COURANT 10

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LECTURE NOTES

Semilinear Schrödinger Equations

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Contents

Preface

This book presents various mathematical aspects of the nonlinear Schrödinger equation. It is based on the notes of three courses, the first one given at the Federal University of Rio de Janeiro and at the IMPA in 1989 [55], the second given at the Federal University of Rio de Janeiro in 1993 [56], and the third one given at the Courant Institute in 1997.

The nonlinear Schrödinger equation received a great deal of attention from mathematicians, in particular because of its applications to nonlinear optics. Indeed, some simplified models lead to certain nonlinear Schrödinger equations. See Bergé [27] and C. Sulem and P.-L. Sulem [330] for the modelization aspects. Nonlinear Schrödinger equations also arise in quantum field theory, and in particular in the Hartree-Fock theory. See, for example, Avron, Herbst, and Simon [5, 6, 7], Bialinycki-Birula and Mycielski [31, 30], Combes, Schrader, and Seiler [93], Eboli and Marques [109], Gogny and Lions [149], Kato [202], Lebowitz, Rose, and Speer [223], Lieb and Simon [229], Reed and Simon [301], B. Simon [313], and C. Sulem and P.-L. Sulem [330]. The nonlinear Schrödinger equation is also a good model dispersive equation, since it is often technically simpler than other dispersive equations like the wave or KdV.

From the mathematical point of view, Schrödinger's equation is a delicate problem, and possesses a mixture of the properties of parabolic and hyperbolic equations. Particularly useful tools are energy and Strichartz's estimates. We study in this book both problems of local nature (local existence of solutions, uniqueness, regularity, smoothing effect) and problems of global nature (finite-time blowup, global existence, asymptotic behavior of solutions). The methods presented apply in principle to a large class of dispersive semilinear equations. On the other hand, we do not study quasilinear Schrödinger equations (with nonlinearities involving derivatives of the solution). They require in general the use of specific linear (and nonlinear) estimates, and most results of global nature are limited to small initial data.

The book is organized as follows. In Chapter 1, we recall some well-known properties of functional analysis concerning integration, Sobolev and Besov spaces, elliptic equations, and linear semigroups that we use throughout the text. We also introduce some useful compactness tools. In Chapter 2, we establish some fundamental properties of the (linear) Schrödinger equation. The case of the whole space $\mathbb{R}^N$ is studied in detail. Chapter 3 contains a few partial results of local existence for the nonlinear Schrödinger equation in a general domain of $\mathbb{R}^N$. The rest of the book is concerned with the case $\Omega = \mathbb{R}^N$. Chapter 4 is devoted to the study of the local Cauchy problem in various spaces, and in Chapter 5 we study the regularity properties and the smoothing effects. Chapter 6 is devoted to the study of global existence and finite-time blowup of solutions. In Chapter 7, we study the asymptotic behavior of solutions in the repulsive case. The main results are the construction of the scattering operator in a weighted Sobolev space and in the energy space. In Chapter 8, we study the stability and instability properties of standing waves in the attractive case. We establish the existence of standing waves, and in particular of ground states, and we show that ground states are stable or unstable, depending on the growth of the nonlinearity. Chapter 9 is devoted to some further results concerning certain nonlinear Schrödinger equations that can be studied either by the methods used in the previous chapters or else by different methods.

Bibliographical references are given in the text. In order to be informed of the latest news, it is advised to have a look at the web page "*Local and global well-posedness for non-linear dispersive and wave equations*"[1] maintained by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Let us also mention a few monographs specialized in the nonlinear Schrödinger equation: Bergé [27], Bourgain [38], Ginibre [128], Kato [204], Strauss [326], and Sulem and Sulem [330].

I am grateful to my colleagues who reported misprints (and more serious mistakes) in previous versions of these notes, and in particular to P. Bégout, F. Castella, J. Ginibre, T. Kato, and G. Velo. I thank my friend Jalal Shatah, who invited me to publish these notes in the Courant Lecture Notes series. Finally, it was a pleasure to collaborate with Paul Monsour and Reeva Goldsmith in their beautiful editing work.

[1] `http://www.math.ucla.edu/~tao/Dispersive`

Notation

a.a.	almost all
a.e.	almost everywhere
iff	if and only if
$\operatorname{Re} z$	real part of the complex number z
$\operatorname{Im} z$	imaginary part of the complex number z
$\overline{E}$	closure of the subset E in the topological space X
$C(E,F)$	space of continuous functions from the topological space E to the topological space F
1_E	characteristic function of E defined by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$
$C_{\mathrm{c}}(E,F)$	space of continuous functions $E \to F$ compactly supported in E
$\mathcal{L}(E,F)$	Banach space of linear, continuous operators from the Banach space E to the Banach space F, equipped with the norm topology
$\mathcal{L}(E)$	$= \mathcal{L}(E,E)$
$X^\star$	(topological) dual of the (topological) space X
$\langle x', x\rangle_{X^\star,X}$	duality product of $x' \in X^\star$ and $x \in X$ (also "$(x',x)_{X^\star,X}$")
$A^\star$	adjoint of the operator A
$X \hookrightarrow Y$	if $X \subset Y$ with continuous injection
Ω	open subset of $\mathbb{R}^N$
$\overline{\Omega}$	closure of Ω in $\mathbb{R}^N$
$\partial\Omega$	boundary of Ω, i.e., $\partial\Omega = \overline{\Omega} \setminus \Omega$
$\omega \Subset \Omega$	if $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact
B_R	$= \{x \in \mathbb{R}^N : \lvert x\rvert < R\}$, ball of radius R and center 0 of $\mathbb{R}^N$
u_t	$= \partial_t u = \dfrac{\partial u}{\partial t} = \dfrac{du}{dt}$
$\partial_i u$	$= u_{x_i} = \dfrac{\partial u}{\partial x_i}$

u_r	$= \partial_r u = \dfrac{\partial u}{\partial r} = \dfrac{1}{r} x \cdot \nabla u$, where $r = \lvert x \rvert$
D^α	$= \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}$ for a multi-index $\alpha \in \mathbb{N}^N$
∇u	$= (\partial_1 u, \dots, \partial_N u)$
Δ	$= \sum_{i=1}^{N} \dfrac{\partial^2}{\partial x_i^2}$
$\mathcal{F}$	Fourier transform[1] $\mathcal{F}u(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x \cdot \xi} u(x) dx$
$\overline{\mathcal{F}}$	$= \mathcal{F}^{-1}$ given by $\overline{\mathcal{F}}v(x) = \int_{\mathbb{R}^N} e^{2\pi i \xi \cdot x} v(\xi) d\xi$
$\widehat{u}$	$= \mathcal{F}u$
$C_\mathrm{c}(\Omega)$	$= C_\mathrm{c}(\Omega, \mathbb{R})$ (or $C_\mathrm{c}(\Omega, \mathbb{C})$)
$C(\overline{\Omega})$	space of continuous functions $\overline{\Omega} \to \mathbb{R}$ (or $\overline{\Omega} \to \mathbb{C}$). When Ω is bounded, $C(\overline{\Omega})$ is a Banach space when equipped with the L^∞ norm
$C_{\mathrm{b,u}}(\overline{\Omega})$	Banach space of uniformly continuous and bounded functions $\overline{\Omega} \to \mathbb{R}$ (or $\overline{\Omega} \to \mathbb{C}$) equipped with the topology of uniform convergence
$C^m_{\mathrm{b,u}}(\overline{\Omega})$	Banach space of functions $u \in C_{\mathrm{b,u}}(\overline{\Omega})$ such that $D^\alpha u \in C_{\mathrm{b,u}}(\overline{\Omega})$ for every multi-index $\alpha \in \mathbb{N}^N$ such that $\lvert\alpha\rvert \le m$. The space $C^m_{\mathrm{b,u}}(\overline{\Omega})$ is equipped with the norm of $W^{m,\infty}(\Omega)$.
$C_0(\Omega)$	closure of $\mathcal{D}(\Omega)$ in $L^\infty(\Omega)$
$C^{m,\alpha}(\overline{\Omega})$	for $0 \le \alpha \le 1$, the Banach space of functions $u \in C^m_{\mathrm{b,u}}(\overline{\Omega})$ such that $\lVert u \rVert_{C^{m,\alpha}} = \lVert u \rVert_{W^{m,\infty}} + \sup_{\substack{x,y \in \Omega \\ \lvert\beta\rvert = m}} \{\lvert x-y\rvert^{-\alpha} \lvert D^\beta u(x) - D^\beta u(y)\rvert\} < \infty$
$\mathcal{D}(\Omega)$	$= C^\infty_\mathrm{c}(\Omega)$, the Fréchet space of C^∞ functions $\Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) compactly supported in Ω, equipped with the topology of uniform convergence of all derivatives on compact subsets of Ω
$\mathcal{D}'(\Omega)$	space of distributions on Ω, i.e., the topological dual of $\mathcal{D}(\Omega)$
$\mathcal{S}(\mathbb{R}^N)$	Schwartz space; i.e., the set of all real- or complex-valued C^∞ functions on $\mathbb{R}^N$ such that for every nonnegative integer m and every multi-index α, $$p_{m,\alpha}(u) = \sup_{x \in \mathbb{R}^N} (1 + \lvert x \rvert^2)^{m/2} \lvert D^\alpha u(x) \rvert < \infty .$$ $\mathcal{S}(\mathbb{R}^N)$ is a Fréchet space when equipped with the seminorms $p_{m,\alpha}$.
$\mathcal{S}'(\mathbb{R}^N)$	space of tempered distributions on $\mathbb{R}^N$; i.e., the topological dual of $\mathcal{S}(\mathbb{R}^N)$. $\mathcal{S}'(\mathbb{R}^N)$ is a subspace of $\mathcal{D}'(\mathbb{R}^N)$.

[1] With this definition of the Fourier transform, $\lVert \mathcal{F} \rVert_{\mathcal{L}(L^2)} = 1$, $\mathcal{F}(u \star v) = \mathcal{F}u\mathcal{F}v$, and $\mathcal{F}(D^\alpha u) = (2\pi i)^{\lvert\alpha\rvert} \prod_{j=1}^N x_j^{\alpha_j} \mathcal{F}u$.

p' — conjugate of $p \in [1, \infty]$ given by $\dfrac{1}{p} + \dfrac{1}{p'} = 1$

$L^p(\Omega)$ — Banach space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) such that $\|u\|_{L^p} < \infty$, with

$$\|u\|_{L^p} = \begin{cases} \left(\displaystyle\int_\Omega |u(x)|^p \, dx \right)^{1/p} & \text{if } p < \infty \\ \operatorname*{ess\,sup}_\Omega |u| & \text{if } p = \infty \end{cases}$$

$W^{m,p}(\Omega)$ — ($m \in \mathbb{N}$, $1 \le p \le \infty$) Banach space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) such that $D^\alpha u \in L^p(\Omega)$ in the sense of distributions, for every multi-index α with $|\alpha| \le m$. $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p}.$$

$W_0^{m,p}(\Omega)$ — ($m \in \mathbb{N}$, $1 \le p < \infty$) closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$

$W^{-m,p'}(\Omega)$ — ($m \in \mathbb{N}$, $1 \le p < \infty$) dual of $W_0^{m,p}(\Omega)$

$H^m(\Omega)$ — $= W^{m,2}(\Omega)$. $H^m(\Omega)$ is equipped with the equivalent norm

$$\|u\|_{H^m} = \left(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha u(x)|^2 \, dx \right)^{1/2}.$$

