unknown x enters only to the first power. All such equations can be rewritten in the form

$$ax + b = 0$$

with solution $x = -b/a$. But in ancient times, with no concept of negative numbers and no symbolic manipulation, finding a solution

was not so straightforward. Even today, many students would struggle with the problem from YBC 4652.

More interesting are *quadratic* equations, in which the unknown can also appear raised to the second power – squared. The modern formulation takes the form

$$ax^2 + bx + c = 0$$

and there is a standard formula to find the value of x. The Babylonian approach is exemplified by a problem on Tablet BM 13901:

'I have added up seven times the side of my square and eleven times the area, [getting] 6;15.' (Here 6;15 is the simplified form of Babylonian sexagesimal notation, and means 6 plus 15/60, or $6^1/_4$ in modern notation.)

The stated solution runs:

'You write down 7 and 11. You multiply 6;15 by 11, [getting] 1,8;45. You break off half of 7, [getting] 3;30 and 3;30. You multiply, [getting] 12;15. You add [this] to 1,8;45 [getting] result 1,21. This is the square of 9. You subtract 3;30, which you multiplied, from 9. Result 5;30. The reciprocal of 11 cannot be found. By what must I multiply 11 to obtain 5;30? [The answer is] 0;30, the side of the square is 0;30.'

Notice that the tablet tells its reader what to do, but not why. It is a recipe. Someone must have understood why it worked, in order to write it down in the first place, but once discovered it could then be used by anyone who was appropriately trained. We don't know whether Babylonian schools merely taught the recipe, or explained why it worked.

The recipe as stated looks very obscure, but it is easier to interpret the recipe than we might expect. The complicated numbers actually help: they make it clearer which rules are being used. To find them, we just have to be systematic. In modern notation, write

$$a = 11, b = 7, c = 6;15 = 6^1/_4$$

Then the equation takes the form

$$ax^2 + bx = c$$

with those particular values for a, b, c. We have to deduce x. The Babylonian solution tells us to:

(1) Multiply c by a, which gives ac.
(2) Divide b by 2, which is $b/2$.
(3) Square $b/2$ to get $b^2/4$.
(4) Add this to ac, which is $ac + b^2/4$.
(5) Take its square root $\sqrt{ac+b^2/4}$.
(6) Subtract $b/2$, which makes $\sqrt{ac+b^2/4} - b/2$.
(7) Divide this by a, and the answer is $x = \dfrac{\sqrt{ac+b^2/4} - b/2}{a}$.

This is equivalent to the formula

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

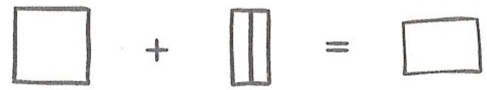
that is taught today because we put the term c on the left hand side, where it becomes $-c$.

It is quite clear that the Babylonians knew that their procedure was a general one. The quoted example is too complex for the solution to be a special one, designed to suit that problem alone.

How did the Babylonians think of their method, and how did they think about it? There had to be some relatively simple idea lying behind such a complicated process. It seems plausible, though there is no direct evidence, that they had a geometric idea, completing the square. An algebraic version of this is taught today, too. We can represent the question, which for clarity we choose to write in the form $x^2 + ax = b$, as a picture:

$$x^2 \quad + \quad ax \quad = \quad b$$

Here the square and the first rectangle have height x; their widths are, respectively, x and a. The smaller rectangle has area b. The Babylonian recipe effectively splits the first rectangle into two pieces,



$$x^2 \quad + \quad 2(a/2 \times x) \quad = \quad b$$

We can then rearrange the two new pieces and stick them on the edge of the square:

$$x^2 + 2(a/2 \times x) \quad = \quad b$$

The left-hand diagram now cries out to be completed to a larger square, by adding the shaded square:

To keep the equation valid, the same extra shaded square is added to the other diagram too. But now we recognize the left-hand diagram as the square of side $(x + a/2)$, and the geometric picture is equivalent to the algebraic statement

$$x^2 + 2(a/2 \times x) + (a/2)^2 = b + (a/2)^2$$

Since the left-hand side is a square, we can rewrite this as

$$(x + a/2)^2 = b + (a/2)^2$$

and then it is natural to take a square root

$$x + a/2 = \sqrt{b+(a/2)^2}$$

and finally rearrange to deduce that

$$x = \sqrt{b+(a/2)^2} - a/2$$

which is exactly how the Babylonian recipe proceeds.

There is no evidence on any tablet to support the view that this geometric picture led the Babylonians to their recipe. However, this suggestion is plausible, and is supported indirectly by various diagrams that do appear on clay tablets.

Al-jabr

The word algebra comes from the Arabic al-jabr, a term employed by Muhammad ibn Musa al-Khwarizmi, who flourished around 820. His work *The Compendious Book on Calculation by al-jabr w'al-muqabala* explained general methods for solving equations by manipulating unknown quantities.

Al-Khwarizmi used words, not symbols, but his methods are recognizably similar to those taught today. *Al-jabr* means 'adding equal amounts to both sides of an equation', which is what we do when we start from

$$x - 3 = 5$$

and deduce that

$$x = 8$$

In effect, we make this deduction by adding 3 to both sides. *Al-muqabala* has two meanings. There is a special meaning: 'subtracting equal amounts from both sides of an equation', which we do to pass from

$$x + 3 = 5$$

to the answer

$$x = 2$$

But it also has a general meaning: 'comparison'.

Al-Khwarizmi gives general rules for solving six kinds of equation, which between them can be used to solve all linear and quadratic equations. In his work, then, we find the ideas of elementary algebra, but not the use of symbols.

Cubic equations

The Babylonians could solve quadratic equations, and their method was essentially the same one that is taught today. Algebraically, it involves nothing more complicated than a square root, beyond the standard operations of arithmetic (add, subtract, multiply, divide). The obvious next step is cubic equations, involving the cube of the unknown. We write such equations as

$$ax^3 + bx^2 + cx + d = 0$$

where x is the unknown and the coefficients a, b, c, d are known numbers. But until the development of negative numbers, mathematicians classified cubic equations into many distinct types – so that, for example, $x^3 + 3x = 7$ and $x^3 - 3x = 7$ were considered to be completely different, and required different methods for their solution.

The Greeks discovered how to use conic sections to solve some cubic equations. Modern algebra shows that if a conic intersects another conic, the points of intersection are determined by an equation of third or fourth degree (depending on the conics). The Greeks did not know this as a general fact, but they exploited its consequences in specific instances, using the conics as a new kind of geometrical instrument.

This line of attack was completed and codified by the Persian Omar Khayyam, best known for his poem the *Rubaiyat*. Around 1075 he classified cubic equations into 14 kinds, and showed how to solve each kind using conics, in his work *On the Proofs of the Problems of Algebra and Muqabala*. The treatise was a geometric *tour de force*, and it polished off the geometric problem almost completely. A modern mathematician would raise a few quibbles – some of Omar's cases are not completely solved because he assumes that certain geometrically constructed points exist when sometimes they do not. That is, he assumes his conics meet when they may fail to do so. But these are minor blemishes.

Geometric solutions of cubic equations were all very well, but could there exist algebraic solutions, involving such things as cube roots, but nothing more complicated? The mathematicians of Renaissance Italy made one of the biggest breakthroughs in algebra when they discovered that the answer is 'yes'.

In those days, mathematicians made their reputation by taking part in public contests. Each contestant would set his opponent problems, and whoever solved the most was adjudged the winner. Members of the audience could place bets on who would win. The contestants often wagered large sums of money – in one recorded instance, the loser had to buy the winner (and his friends) thirty banquets. Additionally, the winner's ability to attract paying students, mostly from the nobility, was likely to be enhanced. So, public mathematical combat was serious stuff.