$H^m(\Omega)$ is a Hilbert space for the scalar product

$$(u, v)_{H^m} = \int_\Omega \mathrm{Re}(u(x)\overline{v(x)}) dx.$$

$H_0^m(\Omega)$ — $= W_0^{m,2}(\Omega)$

$H^{-m}(\Omega)$ — $= W^{-m,2}(\Omega) = \left(H_0^m(\Omega)\right)^\star$

$H^{s,p}(\mathbb{R}^N)$ — ($s \in \mathbb{R}$, $1 \le p \le \infty$) Banach space of elements $u \in \mathcal{S}'(\mathbb{R}^N)$ such that $\mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2}\widehat{u}] \in L^p(\mathbb{R}^N)$. $H^{s,p}(\mathbb{R}^N)$ is equipped with the norm

$$\|u\|_{H^{s,p}} = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{s}{2}}\widehat{u}]\|_{L^p}.$$

$H^s(\mathbb{R}^N)$ — $= H^{s,2}(\mathbb{R}^N)$

$\dot{H}^{s,p}(\mathbb{R}^N)$ — ($s \in \mathbb{R}$, $1 \le p \le \infty$) homogeneous version of the Sobolev space $H^{s,p}(\mathbb{R}^N)$

$\dot{H}^s(\mathbb{R}^N)$ — $= \dot{H}^{s,2}(\mathbb{R}^N)$

$B^s_{p,q}(\mathbb{R}^N)$ — $(s \in \mathbb{R}, 1 \le p,q \le \infty)$ Banach space of elements $u \in \mathcal{S}'(\mathbb{R}^N)$ such that $\|u\|_{B^s_{p,q}} < \infty$ with

$$\|u\|_{B^s_{p,q}} = \|\mathcal{F}^{-1}(\eta\widehat{u})\|_{L^p} + \begin{cases} \Big(\sum_{j=1}^{\infty}(2^{sj}\|\mathcal{F}^{-1}(\varphi_j\widehat{u})\|_{L^p})^q\Big)^{1/q} & \text{if } q < \infty \\ \sup_{j\ge1} 2^{sj}\|\mathcal{F}^{-1}(\varphi_j\widehat{u})\|_{L^p} & \text{if } q = \infty, \end{cases}$$

where $\mathcal{F}^{-1}(\varphi_j\widehat{u})$ is the j^{th} dyadic block of the Littlewood-Paley decomposition of u

$\dot{B}^s_{p,q}(\mathbb{R}^N)$ — $(s \in \mathbb{R}, 1 \le p,q \le \infty)$ homogeneous version of the Besov space $B^s_{p,q}(\mathbb{R}^N)$

$\mathcal{D}(I,X)$ — $= C^\infty_c(I,X)$, the Fréchet space of C^∞ functions $I \to X$ compactly supported in I, equipped with the topology of uniform convergence of all derivatives on compact subintervals of I

$\mathcal{D}'(I,X)$ — space of X-valued distributions on I, i.e., the space of linear, continuous mappings $\mathcal{D}(I) \to X$, where X is equipped with the weak topology

$C_{\mathrm{b,u}}(\overline{I},X)$ — Banach space of uniformly continuous and bounded functions $\overline{I} \to X$, equipped with the topology of uniform convergence

$C^m_{\mathrm{b,u}}(\overline{I},X)$ — Banach space of functions $u : \overline{I} \to X$ whose derivatives of order j belong to $C_{\mathrm{b,u}}(\overline{I},X)$, for all $0 \le j \le m$. $C^m_{\mathrm{b,u}}(\overline{I},X)$ is equipped with the norm of $W^{m,\infty}(I,X)$.

$C^{m,\alpha}(\overline{I},X)$ — for $0 \le \alpha \le 1$, the Banach space of functions $u \in C^m_{\mathrm{b,u}}(\overline{I},X)$ such that

$$\|u\|_{C^{m,\alpha}} = \|u\|_{W^{m,\infty}} + \sup_{s,t\in I}\Big\{|t-s|^{-\alpha}\Big\|\frac{d^m u}{dt^m}(t) - \frac{d^m u}{dt^m}(s)\Big\|\Big\} < \infty$$

$C(\overline{I},X)$ — space of continuous functions $\overline{I} \to X$. When I is bounded, $C(\overline{I},X)$ is a Banach space with the norm of $L^\infty(I,X)$.

$L^p(I,X)$ — Banach space of (classes of) measurable functions $u : I \to X$ such that $\|u\|_{L^p} < \infty$, with

$$\|u\|_{L^p} = \begin{cases} \Big(\int_I \|u(t)\|^p_X\,dt\Big)^{1/p} & \text{if } p < \infty \\ \operatorname*{ess\,sup}_I \|u(t)\|_X & \text{if } p = \infty \end{cases}$$

$W^{m,p}(I,X)$ — Banach space of (classes of) measurable functions $u : I \to X$ such that $\frac{d^j u}{dt^j} \in L^p(I,X)$ for every $0 \le j \le m$. $W^{m,p}(I,X)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{j=1}^{m}\Big\|\frac{d^j u}{dt^j}\Big\|_{L^p}.$$

$\mathcal{T}(t)$ except when otherwise specified, the group of isometries on $L^2(\Omega)$ generated by the skew-adjoint operator iA, where A is the Laplacian with Dirichlet boundary condition on $\partial\Omega$

CHAPTER 1

Preliminaries

In this chapter we recall some basic properties of functional analysis, complex and vector integration, Sobolev spaces, elliptic equations, and linear semigroups that we use in the next chapters.

1.1. Functional Analysis

See, for example, Brezis [43], Brezis and Cazenave [44], Cazenave and Haraux [64, 65], Rudin [304], Strauss [320], and Yosida [366].

We recall that if X and Y are two Banach spaces such that $X \hookrightarrow Y$ with dense embedding e, then $Y^\star \hookrightarrow X^\star$ with embedding $e^\star$. Moreover, if X is reflexive, then the embedding $Y^\star \hookrightarrow X^\star$ is dense.

We will use repeatedly the following elementary properties of weak topologies.

(i) Let $X \hookrightarrow Y$ be two Banach spaces. Consider $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}} \subset X$. If $x_n \rightharpoonup x$ in X as $n \to \infty$, then $x_n \rightharpoonup x$ in Y as $n \to \infty$.

(ii) Let $X \hookrightarrow Y$ be two Banach spaces. Assume X is reflexive and consider $y \in Y$ and a bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$. If $x_n \rightharpoonup y$ in Y as $n \to \infty$, then $y \in X$ and $x_n \rightharpoonup y$ in X as $n \to \infty$.

(iii) Let $X \hookrightarrow Y$ be two Banach spaces and let I be a bounded, open interval of $\mathbb{R}$. Let $u : \overline{I} \to Y$ be weakly continuous. If X is reflexive and if there exists a dense subset E of I such that $u(t) \in X$ for all $t \in E$ and $\sup\{\|u(t)\|_X, t \in E\} = K < \infty$, then $u(t) \in X$ for all $t \in \overline{I}$ and $u : \overline{I} \to X$ is weakly continuous.

(iv) Let X be a uniformly convex Banach space; let I be a bounded, open interval of $\mathbb{R}$; and let $u : \overline{I} \to X$ be weakly continuous. If the function $t \mapsto \|u(t)\|_X$ is continuous $\overline{I} \to \mathbb{R}$, then $u \in C(\overline{I}, X)$.

(v) Let X be a Banach space, let I be a bounded, open interval of $\mathbb{R}$, and let $u : \overline{I} \to X$ be weakly continuous. If there exists a Banach space B such that $X \hookrightarrow B$ with compact embedding, then $u \in C(\overline{I}, B)$.

We will construct solutions of the nonlinear Schrödinger equation either by a fixed point argument, or by a compactness technique. For the first method, we will use Banach's fixed point theorem and for the second, we will use Proposition 1.1.2 below.

THEOREM 1.1.1. (Banach's fixed point theorem) *Let (X, d) be a complete metric space and $F : X \to X$. If there exists a constant $L < 1$ such that $\mathrm{d}(F(x), F(y)) \le L\mathrm{d}(x, y)$ for all $x, y \in X$, then F has a unique fixed point $x_0 \in X$; i.e., there exists a unique $x_0 \in X$ such that $F(x_0) = x_0$.*

PROPOSITION 1.1.2. *Let $X \hookrightarrow Y$ be two Banach spaces and let I be a bounded, open interval of $\mathbb{R}$. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C(\overline{I},Y)$. Assume that $f_n(t) \in X$ for all $(n,t) \in \mathbb{N}\times I$ and that* $\sup\{\|f_n(t)\|_X, (n,t)\in\mathbb{N}\times I\} = K < \infty$. *Assume further that f_n is uniformly equicontinuous in Y (i.e., $\forall\varepsilon>0, \exists\delta>0$, $\forall n,s,t \in \mathbb{N}\times I\times I, \|f_n(t)-f_n(s)\|_Y \le \varepsilon$ if $|t-s|\le\delta$). If X is reflexive, then the following properties hold:*

(i) *There exists a function $f\in C(\overline{I},Y)$ which is weakly continuous $\overline{I}\to X$ and a subsequence n_k such that $f_{n_k}(t) \rightharpoonup f(t)$ in X as $k\to\infty$, for all $t\in\overline{I}$.*

(ii) *If there exists a uniformly convex Banach space B such that $X\hookrightarrow B\hookrightarrow Y$ and if $(f_n)_{n\in\mathbb{N}}\subset C(\overline{I},B)$ and $\|f_{n_k}(t)\|_B \to \|f(t)\|_B$ as $k\to\infty$, uniformly on I, then also $f\in C(\overline{I},B)$ and $f_{n_k}\to f$ in $C(\overline{I},B)$ as $k\to\infty$.*

PROOF. (i) Let $(t_n)_{n\in\mathbb{N}}$ be a representation of $\mathbb{Q}\cap I$. Using the reflexivity of X and the diagonal procedure, we see easily that there exist a subsequence n_k and a function $f:\mathbb{Q}\cap I\to X$ such that $f_{n_k}(t_j)\rightharpoonup f(t_j)$ in X (hence in Y) as $k\to\infty$, for all $j\in\mathbb{N}$. By the uniform equicontinuity of $(f_n)_{n\in\mathbb{N}}$ and the weak lower semicontinuity of the norm, f can be extended to a function of $C(\overline{I},Y)$. Furthermore, $f:\overline{I}\to X$ is weakly continuous and $\sup\{\|f(t)\|_X,\ t\in I\}\le K$. Consider now $t\in\overline{I}$. Let $(t_j)_{j\in\mathbb{N}}\subset\mathbb{Q}\cap I$ converge to t and let $y'\in Y^\star$. We have

$$|\langle y', f_{n_k}(t)-f(t)\rangle_{Y^\star,Y}| \le |\langle y', f_{n_k}(t)-f_{n_k}(t_j)\rangle_{Y^\star,Y}| + |\langle y', f(t)-f(t_j)\rangle_{Y^\star,Y}| + |\langle y', f_{n_k}(t_j)-f(t_j)\rangle_{Y^\star,Y}|.$$

Given $\varepsilon>0$, it follows from the uniform equicontinuity that the first and second terms of the right-hand side are less than $\varepsilon/3$ for j large enough. Given such a j, the third term is less than $\varepsilon/3$ for k large enough; and so

$$|\langle x', f_{n_k}(t)-f(t)\rangle_{Y^\star,Y}| \to 0 \quad \text{as } k\to\infty.$$

Thus $f_{n_k}(t)\rightharpoonup f(t)$ in Y; and so $f_{n_k}(t)\rightharpoonup f(t)$ in X. Hence (i).