In 1535 there was just such a contest, between Antonio Fior and Niccolo Fontana, nicknamed Tartaglia, 'the stammerer'. Tartaglia wiped the floor with Fior, and word of his success spread, coming to the ears of Girolamo Cardano. And Cardano's ears pricked up. He was in the middle of writing a comprehensive algebra text, and the questions that Fior and Tartaglia had posed each other were – cubic equations. At that time, cubic equations were classified into three distinct types, again because negative numbers were not recognized. Fior knew how to solve just one type. Initially, Tartaglia knew how to solve just one different type. In modern symbols, his solution of

a cubic equation of the type $x^3 + ax = b$ is

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}}$$

In a burst of inspired desperation, a week or so before the contest, Tartaglia figured out how to solve the other types too. He then set Fior only the types that he knew Fior could not solve.

Cardano, hearing of the contest, realized that the two combatants had devised methods for solving cubic equations. Wanting to add them to his book, he buttonholed Tartaglia and asked him to reveal his methods. Tartaglia was naturally reluctant, because his livelihood depended on them, but eventually he was persuaded to divulge the secret. According to Tartaglia, Cardano promised never to make the method public. So Tartaglia was understandably peeved when his method appeared in Cardano's *Ars Magna – the Great Art of Algebra*. He complained bitterly and accused Cardano of plagiarism.

Now, Cardano was far from lilywhite. He was an inveterate gambler, who had made and lost considerable sums of money at cards, dice and even chess. He lost the entire family fortune in this manner, and was reduced to penury. He was also a genius, a competent doctor, a brilliant mathematician and an accomplished self-publicist – though his positive attributes were mitigated by frankness that often became offensively blunt and insulting. So Tartaglia can be forgiven for assuming that Cardano had lied to him and stolen his discovery. That Cardano had given full credit to Tartaglia in his book only made things worse; Tartaglia knew that it was the book's author who would be remembered, not some obscure figure given a sentence or so of mention.

However, Cardano had an excuse, quite a good one. And he also had a strong reason to bend his promise to Tartaglia. The reason was that Cardano's student Lodovico Ferrari had found a method for solving quartic equations, those involving the fourth power of the unknown. This was completely new, and of huge importance. So of

Fibonacci Sequence

The third section of the *Liber Abbaci* contains a problem that seems to have originated with Leonardo: 'A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if in every month, each pair begets a new pair, which from the second month onwards becomes productive?'

This rather quirky problem leads to a curious, and famous, sequence of numbers:

1 2 3 5 8 13 21 34 55

and so on. Each number is the sum of the two preceding numbers. This is known as the *Fibonacci Sequence*, and it turns up repeatedly in mathematics and in the natural world. In particular, many flowers have a Fibonacci number of petals. This is not coincidence, but a consequence of the growth pattern of the plant and the geometry of the 'primordia' – tiny clumps of cells at the tip of the growing shoot that give rise to important structures, including petals.

Although Fibonacci's growth rule for rabbit populations is unrealistic, more general rules of a similar kind (called *Leslie models*) are used today for certain problems in population dynamics, the study of how animal populations change in size as the animals breed and die.

course Cardano wanted quartic equations in his book, too. Since it was his student who had made the discovery, this would have been legitimate. But Ferrari's method reduced the solution of any quartic to that of an associated cubic, so it relied on Tartaglia's solution of cubic equations. Cardano could not publish Ferrari's work without also publishing Tartaglia's.

Then news reached him that offered a way out. Fior, who had lost to Tartaglia in public combat, was a student of Scipio del Ferro. Cardano heard that del Ferro had solved all three types of cubic, not

What algebra did for them

Several chapters of the *Liber Abbaci* contain algebraic problems relevant to the needs of merchants. One, not terribly practical, goes like this: 'A man buys 30 birds – partridges, doves and sparrows. A partridge costs 3 silver coins, a dove 2, and a sparrow ½. He pays 30 silver coins. How many birds of each type does he buy?'

In modern notation, if we let x be the number of partridges, y the number of doves, and z the number of sparrows, then we must solve two equations

$$x + y + z = 30$$
$$3x + 2y + \tfrac{1}{2}z = 30$$

In real or rational numbers, these equations would have infinitely many solutions, but there is an extra condition implied by the question: the numbers x, y, z are integers. It turns out that only one solution exists: 3 partridges, 5 doves and 22 sparrows.

Leonardo also mentions a series of problems about buying a horse. One man says to another, 'If you give me one-third of your money, I can buy the horse'. The other says, 'if you give me one-quarter of your money, I can buy the horse'. What is the price of the horse? This time there are many solutions; the smallest one in whole numbers sets the price of the horse at 11 silver coins.

just the one that he had passed on to Fior. And a certain Annibale del Nave was rumoured to possess del Ferro's unpublished papers. So Cardano and Ferrari went to Bologna in 1543 to consult del Nave, viewed the papers – and there, as plain as the nose on your face, were solutions of all three types of cubic. So Cardano could honestly say that he was not publishing Tartaglia's method, but del Ferro's.

Tartaglia didn't see things that way. But he had no real answer to Cardano's point that the solution was not Tartaglia's discovery at all, but del Ferro's. Tartaglia published a long, bitter diatribe about the

Girolamo Cardano

(aka Hieronymus Cardanus, Jerome Cardan)
1501–1576

Girolamo Cardano was the illegitimate son of the Milanese lawyer Fazio Cardano and a young widow named Chiara Micheria who was trying to bring up three children. The children died of the plague in Milan while Chiara was giving birth to Girolamo in nearby Pavia. Fazio was an able mathematician, and he passed on his passion for the subject to Girolamo. Against his father's wishes, Girolamo studied medicine at Pavia University; Fazio had wanted him to study law.

While still a student, Cardano was elected rector of the University of Padua, to which he had moved, by a single vote. Having spent a small legacy from his recently deceased father, Cardano turned to gambling to augment his finances: cards, dice and chess. He always carried a knife and once slashed the face of an opponent whom he believed he had caught cheating.

In 1525 Cardano gained his medical degree, but his application to join the College of Physicians in Milan was rejected, probably because he had a reputation for being difficult. He practised medicine in the village of Sacca, and married Lucia Bandarini, a militia captain's daughter. The practice did not prosper, and in 1533 Girolamo again turned to gambling, but now he lost heavily, and had to pawn his wife's jewellery and some of the family furniture.

Cardano struck lucky, and was offered his father's old position as lecturer in mathematics at the Piatti Foundation. He continued practising medicine on the side, and some miraculous cures enhanced his reputation as a doctor. By 1539, after several attempts, he was finally admitted to the College of Physicians. He began to publish scholarly texts on a variety of topics, including mathematics.

Cardano wrote a remarkable autobiography, *The Book of My Life*, a miscellany of chapters on numerous topics. His fame was at its peak, and he visited Edinburgh to treat the Archbishop of St Andrews, John Hamilton. Hamilton suffered from severe asthma. Under Cardano's care, his health improved dramatically, and Cardano left Scotland 2000 gold crowns the richer.

He became professor at Pavia University, and things were going swimmingly until his eldest son Giambatista secretly married Brandonia di Seroni, 'a worthless, shameless woman' in Cardano's estimation. She and her family publicly humiliated and taunted Giambatista, who poisoned her. Despite Cardano's best efforts, Giambatista was executed. In 1570 Cardano was tried for heresy, having cast the horoscope of Jesus. He was imprisoned, then released, but banned from university employment. He went to Rome, where the Pope unexpectedly gave him a pension and he was admitted to the College of Physicians.