(ii) Note first that $f:\overline{I}\to B$ is weakly continuous. Also, $\|f\|_B:\overline{I}\to\mathbb{R}$ is continuous; and so $f\in C(\overline{I},B)$. It remains to prove that $f_{n_k}\to f$ in $C(\overline{I},B)$. We argue by contradiction, and we assume there exist a sequence $(t_k)_{k\in\mathbb{N}}\subset\overline{I}$ and $\varepsilon>0$ such that $\|f_{n_k}(t_k)-f(t_k)\|_B\ge\varepsilon$, for every $k\in\mathbb{N}$. We may assume that $t_k\to t\in\overline{I}$ as $k\to\infty$. It follows from (i) and the uniform continuity that $f_{n_k}(t_k)\rightharpoonup f(t)$ in Y as $k\to\infty$. Since $(f_n)_{n\in\mathbb{N}}$ is bounded in $C(\overline{I},B)$, we obtain as well that $f_{n_k}(t_k)\rightharpoonup f(t)$ in B as $k\to\infty$. Furthermore,

$$\big|\|f_{n_k}(t_k)\|_B-\|f(t)\|_B\big| \le \big|\|f_{n_k}(t_k)\|_B-\|f(t_k)\|_B\big| + \big|\|f(t_k)\|_B-\|f(t)\|_B\big|.$$

Therefore, $\|f_{n_k}(t_k)\|_B\to\|f(t)\|_B$, and so $f_{n_k}(t_k)\to f(t)$ in B as $k\to\infty$, which is a contradiction. □

Finally, we will use some properties of the intersection and sum of Banach spaces. Consider two Banach spaces X_1 and X_2 that are subsets of a Hausdorff topological vector space $\mathcal{X}$. Let

$$X_1\cap X_2 = \{x\in\mathcal{X} : x\in X_1,\ x\in X_2\}$$

and

$$X_1+X_2 = \{x\in\mathcal{X} : \exists x_1\in X_1,\ \exists x_2\in X_2,\ x=x_1+x_2\}.$$

Set

$$\|x\|_{X_1\cap X_2} = \|x\|_{X_1} + \|x\|_{X_2} \quad \text{for } x \in X_1 \cap X_2\,,$$

and

$$\|x\|_{X_1+X_2} = \inf\{\|x_1\|_{X_1} + \|x_2\|_{X_2} : x = x_1 + x_2\} \quad \text{for } x \in X_1 + X_2\,.$$

We have the following result (see lemma 2.3.1 and theorem 2.7.1 in Bergh and Löfström [28]).

PROPOSITION 1.1.3. *$(X_1 \cap X_2, \|\;\|_{X_1\cap X_2})$ and $(X_1 + X_2, \|\;\|_{X_1+X_2})$ are Banach spaces. If furthermore $X_1 \cap X_2$ is a dense subset of both X_1 and X_2, then $(X_1 \cap X_2)^\star = X_1^\star + X_2^\star$ and $(X_1 + X_2)^\star = X_1^\star \cap X_2^\star$.*

1.2. Integration

For real and complex integration, consult Brezis [43], Dunford and Schwartz [108], Rudin [305], and Yosida [366]. For vector integration, see Brezis and Cazenave [44], Cazenave and Haraux [64, 65], Diestel and Uhl [105], Dinculeanu [106], Dunford and Schwartz [108], J. Simon [314], Yosida [366], and the appendix of Brezis [42].

Throughout these notes, we consider L^p spaces of complex-valued functions. Ω being an open subset of $\mathbb{R}^N$, $L^p(\Omega)$ (or L^p, when there is no risk of confusion) denotes the space of (classes of) measurable functions $u : \Omega \to \mathbb{C}$ such that $\|u\|_{L^p} < \infty$ with

$$\|u\|_{L^p} = \begin{cases} \left(\displaystyle\int_\Omega \|u(x)\|^p\,dx\right)^{1/p} & \text{if } p \in [1,\infty) \\ \operatorname*{ess\,sup}_\Omega \|u\| & \text{if } p = \infty. \end{cases}$$

$L^p(\Omega)$ is a Banach space and $L^2(\Omega)$ is a *real* Hilbert space when equipped with the scalar product

$$(u,v)_{L^2} = \operatorname{Re}\int_\Omega u(x)\overline{v(x)}\,dx\,.$$

Below is a useful result of Strauss [321].

PROPOSITION 1.2.1. *Let Ω be an open subset of $\mathbb{R}^N$ and let $1 < p \le \infty$. Consider $u : \Omega \to \mathbb{R}$ and a bounded sequence $(u_n)_{n\in\mathbb{N}}$ of $L^p(\Omega)$. If $u_n \to u$ a.e. in Ω as $n \to \infty$, then $u \in L^p(\Omega)$ and $u_n \to u$ as $n \to \infty$ in $L^q(\Omega')$, for every $\Omega' \subset \Omega$ of finite measure and every $q \in [1,p)$. In particular, $u_n \to u$ as $n \to \infty$, in $L^p(\Omega)$ weak if $p < \infty$, and in $L^\infty(\Omega)$ weak-$\star$ if $p = \infty$.*

Consider now an open interval $I \subset \mathbb{R}$ and a Banach space X equipped with the norm $\|\cdot\|$. A function $f : I \to X$ is measurable if there exist a set $N \subset I$ of measure 0 and a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(I,X)$ such that

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \text{for all } t \in I \setminus N\,.$$

We deduce easily from the definition that if $f : I \to X$ is measurable, then $\|f\| : I \to \mathbb{R}$ is also measurable. Also, if $f : I \to X$ is measurable and if Y is a Banach space such that $X \hookrightarrow Y$, then $f : I \to Y$ is measurable. More generally, if $f : I \to X$ is measurable, Y is a Banach space, and $\Phi : X \to Y$ is continuous, then $\Phi \circ f : I \to Y$ is measurable.

REMARK 1.2.2. Pettis' theorem asserts that a function f is measurable if and only if f is weakly measurable (i.e., for every $x' \in X^\star$, the function $t \mapsto \langle x', f(t) \rangle_{X^\star, X}$ is measurable $I \to \mathbb{R}$) and there exists a set $N \subset I$ of measure 0 such that $f(I \setminus N)$ is separable. One deduces the following properties:

(i) If $f : I \to X$ is weakly continuous (i.e., continuous from I to X equipped with its weak topology), then f is measurable.
(ii) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions $I \to X$ and let $f : I \to X$. If $f_n(t) \rightharpoonup f(t)$ in X as $n \to \infty$, for a.a. $t \in I$, then f is measurable.
(iii) Let $X \hookrightarrow Y$ be two Banach spaces and let $f : I \to Y$ be a measurable function. If $f(t) \in X$ for a.a. $t \in I$ and if X is reflexive, then $f : I \to X$ is measurable.

A measurable function $f : I \to X$ is integrable if there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset C_c(I, X)$ such that

$$\lim_{n \to \infty} \int_I \|f_n(t) - f(t)\| dt = 0 . \tag{1.2.1}$$

If $f : I \to X$ is integrable, then there exists $x(f) \in X$ such that for any sequence $(f_n)_{n \in \mathbb{N}} \subset C_c(I, X)$ satisfying (1.2.1), one has

$$\lim_{n \to \infty} \int_I f_n(t) dt = x(f) ,$$

the above limit being for the strong topology of X. The element $x(f)$ is called the integral of f on I. We write

$$x(f) = \int f = \int_I f = \int_I f(t) dt .$$

If $I = (a, b)$, we also note

$$x(f) = \int_a^b f = \int_a^b f(t) dt .$$

As for real-valued functions, it is convenient to set

$$\int_\alpha^\beta f(t) dt = - \int_\beta^\alpha f(t) dt$$

if $\beta < \alpha$. Bochner's theorem asserts that if $f : I \to X$ is measurable, then f is integrable if and only if $\|f\| : I \to \mathbb{R}$ is integrable. In addition,

$$\left\| \int_I f(t) dt \right\| \le \int_I \|f(t)\| dt .$$

Bochner's theorem allows one to deal with vector-valued integrable functions like one deals with real-valued integrable functions. It suffices in general to apply the usual convergence theorems to $\|f\|$. For example, one can easily establish the following result (the dominated convergence theorem). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions $I \to X$, let $g \in L^1(I)$, and let $f : I \to X$. Assume that

$$\begin{cases} \|f_n(t)\| \le g(t) & \text{for a.a. } t \in I \text{ and all } n \in \mathbb{N} \\ \lim_{n\to\infty} f_n(t) = f(t) & \text{for a.a. } t \in I. \end{cases}$$

It follows that f is integrable and

$$\int_I f(t)dt = \lim_{n\to\infty} \int_I f_n(t)dt\,.$$

For $p \in [1,\infty]$, one denotes by $L^p(I,X)$ the set of (classes of) measurable functions $f : I \to X$ such that the function $t \mapsto \|f(t)\|$ belongs to $L^p(I)$. For $f \in L^p(I,X)$, one defines

$$\|f\|_{L^p(I,X)} = \begin{cases} \left(\int_I \|f(t)\|^p \, dt\right)^{1/p} & \text{if } p < \infty \\ \operatorname*{ess\,sup}_{t\in I} \|f(t)\| & \text{if } p = \infty. \end{cases}$$

When there is no risk of confusion, we denote $\| \ \|_{L^p(I,X)}$ by $\| \ \|_{L^p(I)}$ or $\| \ \|_{L^p}$ or $\| \ \|_p$.

REMARK 1.2.3. The space $L^p(I,X)$ enjoys most of the properties of the space $L^p(I) = L^p(I,\mathbb{R})$, with essentially the same proofs. In particular, one obtains easily the following results:

(i) $\| \ \|_{L^p(I,X)}$ is a norm on the space $L^p(I,X)$. $L^p(I,X)$ equipped with that norm is a Banach space. If $p < \infty$, then $\mathcal{D}(I,X)$ is dense in $L^p(I,X)$ (apply the classical procedure by truncation and regularization).

(ii) A measurable function $f : I \to X$ belongs to $L^p(I,X)$ if and only if there exists a function $g \in L^p(I)$ such that $\|f\| \le g$ a.e. on I.

(iii) Suppose $f : I \to X$ is measurable. If $f \in L^p(J,X)$ for all $J \Subset I$ and if $\|f\|_{L^p(J,X)} \le C$ for some C independent of J, then $f \in L^p(I,X)$ and $\|f\|_{L^p(I,X)} \le C$.

(iv) If $f \in L^p(I,X)$ and $\varphi \in L^q(I)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \le 1$, then $\varphi f \in L^r(I,X)$ and

$$\|\varphi f\|_{L^r(I,X)} \le \|f\|_{L^p(I,X)} \|\varphi\|_{L^q(I)}\,.$$

In particular, if $f \in L^p(I,X)$ and if J is an open subinterval of I, then $f|_J \in L^p(J,X)$.

(v) If $f \in L^p(I,X)$ and $g \in L^q(I,X^\star)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \le 1$, and if $h(t) = \langle g(t), f(t)\rangle_{X^\star,X}$, then

$$h \in L^r(I)) \quad \text{and} \quad \|h\|_{L^r(I)} \le \|f\|_{L^p(I,X)} \|g\|_{L^q(I,X^\star)}\,.$$

(vi) If $f \in L^p(I,X) \cap L^q(I,X)$ with $p < q$, then $f \in L^r(I,X)$ for every $r \in [p,q]$, and

$$\|f\|_{L^r(I,X)} \le \|f\|^{\theta}_{L^p(I,X)} \|f\|^{1-\theta}_{L^q(I,X)} \quad \text{where} \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}\,.$$

(vii) If $|I| < \infty$ and $p \le q$, then

$$\|f\|_{L^p(I,X)} \le |I|^{\frac{q-p}{pq}} \|f\|_{L^q(I,X)} \quad \text{for all } f \in L^q(I,X).$$

(viii) If Y is a Banach space and if $A \in \mathcal{L}(X,Y)$, then $Af \in L^p(I,Y)$ for every $f \in L^p(I,X)$, and $\|Af\|_{L^p(I,Y)} \le \|A\|_{\mathcal{L}(X,Y)} \|f\|_{L^p(I,X)}$. In particular, if $X \hookrightarrow Y$ and if $f \in L^p(I,X)$, then $f \in L^p(I,Y)$ (let A be the embedding).

(ix) If Y is a Banach space and if $A \in \mathcal{L}(X,Y)$, then

$$\int_I Af(t)dt = A\Big(\int_I f(t)dt\Big)$$

for every $f \in L^1(I,X)$. In particular, if $X \hookrightarrow Y$ and if $f \in L^1(I,X)$, then the integral of f in the sense of X is also the integral of f in the sense of Y (let A be the embedding).

(x) If I is an interval of $\mathbb{R}$, one defines the space $L^p_{\text{loc}}(I,X)$ as the set of functions $f : I \to X$ such that $f|_J \in L^p(J,X)$ for all open, bounded intervals $J \subset I$.

We end this section by two useful criteria.

THEOREM 1.2.4. *Let $1 \le p \le \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(I,X)$. If there exists $f : I \to X$ such that for a.a. $t \in I$, $f_n(t) \rightharpoonup f(t)$ in X as $n \to \infty$, then $f \in L^p(I,X)$ and $\|f\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{L^p(I,X)}$.*

THEOREM 1.2.5. *Consider two Banach spaces $X \hookrightarrow Y$ and $1 < p,q \le \infty$. Let $(f_n)_{n\ge 0}$ be a bounded sequence in $L^q(I,Y)$ and let $f : I \to Y$ be such that $f_n(t) \rightharpoonup f(t)$ in Y as $n \to \infty$, for a.a. $t \in I$. If $(f_n)_{n\ge 0}$ is bounded in $L^p(I,X)$ and if X is reflexive, then $f \in L^p(I,X)$ and $\|f\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{L^p(I,X)}$.*

1.3. Sobolev Spaces

For Sobolev spaces of real- (or complex-) valued functions, see, for example, Adams [3], Bergh and Löfström [28], Brezis [43], Gilbarg and Trudinger [127], J.-L. Lions [231], Lions and Magenes [232], and Triebel [338]. For vector-valued Sobolev spaces, see the appendix of Brezis [42], Brezis and Cazenave [44], Cazenave and Haraux [64, 65], J.-L. Lions [231], and Lions and Magenes [232].

Consider an open subset Ω of $\mathbb{R}^N$. We recall that $\mathcal{D}(\Omega)$ $(= \mathcal{D}(\Omega,\mathbb{C}))$ is equipped with the topology induced by the family of seminorms $d_{K,m}$, where K is a compact subset of Ω and $m \in \mathbb{N}$, defined by

$$d_{K,m}(\varphi) = \sup_{x\in K} \sum_{|\alpha|=m} |D^\alpha \varphi(x)| \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

The set of distributions on Ω, $\mathcal{D}'(\Omega)$, is the dual space of $\mathcal{D}(\Omega)$. If $T \in \mathcal{D}'(\Omega)$ and if $\alpha \in \mathbb{N}^N$ is a multi-index, one defines the distribution

$$D^\alpha T = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} T \in \mathcal{D}'(\Omega)$$

by

$$\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi\rangle \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

A function $f \in L^1_{\rm loc}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$ by

$$\langle T_f, \varphi\rangle = \mathrm{Re}\left(\int_\Omega f(x)\overline{\varphi(x)}\,dx\right) \quad \text{for all } \varphi \in \mathcal{D}(\Omega)\,.$$

It is well known that if $T_f = T_g$, then $f = g$ a.e. A distribution $T \in \mathcal{D}'(\Omega)$ is said to belong to $L^p(\Omega)$ if there exists $f \in L^p(\Omega)$ such that $T = T_f$. In this case, f is unique.

For $m \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \le m\}\,.$$

$W^{m,p}(\Omega)$ is a Banach space when equipped with the norm $\|\ \|_{W^{m,p}} = \|\ \|_{W^{m,p}(\Omega)}$ defined by

$$\|u\|_{W^{m,p}} = \sum_{0\le|\alpha|\le m} \|D^\alpha u\|_{L^p(\Omega)}\,.$$

If $p < \infty$, one defines the closed subset $W^{m,p}_0(\Omega)$ of $W^{m,p}(\Omega)$ as the closure in $W^{m,p}(\Omega)$ of $\mathcal{D}(\Omega)$.

When $p = 2$, set $W^{m,p}(\Omega) = H^m(\Omega)$ and $W^{m,p}_0(\Omega) = H^m_0(\Omega)$ and equip $H^m(\Omega)$ with the equivalent norm

$$\|u\|_{H^m(\Omega)} = \|u\|_{H^m} = \Big(\sum_{0\le|\alpha|\le m} \int_\Omega \|D^\alpha u(x)\|^2\,dx\Big)^{1/2}\,.$$

The space $H^m(\Omega)$ (hence $H^m_0(\Omega)$) is then a Hilbert space with the scalar product

$$(u,v)_{H^m} = \sum_{0\le|\alpha|\le m} \mathrm{Re}\int_\Omega D^\alpha u(x)\overline{D^\alpha v(x)}\,dx\,.$$

REMARK 1.3.1. The following properties are well known:

(i) If $1 < p < \infty$, then the spaces $W^{m,p}(\Omega)$ and $W^{m,p}_0(\Omega)$ are reflexive.

(ii) If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $W^{1,p}(\Omega)$, $1 \le p \le \infty$, then $(u_n|_\omega)_{n\in\mathbb{N}}$ is a relatively compact subset of $L^1(\omega)$ for every $\omega \Subset \Omega$. In particular, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging a.e. in ω. Therefore, one constructs easily a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging a.e. in Ω.

(iii) Assume $m \ge 1$ and $1 < p \le \infty$. If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $W^{m,p}(\Omega)$, then there exist $u \in W^{m,p}(\Omega)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e. as $k \to \infty$, and

$$\|u\|_{W^{m,p}} \le \liminf_{n\to\infty} \|u_n\|_{W^{m,p}}\,.$$

If $p < \infty$, then also $u_{n_k} \rightharpoonup u$ in $W^{m,p}$. If $p < \infty$ and $(u_n)_{n\in\mathbb{N}} \subset W^{m,p}_0(\Omega)$, then $u \in W^{m,p}_0(\Omega)$.

(iv) Let $m \ge 0$ and $1 < p \le \infty$. Consider a bounded sequence $(u_n)_{n\in\mathbb{N}}$ of $W^{m,p}(\Omega)$ and assume that there exits $u : \Omega \to \mathbb{R}$ such that $u_n \to u$ a.e. as $n \to \infty$. It follows that $u \in W^{m,p}(\Omega)$ and

$$\|u\|_{W^{m,p}} \le \liminf_{n\to\infty} \|u_n\|_{W^{m,p}}\,.$$

If $p < \infty$, then also $u_n \rightharpoonup u$ in $W^{m,p}$. If $p < \infty$ and $(u_n)_{n\in\mathbb{N}} \subset W_0^{m,p}(\Omega)$, then $u \in W_0^{m,p}(\Omega)$.

(v) Let $F : \mathbb{C} \to \mathbb{C}$ be a Lipschitz continuous function such that $F(0) = 0$. We may consider F as a function $\mathbb{R}^2 \to \mathbb{R}^2$, so that $F'(u) = DF(u)$ (which is defined for a.a. $u \in \mathbb{C}$) is a 2×2 real matrix, hence a linear operator $\mathbb{C} \to \mathbb{C}$. Let $p \in [1, \infty]$. For every $u \in W^{1,p}(\Omega)$, $F(u) \in W^{1,p}(\Omega)$ and $|\partial_i F(u)| \le L|\partial_i u|$ a.e. for every $1 \le i \le N$, where L is the Lipschitz constant of F. In particular, $\|\nabla F(u)\|_{L^p} \le L\|\nabla u\|_{L^p}$. If $p < \infty$ and if $u \in W_0^{1,p}(\Omega)$, then $F(u) \in W_0^{1,p}(\Omega)$. If we assume furthermore that F is C^1 except at a finite number of points, then $\nabla F(u) = DF(u)\nabla u$ a.e. for every $u \in W^{1,p}(\Omega)$ and the mapping $u \mapsto F(u)$ is continous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ for every $p < \infty$. On these questions, see Marcus and Mizel [237, 238, 239] and the appendix of Brezis and Cazenave [44].

(vi) In particular, if $p \in [1, \infty]$ and $u \in W^{1,p}(\Omega)$, then $|u| \in W^{1,p}(\Omega)$ and $|\nabla |u|| \le |\nabla u|$ a.e. If $p < \infty$ and $u \in W_0^{1,p}(\Omega)$, then $|u| \in W_0^{1,p}(\Omega)$. Moreover, the mapping $u \mapsto |u|$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ if $p < \infty$.

(vii) Let $F : \mathbb{C} \to \mathbb{C}$ satisfy $F(0) = 0$, and assume that there exists $\alpha \ge 0$ such that $|F(v) - F(u)| \le L(|v|^\alpha + |u|^\alpha)|v - u|$ for all $u, v \in \mathbb{C}$. Let $1 \le p, q, r \le \infty$ be such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1}{q}$. Let $u \in L^p(\Omega)$ be such that $\nabla u \in L^q(\Omega)$. It follows that $|\nabla F(u)| \le 2L|u|^\alpha |\nabla u|$ a.e., thus $\nabla F(u) \in L^r(\Omega)$ and $\|\nabla F(u)\|_{L^r} \le L\|u\|_{L^p}^\alpha \|\nabla u\|_{L^q}$. In particular, if $p = \alpha + 2$, then $F(u) \in W^{1,p'}(\Omega)$ for every $u \in W^{1,p}(\Omega)$ (respectively, $F(u) \in W_0^{1,p'}(\Omega)$ for every $u \in W_0^{1,p}(\Omega)$), and $\|\nabla F(u)\|_{L^{p'}} \le L\|u\|_{L^p}^\alpha \|\nabla u\|_{L^p}$.

(viii) If $1 \le p, q < \infty$ and m, j are nonnegative integers, then $\mathcal{D}(\mathbb{R}^N)$ is a dense subset of $W^{m,p}(\mathbb{R}^N) \cap W^{j,q}(\mathbb{R}^N)$. In particular, $W_0^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$.

We recall below some well-known inequalities and embedding results.

THEOREM 1.3.2. (Poincaré's inequality) *Assume $|\Omega| < \infty$ (or Ω is bounded in one direction) and let $1 \le p \le \infty$. There exists a constant C such that*

$$\|u\|_{L^p} \le C\|\nabla u\|_{L^p} \quad \textit{for every } u \in W_0^{1,p}(\Omega).$$

In particular, $\|\nabla u\|_{L^p(\Omega)}$ is an equivalent norm to $\|u\|_{W^{1,p}(\Omega)}$ on $W_0^{1,p}(\Omega)$.