He forecast the date of his own death, and allegedly made sure he was right by committing suicide. Despite many tribulations, he remained an optimist to the end.

affair, and was challenged to a public debate by Ferrari, defending his master. Ferrari won hands down, and Tartaglia never really recovered from the setback.

Algebraic symbolism

The mathematicians of Renaissance Italy had developed many algebraic methods, but their notation was still rudimentary. It took hundreds of years for today's algebraic symbolism to develop.

One of the first to use symbols in place of unknown numbers was Diophantus of Alexandria. His *Arithmetica*, written around 250, originally consisted of 13 books, of which six have survived as later copies. Its focus is the solution of algebraic equations, either in whole numbers or in rational numbers – fractions p/q where p and q are whole numbers. Diophantus's notation differs considerably from what we use today. Although the *Arithmetica* is the only surviving document on this topic, there is fragmentary evidence that Diophantus was part of a wider tradition, and not just an isolated figure. Diophantus's notation is not very well suited to calculations, but it does summarize them in a compact form.

The Arabic mathematicians of the Medieval period developed sophisticated methods for solving equations, but expressed them in words, not symbols.

Diophantus's Notation and Ours

Meaning	Modern symbol	Diophantus's symbol
The unknown	x	γ
Its square	x^2	$\Delta\gamma$
Its cube	x^3	$K\gamma$
Its fourth power	x^4	$\Delta\gamma\Delta$
Its fifth power	x^5	$\Delta K\gamma$
Its sixth power	x^6	$K\gamma K$
Addition	$+$	Juxtapose terms (use AB for A+B)
Subtraction	$-$	⋔
Equality	$=$	ι^{σ}

The move to symbolic notation gained momentum in the Renaissance period. The first of the great algebraists to start using symbols was François Vieta, who stated many of his results in symbolic form, but his notation differed considerably from the modern one. He did, however, use letters of the alphabet to represent known quantities, as well as unknown ones. To distinguish these, he adopted the convention that consonants B, C, D, F, G... represented known quantities, whereas vowels *A*, *E*, *I*,... represented unknowns.

In the 15th century, a few rudimentary symbols made their appearance, notably the letters p and m for addition and subtraction: plus and minus. These were abbreviations rather than true symbols. The symbols + and – also appeared around this time. They arose in commerce, where they were used by German merchants to distinguish overweight and underweight items. Mathematicians quickly began to employ them too, the first written examples appearing in 1481. William Oughtred introduced the symbol × for multiplication, and was roundly (and rightly) criticized by Leibniz on the grounds that this was too easily confused with the letter x.

In 1557, in his *The Whetstone of Witte*, the English mathematician Robert Recorde invented the symbol = for equality, in use ever since. He wrote that he could think of no two things that were more alike than two parallel lines of the same length. However, he used much longer lines than we do today, more like ═══════. Vieta initially wrote the word 'aequalis' for equality, but later replaced it by the symbol ~. René Descartes used a different symbol, ∝.

The current symbols > and < for 'greater than' and 'less than' are due to Thomas Harriot. Round brackets ( ) show up in 1544, and square [ ] and curly { } brackets were used by Vieta around 1593. Descartes used the square root symbol √, which is an elaboration on the letter r for radix, or root; but he wrote √c for the cube root.

To see how different Renaissance algebraic notation was from ours, here is a short extract from Cardano's *Ars Magna*:

5p: R̷ m:15

5m: R̷ m:15

25m:m:15 *qd. est* 40

In modern notation this would read:

$$(5 + \sqrt{-15})(5 - \sqrt{-15}) = 25 - (-15) = 40$$

So here we see p: and m: for plus and minus, R̷ for 'square root', and '*qd. est*' abbreviating the Latin phrase 'which is'. He wrote

qdratu aeqtur 4 *rebus* p:32

where we would write

$$x^2 = 4x + 32$$

and therefore used separate abbreviations '*rebus*' and '*qdratu*' for the unknown (thing) and its square. Elsewhere he used R for the unknown, Z for its square and C for its cube.