THEOREM 1.3.3. (Sobolev's embedding theorem) *If Ω has a Lipschitz continuous boundary, then the following properties hold:*

(i) *If $1 \le p < N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [p, \frac{Np}{N-p}]$.*
(ii) *If $p = N > 1$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [p, \infty)$.*
(iii) *If $p = N = 1$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in [p, \infty]$.*
(iv) *If $p > N$, then $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.*

If Ω has a uniformly Lipschitz continuous boundary, then:

(v) *If $p > N$, then $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$, where $\alpha = \frac{p-N}{p}$.*

THEOREM 1.3.4. (Rellich's compactness theorem) *If Ω is bounded and has a Lipschitz continuous boundary, then the following properties hold:*

(i) *If $1 \le p \le N$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $q \in [p, \frac{Np}{N-p})$.*

(ii) *If $p > N$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact.*

If we assume further that Ω has a uniformly Lipschitz continuous boundary, then:

(iii) *If $p > N$, then the embedding $W^{1,p}(\Omega) \hookrightarrow C^{0,\lambda}(\overline{\Omega})$ is compact for every $\lambda \in (0, \frac{p-N}{p})$.*

THEOREM 1.3.5. *The conclusions of Theorems 1.3.3 and 1.3.4 remain valid without any smoothness assumption on Ω if one replaces $W^{1,p}(\Omega)$ by $W_0^{1,p}(\Omega)$ (note that Ω still needs to be bounded for the compact embedding).*

REMARK 1.3.6. If $p = N > 1$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $p \le q < \infty$, but $W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega)$. However, Sobolev's embedding theorem can be improved by Trudinger's inequality. In particular, if $N = 2$, then for every $M < \infty$, there exist $\mu > 0$ and $K < \infty$ such that

$$\int_\Omega (e^{\mu|u|^2} - 1) \le K$$

for every $u \in H_0^1(\Omega)$ with $\|u\|_{H^1} \le M$ (see Adams [3]).

THEOREM 1.3.7. (Gagliardo-Nirenberg's inequality) *Let $1 \le p, q, r \le \infty$ and let j, m be two integers, $0 \le j < m$. If*

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{(1-a)}{q}$$

for some $a \in [j/m, 1]$ ($a < 1$ if $r > 1$ and $m - j - \frac{N}{r} = 0$), then there exists $C(N, m, j, a, q, r)$ such that

$$\sum_{|\alpha|=j} \|D^\alpha u\|_{L^p} \le C\Big(\sum_{|\alpha|=m} \|D^\alpha u\|_{L^r}\Big)^a \|u\|_{L^q}^{1-a} \quad \textit{for every } u \in \mathcal{D}(\mathbb{R}^N).$$

For $1 \le p < \infty$ and $m \in \mathbb{N}$, one defines $W^{-m,p'}(\Omega)$ as the (topological) dual of $W_0^{m,p}(\Omega)$. One defines $H^{-m}(\Omega) = W^{-m,2}(\Omega)$, so that $H^{-m}(\Omega) = (H_0^m(\Omega))^\star$.

REMARK 1.3.8. Here are some useful properties of the spaces $W^{-m,p'}(\Omega)$.

(i) From the dense embedding $\mathcal{D}(\Omega) \hookrightarrow W_0^{m,p}(\Omega)$, we deduce that $W^{-m,p'}(\Omega)$ is a space of distributions on Ω. Furthermore, it follows from the dense embedding $W_0^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$ that $L^{p'}(\Omega) \hookrightarrow W^{-m,p'}(\Omega)$. If $p > 1$, then the embedding is dense. In particular, $\mathcal{D}(\Omega)$ is dense in $W^{-m,p'}(\Omega)$.

(ii) Assume that that $1 \le q \le \infty$ is such that $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$. It follows that $L^{q'}(\Omega) \hookrightarrow W^{-m,p'}(\Omega)$. Furthermore, if $p, q > 1$, then the embedding is dense.

(iii) Even though $H_0^m(\Omega)$ is a Hilbert space, one generally does not identify $H^{-m}(\Omega)$ with $H_0^m(\Omega)$. One rather identifies $L^2(\Omega)$ with its dual, so that

$H^{-m}(\Omega)$ becomes a subspace of $\mathcal{D}'(\Omega)$ containing $L^2(\Omega)$. In particular, if $u \in H_0^m(\Omega)$ and $v \in L^2(\Omega)$, then

$$\langle u, v\rangle_{H_0^m, H^{-m}} = \mathrm{Re} \int_\Omega u(x)\overline{v(x)}\, dx .$$

It follows that $\|u\|_{L^2}^2 \le \|u\|_{H_0^m} \|u\|_{H^{-m}}$ for all $u \in H_0^m(\Omega)$.

(iv) Like any distribution, an element of $H^{-m}(\Omega)$ can be localized. Indeed, if $T \in H^{-m}(\Omega)$ and Ω' is an open subset of Ω, then one defines $T|_{\Omega'}$ as follows. Let $\varphi \in \mathcal{D}(\Omega')$ and let $\widetilde{\varphi} \in \mathcal{D}(\Omega)$ be equal to φ on Ω' and to 0 on $\Omega \setminus \Omega'$. It follows that

$$\Psi(\varphi) = \langle \widetilde{\varphi}, T\rangle_{H_0^m(\Omega), H^{-m}(\Omega)}$$

defines a distribution $\Psi \in \mathcal{D}'(\Omega')$. Since $\|\widetilde{\varphi}\|_{H_0^m(\Omega')} \le \|\varphi\|_{H_0^m(\Omega)}$, it follows that $\Psi \in H^{-m}(\Omega')$, and one sets $T|_{\Omega'} = \Psi$. It is clear that the operator

$$P_{\Omega'} : \begin{cases} H^{-m}(\Omega) \to H^{-m}(\Omega') \\ T \mapsto T|_{\Omega'} \end{cases}$$

is linear and continuous, and is consistent with the usual restriction of functions.

(v) For every multi-index α of length j, D^α is a bounded operator from $H^{-m}(\Omega)$ to $H^{-m-j}(\Omega)$ for every $m \in \mathbb{N}$. Since also D^α is bounded from $H^k(\Omega)$ to $H^{k-j}(\Omega)$ for every $k \ge j$, it follows easily that for every $k \in \mathbb{Z}$, D^α is bounded from $H^k(\Omega)$ to $H^{k-j}(\Omega)$.

(vi) In particular, Δ defines a linear, continuous operator $H^1(\Omega) \to H^{-1}(\Omega)$. Note that for $u \in H^1(\Omega)$, the linear form $\Delta u \in H^{-1}(\Omega)$ on $H_0^1(\Omega)$ is defined by

$$\langle \Delta u, v\rangle = -\,\mathrm{Re} \int_\Omega \nabla u(x)\overline{\nabla v(x)}\, dx \quad \text{for } v \in H_0^1(\Omega) .$$

This is clear for $v \in \mathcal{D}(\Omega)$ and follows by density for $v \in H_0^1(\Omega)$.

Consider now an open interval $I \subset \mathbb{R}$ and a Banach space X, equipped with the norm $\|\ \|$. We denote by $\mathcal{D}'(I, X)$ the space of linear, continuous mappings $\mathcal{D}(I) \to X$, where X is equipped with the *weak* topology. It is called the space of X-valued distributions on I. An element $f \in L^1_{\mathrm{loc}}(I, X)$ defines a distribution $T_f \in \mathcal{D}'(I, X)$ by the formula

$$\langle T_f, \varphi\rangle = \int_I f(t)\varphi(t) dt \quad \text{for every } \varphi \in \mathcal{D}(I) .$$

One defines the n^{th} derivative $T^{(n)}$ (or $\frac{d^nT}{dt^n}$) of a distribution T by the formula

$$\langle T^n, \varphi\rangle = (-1)^n \int_I f(t) \frac{d^n\varphi(t)}{dt^n}\, dt \quad \text{for every } \varphi \in \mathcal{D}(I) .$$

For $1 \le p \le \infty$, we denote by $W^{1,p}(I, X)$ the set of (classes of) functions $f \in L^p(I, X)$ such that $f' \in L^p(I, X)$, in the sense of $\mathcal{D}'(I, X)$. For $f \in W^{1,p}(I, X)$, we set

$$\|f\|_{W^{1,p}(I,X)} = \|f\|_{L^p(I,X)} + \|f'\|_{L^p(I,X)} .$$

When there is no risk of confusion, we denote $\| \ \|_{W^{1,p}(I,X)}$ by $\| \ \|_{W^{1,p}(I)}$ or $\| \ \|_{W^{1,p}}$.

REMARK 1.3.9. The space $W^{1,p}(I,X)$ enjoys many properties of the space $W^{1,p}(I) = W^{1,p}(I,\mathbb{R})$, with essentially the same proofs. Here are some of them.

(i) $\| \ \|_{W^{1,p}(I,X)}$ is a norm on the space $W^{1,p}(I,X)$. The space $W^{1,p}(I,X)$ equipped with the norm $\| \ \|_{W^{1,p}(I,X)}$ is a Banach space.

(ii) Let $f \in L^p(I,X)$. If $f \in W^{1,p}(J,X)$ for all $J \Subset I$ and if $\|f'\|_{L^p(J,X)} \le C$ for some C independent of J, then $f \in W^{1,p}(I,X)$ and $\|f'\|_{L^p(I,X)} \le C$.

(iii) If Y is a Banach space and if $A \in \mathcal{L}(X,Y)$, then for every $f \in W^{1,p}(I,X)$, $Af \in W^{1,p}(I,Y)$, and

$$\|Af\|_{W^{1,p}(I,Y)} \le \|A\|_{\mathcal{L}(X,Y)} \|f\|_{W^{1,p}(I,X)}\,.$$

In particular, if $X \hookrightarrow Y$ and if $f \in W^{1,p}(I,X)$, then $f \in W^{1,p}(I,Y)$ (let A be the embedding).

If I is an interval of $\mathbb{R}$, one defines the space $W^{1,p}_{\mathrm{loc}}(I,X)$ as the set of functions $f : I \to X$ such that $f|_J \in W^{1,p}(J,X)$ for all open, bounded intervals $J \subset I$.

THEOREM 1.3.10. *If $1 \le p \le \infty$ and $f \in L^p(I,X)$, then the following properties are equivalent.*

(i) $f \in W^{1,p}(I,X)$.

(ii) *There exists $g \in L^p(I,X)$ such that $f(t) = f(s) + \int_s^t g(\sigma)d\sigma$ for a.a. $s,t \in I$.*

(iii) *f is weakly absolutely continuous (hence weakly differentiable a.e.) and f' (in the sense of the a.e. weak derivative) is in $L^p(I,X)$.*

In addition, if f satisfies these properties, then the derivatives of f in the senses of $\mathcal{D}'(I,X)$ and almost everywhere coincide and one may let $g = f'$ in (ii).

REMARK 1.3.11. It follows easily from the above result that

$$W^{1,1}(I,X) \hookrightarrow C_{\mathrm{b,u}}(\overline{I},X)$$

and that if $p > 1$, then $W^{1,p}(I,X) \hookrightarrow C^{0,\alpha}(\overline{I},X)$ with $\alpha = \frac{p-1}{p}$.

The following result is also quite useful.

PROPOSITION 1.3.12. *Assume X is reflexive and let $f \in L^p(I,X)$. It follows that $f \in W^{1,p}(I,X)$ iff there exist $\varphi \in L^p(I)$ and a set N of measure 0 such that*

$$\|f(t) - f(s)\| \le \left| \int_s^t \varphi(\sigma)d\sigma \right| \quad \textit{for all } t,s \in I \setminus N\,.$$

In this case, $\|f'\|_{L^p(I,X)} \le \|\varphi\|_{L^p(I)}$.

REMARK 1.3.13. Applying Proposition 1.3.12, one can show the following results:

(i) Assume that X is reflexive and let $f : I \to X$ be Lipschitz continuous and bounded. It follows that $f \in W^{1,\infty}(I,X)$ and $\|f'\|_{L^\infty(I,X)} \le L$, where L is the Lipschitz constant of f.

(ii) Assume that X is reflexive and that $1 < p \le \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of $W^{1,p}(I,X)$ and let $f: I \to X$ be such that $f_n(t) \rightharpoonup f(t)$ in X as $n \to \infty$ for a.a. $t \in I$. It follows that $f \in W^{1,p}(I,X)$ and $\|f\|_{W^{1,p}(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{W^{1,p}(I,X)}$.

(iii) Assume that X is reflexive, $1 < p \le \infty$, and let $f \in L^p(I,X)$. If $\exists K$ such that for all $J \Subset I$ and all $|h| < \operatorname{dist}(\overline{J}, \partial I)$, $\|f(\cdot + h) - f(\cdot)\|_{L^p(J,X)} \le K|h|$, then $f \in W^{1,p}(I,X)$ and $\|f'\|_{L^p(I,X)} \le K$.

PROPOSITION 1.3.14. *Let I be a bounded interval of $\mathbb{R}$, let m be a nonnegative integer, let Ω be an open subset of $\mathbb{R}^N$, and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(I,H_0^1(\Omega)) \cap W^{1,\infty}(I,H^{-m}(\Omega))$.*

(i) *There exist $f \in L^\infty(I,H_0^1(\Omega)) \cap W^{1,\infty}(I,H^{-m}(\Omega))$ and some subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for every $t \in \overline{I}$, $f_{n_k}(t) \rightharpoonup f(t)$ in $H_0^1(\Omega)$ as $k \to \infty$.*

(ii) *If $\|f_{n_k}(t)\|_{L^2} \to \|f(t)\|_{L^2}$ as $k \to \infty$, uniformly on I, then also $f_{n_k} \to f$ in $C(\overline{I}, L^2(\Omega))$ as $k \to \infty$.*

(iii) *If $(f_n)_{n\in\mathbb{N}} \subset C(\overline{I},H_0^1(\Omega))$ and $\|f_{n_k}(t)\|_{H^1} \to \|f(t)\|_{H^1}$ as $k \to \infty$, uniformly on I, then also $f \in C(\overline{I},H_0^1(\Omega))$ and $f_{n_k} \to f$ in $C(\overline{I},H_0^1(\Omega))$ as $k \to \infty$.*

PROOF. Part (i) follows from Proposition 1.1.2(i) applied with $X = H_0^1(\Omega)$ and $Y = H^{-m}(\Omega)$ and from Remark 1.3.13(ii) (note that $(f_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous in Y by Remark 1.3.11). Part (ii) follows from Proposition 1.1.2(ii) applied with $X = H_0^1(\Omega)$, $Y = H^{-m}(\Omega)$, and $B = L^2(\Omega)$. Part (iii) follows from Proposition 1.1.2(ii) applied with $X = B = H_0^1(\Omega)$ and $Y = H^{-m}(\Omega)$. □

One can define higher-order vector-valued Sobolev spaces as follows: For $m \in \mathbb{N}$, set

$$W^{m,p}(I,X) = \left\{ f \in L^p(I,X) : \frac{d^j f}{dt^j} \in L^p(I,X) \text{ for all } j \in \{1,\dots,m\} \right\}.$$

It is clear that

$$W^{m,p}(I,X) = \left\{ f \in W^{1,p}(I,X) : \frac{d^j f}{dt^j} \in W^{1,p}(I,X) \text{ for all } j \in \{1,\dots,m-1\} \right\},$$

so that $W^{m,1}(I,X) \hookrightarrow C^{m-1}_{\mathrm{b,u}}(\overline{I},X)$ and $W^{m,p}(I,X) \hookrightarrow C^{m-1,\alpha}(\overline{I},X)$ with $\alpha = \frac{p-1}{p}$, if $p > 1$.

1.4. Sobolev and Besov Spaces on $\mathbb{R}^N$

For more detail, see, for example, Adams [3], Bergh and Löfström [28], the appendix to Ginibre and Velo [140], Lemarié-Rieusset [225], Shatah and Struwe [312], and Triebel [337, 338].

It is convenient to consider a function $\eta \in C_c^\infty(\mathbb{R}^N)$ such that

$$\eta(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1 \\ 0 & \text{if } |\xi| \ge 2, \end{cases} \tag{1.4.1}$$

and to define the sequence $(\psi_j)_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^N)$ by

$$\psi_j(\xi) = \eta\Big(\frac{\xi}{2^j}\Big) - \eta\Big(\frac{\xi}{2^{j-1}}\Big) \tag{1.4.2}$$

in order to define the Littlewood-Paley decomposition. We see that

$$\operatorname{supp} \psi_j \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\}$$

and that

$$\sum_{j=-\infty}^{+\infty} \psi_j(\xi) = \begin{cases} 1 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0, \end{cases}$$

where the above sum contains at most two nonzero terms.

Given $s \in \mathbb{R}$, one defines

$$H^s(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N) : (1+|\xi|^2)^{\frac{s}{2}} \widehat{u} \in L^2(\mathbb{R}^N)\}$$

and

$$\|u\|_{H^s} = \|(1+|\xi|^2)^{\frac{s}{2}} \widehat{u}\|_{L^2}.$$

It is clear that $H^s(\mathbb{R}^N)$ is a Hilbert space and it follows easily from Plancherel's formula that the above definitions are consistent with those of Section 1.3. More generally, one defines

$$H^{s,p}(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N) : \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}} \widehat{u}] \in L^p(\mathbb{R}^N)\}$$

and

$$\|u\|_{H^{s,p}} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}} \widehat{u}]\|_{L^p}$$

for $1 \le p \le \infty$ and $s \in \mathbb{R}$, so that $H^{s,p}(\mathbb{R}^N)$ is a Banach space (reflexive if $1 < p < \infty$).

REMARK 1.4.1. Here are some fundamental properties of the space $H^{s,p}(\mathbb{R}^N)$.

(i) $H^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$ and $H^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ (same norms).

(ii) $H^{s_1,p}(\mathbb{R}^N) \hookrightarrow H^{s_2,p}(\mathbb{R}^N)$ if $s_1 \ge s_2$.

(iii) If $p < \infty$, then $[H^{s,p}(\mathbb{R}^N)]^\star = H^{-s,p'}(\mathbb{R}^N)$ (see corollary 6.2.8 in [28]).

(iv) It follows from Mihlin's multiplier theorem (see theorem 6.1.6 in [28]) that if m is a nonnegative integer and $1 < p < \infty$, then $W^{m,p}(\mathbb{R}^N) = H^{m,p}(\mathbb{R}^N)$ with equivalent norms. By (iii), we also have $W^{m,p}(\mathbb{R}^N) = H^{m,p}(\mathbb{R}^N)$ when m is a negative integer.

(v) Sobolev's embedding: $H^{s,p}(\mathbb{R}^N) \hookrightarrow H^{s_1,p_1}(\mathbb{R}^N)$ if $s - N/p = s_1 - N/p_1$ and $1 < p \le p_1 < \infty$, $s_1, s_2 \in \mathbb{R}$ (see theorem 6.5.1 in [28]). In particular, if $1 \le p < \infty$ and $0 < s < N/p$, then

$$H^{s,p}(\mathbb{R}^N) \hookrightarrow L^{\frac{pN}{N-sp}}(\mathbb{R}^N).$$

Moreover, $H^{s,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ if $p > 1$ and $s > N/p$. See remark 2, p. 206 in [337].

We now define the Besov space $B^s_{p,q}(\mathbb{R}^N)$ for $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ by

$$B^s_{p,q}(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N) : \|u\|_{B^s_{p,q}} < \infty\},$$

with

$$\|u\|_{B^s_{p,q}} = \|\mathcal{F}^{-1}(\eta \widehat{u})\|_{L^p(\mathbb{R}^N)} + \begin{cases} \left(\sum_{j=1}^{\infty} \left(2^{sj} \|\mathcal{F}^{-1}(\psi_j \widehat{u})\|_{L^p(\mathbb{R}^N)}\right)^q \right)^{1/q} & \text{if } q < \infty \\ \sup_{j \ge 1} 2^{sj} \|\mathcal{F}^{-1}(\psi_j \widehat{u})\|_{L^p(\mathbb{R}^N)} & \text{if } q = \infty, \end{cases}$$

where the functions η and ψ_j satisfy (1.4.1)–(1.4.2). It is understood that if $f \in \mathcal{S}'(\mathbb{R}^N)$ and $1 \le p \le \infty$, then $\|f\|_{L^p}$ is the L^p norm of f if $f \in L^p(\mathbb{R}^N)$ and is $+\infty$ otherwise.

REMARK 1.4.2. Here are some fundamental properties of the space $B^s_{p,q}(\mathbb{R}^N)$.

(i) $B^s_{p,q}(\mathbb{R}^N)$ is independent of the choice of the function η satisfying (1.4.1) and two different choices of η yield two equivalent norms.
(ii) $B^s_{p,q}(\mathbb{R}^N)$ is a Banach space. Moreover, $B^s_{p,q}(\mathbb{R}^N)$ is isomorphic to a closed subspace of $\ell^q(\mathbb{N}, L^p(\mathbb{R}^N))$, so that $B^s_{p,q}(\mathbb{R}^N)$ is reflexive if $1 < p, q < \infty$.
(iii) $B^{s_1}_{p,q}(\mathbb{R}^N) \hookrightarrow B^{s_2}_{p,q}(\mathbb{R}^N)$ if $s_1 \ge s_2$; and $B^s_{p,q_1}(\mathbb{R}^N) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^N)$ if $1 \le q_1 \le q_2 \le \infty$.
(iv) If $p, q < \infty$, then $[B^s_{p,q}(\mathbb{R}^N)]^\star = B^{-s}_{p',q'}(\mathbb{R}^N)$ (see Corollary 6.2.8 in [28]).
(v) Sobolev's embedding (see theorem 6.5.1 in [28]): $B^s_{p,q}(\mathbb{R}^N) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^N)$ if $s - N/p = s_1 - N/p_1$ and $1 \le p \le p_1 \le \infty$, $1 \le q \le q_1 \le \infty$, $s_1, s_2 \in \mathbb{R}$.

REMARK 1.4.3. Here are some relations between the spaces $H^{s,p}(\mathbb{R}^N)$ and $B^s_{p,q}(\mathbb{R}^N)$ (see theorem 6.4.5 in [28]).

(i) If $1 < p \le 2$, then $B^s_{p,p}(\mathbb{R}^N) \hookrightarrow H^{s,p}(\mathbb{R}^N) \hookrightarrow B^s_{p,2}(\mathbb{R}^N)$.
(ii) If $2 \le p < \infty$, then $B^s_{p,2}(\mathbb{R}^N) \hookrightarrow H^{s,p}(\mathbb{R}^N) \hookrightarrow B^s_{p,p}(\mathbb{R}^N)$.
(iii) In particular, $B^s_{2,2}(\mathbb{R}^N) = H^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$.

We now introduce the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^N)$ and $\dot{H}^{s,p}(\mathbb{R}^N)$ and the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^N)$. In fact, they are rather delicate to define, since they can be considered either as seminormed spaces or as quotient spaces. It will be sufficient for our purpose to define only the (semi-) norms.