An influential but little-known figure was the Frenchman Nicolas Chuquet, whose book *Triparty en la Science de Nombres* of 1484 discussed three main mathematical topics: arithmetic, roots and unknowns. His notation for roots was much like Cardano's, but he started to systematize the treatment of powers of the unknown, by using superscripts for exponents. He referred to the first four powers of the unknown as *premier*, *champs*, *cubiez* and *champs de champs*. For what we would now write as $6x$, $4x^2$ and $5x^3$ he used .6.1, .4.2 and .5.3. He also used zeroth and negative powers, writing .2.0 and .3.$^{1.m.}$ where we would write $2x^0$ and $3x^{-1}$. In short: he used exponential notation (superscripts) for powers of the unknown, but had no explicit symbol for the unknown itself.

That omission was supplied by Descartes. His notation was very similar to what we use nowadays, with one exception. Where we would write

$$5 + 4x + 6x^2 + 11x^3 + 3x^4$$

say, Descartes wrote

$$5 + 4x + 6xx + 11x^3 + 3x^4$$

That is, he used *xx* for the square. Occasionally, though he used x^2. Newton wrote powers of the unknown exactly as we do now, including fractional and negative exponents, such as $x^{3/2}$ for the square root of x^3. It was Gauss who finally abolished *xx* in favour of x^2; once the Grand Master had done this, everyone else followed suit.

The logic of species

Algebra began as a way to systematize problems in arithmetic, but by the time of Vieta it had acquired a life of its own. Before Vieta, algebraic symbolism and manipulation were viewed as ways to state and carry out arithmetical procedures, but numbers were still the main point. Vieta made a crucial distinction between what he called the logic of species and the logic of numbers. In his view, an algebraic expression represented an entire class (species) of arithmetical expressions. It was a different concept. In his 1591 *In Artem Analyticam Isagoge* (Introduction to the Analytic Art) he explained that algebra is a method for operating on general forms, whereas arithmetic is a method for operating on specific numbers.

This may sound like logical hair-splitting, but the difference in the point of view was significant. To Vieta, an algebraic calculation such as (in our notation)

$$(2x + 3y) - (x + y) = x + 2y$$

What algebra does for us

The leading consumers of algebra in the modern world are scientists, who represent nature's regularities in terms of algebraic equations. These equations can be solved to represent unknown quantities in terms of known ones. The technique has become so routine that no one notices they're using algebra.

Algebra was very nearly applied to archaeology in one episode of *Time Team*, when the intrepid TV archaeologists wanted to work out how deep a mediaeval well was. The first idea was to drop something down it, and time how long it took to reach the bottom. It took six seconds. The relevant algebraic formula here, neglecting the speed of sound, is

$$s = \tfrac{1}{2}gt^2$$

where s is the depth, t is the time taken to hit the bottom, and g is the acceleration due to gravity, roughly 10 metres per second^2. Taking $t = 6$, the formula tells us that the well is roughly 180 metres deep.

Because of some uncertainty about the formula – which in fact they had remembered correctly – the *Time Team* used three long tape-measures tied together.

The measured depth was in fact very close to 180 metres.

Algebra enters more obviously if we know the depth and want to calculate the time. Now we have to solve the equation for t in terms of s, leading to the answer

$$t = \sqrt{\frac{2s}{g}}$$

Knowing that $s = 180$ metres for instance, lets us predict that t is the square root of 360/10, that is, the square root of 36 – which is 6 seconds.

expresses a way to manipulate symbolic expressions. The individual terms $2x + 3y$ and so on are themselves mathematical objects. They can be added, subtracted, multiplied and divided without ever considering them as representations of specific numbers. To Vieta's predecessors, however, the same equation was simply a numerical relationship that was valid whenever specific numbers were substituted for the symbols x and y. So algebra took on a life of its own, as the mathematics of symbolic expressions. It was the first step towards freeing algebra from the shackles of arithmetical interpretation.