Let η and ψ_j satisfy (1.4.1)–(1.4.2) and let $1 \le p \le \infty$ and $s \in \mathbb{R}$. Given $u \in \mathcal{S}'(\mathbb{R}^N)$, we set

$$\|u\|_{\dot{H}^{s,p}} = \Big\| \sum_{j=-\infty}^{+\infty} \mathcal{F}^{-1}(|\xi|^s \psi_j \widehat{u}) \Big\|_{L^p(\mathbb{R}^N)}$$

if the series $\sum_{j=-\infty}^{+\infty} \mathcal{F}^{-1}(|\xi|^s \psi_j \widehat{u})$ is convergent in $\mathcal{S}'(\mathbb{R}^N)$ to a function of $L^p(\mathbb{R}^N)$, and $\|u\|_{\dot{H}^{s,p}} = \infty$ otherwise. We define

$$\|u\|_{\dot{H}^s} = \|u\|_{\dot{H}^{s,2}}\,.$$

We note that $|\xi|^s \psi_j \widehat{u} \in \mathcal{S}'(\mathbb{R}^N)$ so that the definition makes sense. Moreover, if $\widehat{u}$ vanishes in a neighborhood of the origin, then $|\xi|^s \widehat{u} \in \mathcal{S}'(\mathbb{R}^N)$, and so $\|u\|_{\dot{H}^{s,p}} = \|\mathcal{F}^{-1}(|\xi|^s \widehat{u})\|_{L^p}$. Finally, we note that $\|u\|_{\dot{H}^{s,p}} = 0$ if and only if $\operatorname{supp} \widehat{u} = \{0\}$; i.e., u is a polynomial.

Next, given $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we define for $u \in \mathcal{S}'(\mathbb{R}^N)$,

$$\|u\|_{\dot{B}^s_{p,q}} = \begin{cases} \Big(\displaystyle\sum_{j=-\infty}^{+\infty} (2^{sj} \|\mathcal{F}^{-1}(\psi_j \widehat{u})\|_{L^p(\mathbb{R}^N)})^q \Big)^{1/q} & \text{if } q < \infty \\ \displaystyle\sup_{j \in \mathbb{Z}} 2^{sj} \|\mathcal{F}^{-1}(\psi_j \widehat{u})\|_{L^p(\mathbb{R}^N)} & \text{if } q = \infty. \end{cases}$$

We again note that $\|u\|_{\dot{B}^s_{p,q}} = 0$ if and only if $\operatorname{supp} \widehat{u} = \{0\}$; i.e., u is a polynomial.

REMARK 1.4.4. Here are some fundamental properties of the semi-norms of $H^{s,p}(\mathbb{R}^N)$ and $B^s_{p,q}(\mathbb{R}^N)$.

(i) The semi-norms $\|\cdot\|_{\dot{H}^{s,p}}$ and $\|\cdot\|_{\dot{B}^s_{p,q}}$ are independent of the choice of the function η satisfying (1.4.1) in the sense that two different choices of η yield two equivalent semi-norms.

(ii) If $s > 0$, then $\|u\|_{B^s_{p,q}} \approx \|u\|_{L^p} + \|u\|_{\dot{B}^s_{p,q}}$ (see theorem 6.3.2 in [28]).

(iii) If $0 < s < 1$, then

$$\|u\|_{\dot{B}^s_{p,q}} \approx \begin{cases} \left(\int_0^\infty (t^{-s} \sup_{|y|\le t} \|u(\cdot - y) - u(\cdot)\|_{L^p(\mathbb{R}^N)})^q \frac{dt}{t} \right)^{1/q} & \text{if } q < \infty \\ \sup_{t>0} t^{-s} \sup_{|y|\le t} \|u(\cdot - y) - u(\cdot)\|_{L^p(\mathbb{R}^N)} & \text{if } q = \infty. \end{cases}$$

There are also similar (though more complicated) formulas for $s \ge 1$ (see theorem 6.3.1 in [28]).

1.5. Elliptic Equations

Consult, for example, Agmon, Douglis, and Nirenberg [4], Brezis [43], Brezis and Cazenave [44], Gilbarg and Trudinger [127], J.-L. Lions [231], Lions and Magenes [232], and Nirenberg [272].

We recall below some of the results that we will use in the following sections. In all this section we consider an open subset $\Omega \subset \mathbb{R}^N$. We equip $H^{-1}(\Omega)$ with the dual norm, that is

$$\|u\|_{H^{-1}} = \sup \left\{ \langle u, v \rangle, v \in H^1_0(\Omega), \|v\|_{H^1_0} = 1 \right\}.$$

We recall that (by Lax-Milgram's lemma) for every $f \in H^{-1}(\Omega)$, there exists a unique element $u \in H^1_0(\Omega)$ such that

$$-\Delta u + u = f \quad \text{in } H^{-1}(\Omega).$$

In addition,

$$\|f\|_{H^{-1}} = \|u\|_{H^1_0}.$$

It follows in particular that $\Delta - I$ defines an isometry from $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$.

By the same method, one shows also that for every $\lambda > 0$ and every $f \in H^{-1}(\Omega)$, there exists a unique element $u \in H^1_0(\Omega)$ such that

$$-\Delta u + \lambda u = f \quad \text{in } H^{-1}(\Omega).$$

$|||f||| = \|u\|_{H^1_0(\Omega)}$ defines an equivalent norm on $H^{-1}(\Omega)$ and $\lambda \|u\|_{H^{-1}} \le \|f\|_{H^{-1}}$. If $f \in L^2(\Omega)$, then $\Delta u \in L^2(\Omega)$, the equation makes sense in $L^2(\Omega)$, and $\lambda \|u\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}$.

One shows also that if Ω has a C^2 boundary and if $f \in L^2(\Omega)$, then $u \in H^2(\Omega)$ and $\|u\|_{H^2} \le C\|f\|_{L^2}$. In particular, $-\Delta + I$ is an isomorphism from $H^2(\Omega) \cap H^1_0(\Omega)$ onto $L^2(\Omega)$. Concerning L^p estimates, we have the following result.

PROPOSITION 1.5.1. *Let $\lambda > 0$, $u \in H^1_0(\Omega)$, and $f \in H^{-1}(\Omega)$ satisfy $-\Delta u + \lambda u = f$. If $f \in L^p(\Omega)$ for some $p \in [1, \infty)$, then $u \in L^p(\Omega)$ and $\lambda \|u\|_{L^p} \le \|f\|_{L^p}$.*

PROOF. Let $\varphi : (0,\infty) \to [0,\infty)$ be smooth. Assume further that $\varphi(s)$ and $s\varphi'(s)$ are bounded on $(0,\infty)$ and that $\varphi, \varphi' \geq 0$. It follows that $\varphi(|u|)u \in H_0^1(\Omega)$ for all $H_0^1(\Omega)$ (see Remark 1.3.1 (v)). Moreover, one easily verifies the following identity.

$$\operatorname{Re}(\nabla u \cdot \nabla(\varphi(|u|)\overline{u})) = \varphi(|u|)|\nabla u|^2 + \frac{\varphi'(|u|)}{|u|}|\operatorname{Re}(u\overline{\nabla u})|^2 \quad \text{a.e.}$$

In particular, $\langle -\Delta u, \varphi(|u|)u\rangle_{H^{-1},H_0^1} \geq 0$. Taking the $H^{-1} - H^1$ product of the equation $-\Delta u + \lambda u = f$ with $\varphi(|u|)u$, we deduce that

$$\lambda \int_\Omega |u|^2\varphi(|u|) \leq \int_\Omega |f||u|\varphi(|u|).$$

If $\varphi(s) \leq s^{p-2}$ for some $p \in [1,\infty)$, then $|u|\varphi(|u|) \leq (|u|^2\varphi(|u|))^{\frac{p-1}{p}}$; and so, by Hölder's inequality,

$$\lambda^p \int_\Omega |u|^2\varphi(|u|) \leq \int_\Omega |f|^p. \tag{1.5.1}$$

For $p \leq 2$ and $\varepsilon > 0$, let $\varphi(s) = (\varepsilon + s^2)^{\frac{p-2}{2}}$. By (1.5.1),

$$\lambda^p \int_\Omega |u|^2(\varepsilon + |u|^2)^{\frac{p-2}{2}} \leq \int_\Omega |f|^p.$$

Letting $\varepsilon \downarrow 0$ and applying Fatou's lemma, we see that $u \in L^p$ and $\lambda\|u\|_{L^p} \leq \|f\|_{L^p}$. For $2 \leq p < \infty$ and $\varepsilon > 0$, let

$$4\varphi(s) = \left(\frac{s^2}{1+\varepsilon s^2}\right)^{\frac{p-2}{2}}.$$

It follows from (1.5.1) that

$$\lambda^p \int_\Omega \frac{|u|^p}{(1+\varepsilon|u|^2)^{\frac{p-2}{2}}} \leq \int_\Omega |f|^p.$$

Letting $\varepsilon \downarrow 0$ and applying Fatou's lemma, we obtain $u \in L^p$ and $\lambda\|u\|_{L^p} \leq \|f\|_{L^p}$. □

Next, we recall some convergence results. Given $\varepsilon > 0$, we define the operator J_ε on $H^{-1}(\Omega)$ by

$$J_\varepsilon u = (I - \varepsilon\Delta)^{-1}.$$

In other words, for every $f \in H^{-1}(\Omega)$, $u_\varepsilon = J_\varepsilon f \in H_0^1(\Omega)$ is the unique solution of $u_\varepsilon - \varepsilon\Delta u_\varepsilon = f$. We deduce from what precedes that $\|J_\varepsilon f\|_X \leq \|f\|_X$ whenever $X = H_0^1(\Omega), L^2(\Omega), H^{-1}(\Omega)$, or $X = L^p(\Omega)$ for $1 \leq p < \infty$. In particular, J_ε can be extended by continuity to an operator of $\mathcal{L}(X)$ with $\|J_\varepsilon\|_{\mathcal{L}(X)} \leq 1$. Furthermore, we have the following result.

PROPOSITION 1.5.2. *If X is either of the spaces $H_0^1(\Omega)$, $L^2(\Omega)$, $H^{-1}(\Omega)$, or $L^p(\Omega)$ for $1 < p < \infty$, then:*

(i) *$\langle J_\varepsilon f, g\rangle_{X,X^\star} = \langle f, J_\varepsilon g\rangle_{X,X^\star}$ for all $f \in X$, $g \in X^\star$.*
(ii) *$J_\varepsilon f \to f$ in X as $\varepsilon \downarrow 0$ for every $f \in X$.*
(iii) *If f_ε is bounded in X as $\varepsilon \downarrow 0$, then $J_\varepsilon f_\varepsilon - f_\varepsilon \rightharpoonup 0$ in X as $\varepsilon \downarrow 0$.*

PROOF. (i) Let $f, g \in \mathcal{D}(\Omega)$ and let $u = J_\varepsilon f$, $v = J_\varepsilon g$. It follows that

$$\begin{aligned}\langle J_\varepsilon f, g\rangle_{X,X^\star} &= \langle J_\varepsilon f, g\rangle_{L^2} = \langle u, -\varepsilon\Delta v + v\rangle_{L^2} \\ &= \langle -\varepsilon\Delta u + u, v\rangle_{L^2} = \langle f, J_\varepsilon g\rangle_{L^2} = \langle f, J_\varepsilon g\rangle_{X,X^\star}.\end{aligned}$$

(i) follows by density of $\mathcal{D}(\Omega)$ in X.

(ii) By density, we may assume $f \in \mathcal{D}(\Omega)$. Let $u_\varepsilon = J_\varepsilon f$. One easily verifies that

$$u_\varepsilon - f = J_\varepsilon(f - (I - \varepsilon\Delta)f) = \varepsilon J_\varepsilon \Delta f\,,$$

and so

$$\|u_\varepsilon - f\|_{H^{-1}} \le \varepsilon\|\Delta f\|_{H^{-1}} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0\,.$$

Hence (ii) is proven for $X = H^{-1}(\Omega)$. Furthermore, u_ε is bounded in $H^1_0(\Omega)$, and it follows from Remark 1.3.8 (iii) that

$$\|u_\varepsilon - f\|^2_{L^2} \le \|u_\varepsilon - f\|_{H^1}\|u_\varepsilon - f\|_{H^{-1}} \le C\varepsilon^{\frac{1}{2}} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0\,.$$

Hence (ii) is proven for $X = L^2(\Omega)$. Finally, one easily verifies that $(I - \Delta)u_\varepsilon = J_\varepsilon(I-\Delta)f$. Therefore, from the result for $X = H^{-1}(\Omega)$, we deduce that $(I-\Delta)u_\varepsilon \to (I-\Delta)f$ in $H^{-1}(\Omega)$, which implies that $u_\varepsilon \to u$ in $H^1_0(\Omega)$. Hence (ii) is established for $X = H^1_0(\Omega)$. Finally, let $1 \le q < p < r < \infty$. It follows from Hölder's inequality that

$$\|u_\varepsilon - f\|_{L^p} \le \|u_\varepsilon - f\|_{L^q}^{\frac{q(r-p)}{p(r-q)}}\|u_\varepsilon - f\|_{L^r}^{\frac{r(p-q)}{p(r-q)}}\,.$$

Note that $\|u_\varepsilon - f\|_{L^r} \le \|u_\varepsilon\|_{L^r} + \|f\|_{L^r} \le 2\|f\|_{L^r}$. If $p > 2$, we let $q = 2$ and we obtain

$$\|u_\varepsilon - f\|_{L^p} \le \|u_\varepsilon - f\|_{L^2}^{\frac{2(r-p)}{p(r-2)}}(2\|f\|_{L^r})^{\frac{r(p-2)}{p(r-2)}} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0\,.$$

If $p < 2$, we let $r = 2$ and we obtain a similar conclusion. This completes the proof of (ii)

(iii) Let $u_\varepsilon = J_\varepsilon f_\varepsilon$. We know that u_ε is bounded in X; and so it suffices to show that $u_\varepsilon - f_\varepsilon \to 0$ in $\mathcal{D}'(\Omega)$. Given $\varphi \in \mathcal{D}(\Omega)$,

$$\langle u_\varepsilon - f_\varepsilon, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \varepsilon\langle u_\varepsilon, \Delta\varphi\rangle_{\mathcal{D}',\mathcal{D}} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0\,.$$

Hence the result follows. □

Suppose now $\Omega = \mathbb{R}^N$. Applying the Fourier transform, we see that $u = J_\varepsilon f$, being the solution of $u - \varepsilon\Delta u = f$, is given by $(1 + 4\varepsilon\pi^2|\xi|^2)\widehat{u} = \widehat{f}$. In particular, J_ε can be extended to an operator $\mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$. We have the following result.

PROPOSITION 1.5.3. *Suppose $\Omega = \mathbb{R}^N$ and let $J_\varepsilon = (I - \varepsilon\Delta)^{-1}$ for $\varepsilon > 0$. Given any $s \in \mathbb{R}$, it follows that J_ε is a contraction of $H^s(\mathbb{R}^N)$ and that $J_\varepsilon \in \mathcal{L}(H^s(\mathbb{R}^N), H^{s+2}(\mathbb{R}^N))$ with $\|J_\varepsilon\|_{\mathcal{L}(H^s,H^{s+2})} \le \max\{1, (4\varepsilon\pi^2)^{-1}\}$.*

PROOF. Let $f \in \mathcal{S}'(\mathbb{R}^N)$ and $u = J_\varepsilon f$, so that $\widehat{u} = (1 + 4\varepsilon\pi^2|\xi|^2)^{-1}\widehat{f}$. We see that $|\widehat{u}| \le |\widehat{f}|$ and $(1 + |\xi|^2)|\widehat{u}| \le (1 + |\xi|^2)(1 + 4\varepsilon\pi^2|\xi|^2)^{-1}|\widehat{f}| \le \max\{1, (4\varepsilon\pi^2)^{-1}\}|\widehat{f}|$. The result follows from the definition of $H^s(\mathbb{R}^N)$ (see Section 1.4). □

1.6. Semigroups of Linear Operators

Consult, for example, Brezis [43], Brezis and Cazenave [44], Cazenave and Haraux [64, 65], Haraux [158], and Pazy [294].

Let X be a complex Hilbert space with norm $\|\cdot\|_X$ and sesquilinear form $\langle\cdot,\cdot\rangle_X$. We consider X as a *real* Hilbert space with the scalar product $(x,y)_X = \operatorname{Re}\langle x,y\rangle_X$.

Let $A : D(A) \subset X \to X$ be a $\mathbb{C}$-linear operator. Assume that A is self-adjoint (so that $D(A)$ is a dense subset of X) and that $A \le 0$ (i.e., $(Ax,x) \le 0$ for all $x \in D(A)$). A generates a self-adjoint semigroup of contractions $(S(t))_{t\ge 0}$ on X. $D(A)$ is a Hilbert space when equipped with the scalar product

$$(x,y)_{D(A)} = (Ax,Ay)_X + (x,y)_X$$

corresponding to the norm $\|u\|^2_{D(A)} = \|Au\|^2_X + \|u\|^2_X$ and $D(A) \hookrightarrow X \hookrightarrow (D(A))^\star$, all the embeddings being dense. We denote by X_A the completion of $D(A)$ for the norm $\|x\|^2_{X_A} = \|x\|^2_X - (Ax,x)_X$. X_A is also a Hilbert space with the scalar product defined by $(x,y)_A = (x,y)_X - (Ax,y)_X$ for $x,y \in D(A)$. It follows that

$$D(A) \hookrightarrow X_A \hookrightarrow X \hookrightarrow X_A^\star \hookrightarrow (D(A))^\star\,,$$

all the embeddings being dense. Furthermore, it is easily shown that A can be extended to a self-adjoint operator $\overline{A}$ on $(D(A))^\star$ with domain X. We have $\overline{A}|_{D(A)} = A$, $\overline{A}|_{D(A)} \in \mathcal{L}(D(A),X)$, $\overline{A}|_{X_A} \in \mathcal{L}(X_A, X_A^\star)$, and $\overline{A}|_X \in \mathcal{L}(X,(D(A))^\star)$.

Since A is self-adjoint, $iA : D(A) \subset X \to X$ defined by $(iA)x = iAx$ for $x \in D(A)$ is also $\mathbb{C}$-linear and is skew-adjoint. In particular, iA generates a group of isometries $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on X. We deduce easily from the skew-adjointness of iA that

$$\mathcal{T}(t)^\star = \mathcal{T}(-t) \quad \text{for every } t \in \mathbb{R}\,.$$

We know that for every $x \in D(A)$, $u(t) = \mathcal{T}(t)x$ is the unique solution of the problem

$$\begin{cases} u \in C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, X), \\ i\dfrac{du}{dt} + Au = 0 \quad \text{for all } t \in \mathbb{R}, \\ u(0) = x. \end{cases}$$

Moreover,

$$\|u(t+h) - u(t)\| \le |h|\|Ax\| \quad \text{for all } t, x \in \mathbb{R}\,.$$

Next, it follows easily from the preceding observations that $(\mathcal{T}(t))_{t\in\mathbb{R}}$ can be extended to a group of isometries $(\overline{\mathcal{T}}(t))_{t\in\mathbb{R}}$ on $(D(A))^\star$, which is the group generated by the skew-adjoint operator $i\overline{A}$. $\overline{\mathcal{T}}(t)$ coincides with $\mathcal{T}(t)$ on X, and $(\overline{\mathcal{T}}(t))_{t\in\mathbb{R}}$ restricted to any of the spaces $X_A^\star$, X, X_A, $D(A)$ is a group of isometries. For convenience, we use the same notation for $\mathcal{T}(t)$ and $\overline{\mathcal{T}}(t)$. We know that for every $x \in X$, $u(t) = \mathcal{T}(t)x$ is the unique solution of the problem

$$\begin{cases} u \in C(\mathbb{R}, X) \cap C^1(\mathbb{R}, (D(A))^\star), \\ i\dfrac{du}{dt} + \overline{A}u = 0 \quad \text{for all } t \in \mathbb{R}, \\ u(0) = x. \end{cases}$$

In addition, the following regularity properties hold:

$$\begin{aligned} &\text{If } x \in X_A, && \text{then } u \in C(\mathbb{R}, X_A) \cap C^1(\mathbb{R}, X_A^\star)\,; \\ &\text{if } x \in D(A), && \text{then } u \in C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, X)\,. \end{aligned}$$

Concerning the nonhomogeneous problem, we recall that for every $x \in X$ and every $f \in C([0,T], X)$ (where $T \in \mathbb{R}$), there exists a unique solution of the problem

$$\begin{cases} u \in C([0,T], X) \cap C^1([0,T], (D(A))^\star), \\ i\dfrac{du}{dt} + \overline{A}u + f = 0 \quad \text{for all } t \in [0,T], \\ u(0) = x. \end{cases} \tag{1.6.1}$$

Indeed, $u \in C([0,T], X)$ is a solution of the above problem if and only if u satisfies

$$u(t) = \mathcal{T}(t)x + i\int_0^t \mathcal{T}(t-s)f(s)ds \quad \text{for all } t \in [0,T]\,. \tag{1.6.2}$$

Formula (1.6.2) is known as Duhamel's formula. It is well known that if, in addition, $f \in W^{1,1}((0,T), X)$ or $f \in L^1((0,T), D(A))$, then

$$u \in C([0,T], D(A)) \cap C^1([0,T], X)\,.$$

REMARK 1.6.1. For every $x \in (D(A))^\star$ and $f \in L^1((0,T), (D(A))^\star)$, (1.6.2) defines a function $u \in C([0,T], (D(A))^\star)$. A natural question to ask is under what additional conditions u satisfies an equation of the type (1.6.1). Here are some answers.

(i) If, in addition, $x \in X$ and $u \in W^{1,1}((0,T), (D(A))^\star))$ or $u \in L^1((0,T), X)$, then u satisfies (1.6.2) if and only if u satisfies

$$\begin{cases} u \in L^1((0,T), X) \cap W^{1,1}((0,T), (D(A))^\star), \\ i\dfrac{du}{dt} + \overline{A}u + f = 0 \quad \text{a.e. on } [0,T], \\ u(0) = x. \end{cases}$$

(ii) If $x \in X$ and $f \in C([0,T], (D(A))^\star)$, and if $u \in C^1([0,T], (D(A))^\star))$ or $u \in C([0,T], X)$, then u satisfies (1.6.2) if and only if u satisfies (1.6.1).

(iii) Similarly, if $x \in X_A$, $f \in L^1([0,T], X_A^\star)$ and if $u \in W^{1,1}((0,T), X_A^\star))$ or if $u \in L^1((0,T), X_A)$, then u satisfies (1.6.2) if and only if u satisfies

$$\begin{cases} u \in L^1((0,T), X_A) \cap W^{1,1}((0,T), X_A^\star), \\ i\dfrac{du}{dt} + \overline{A}u + f = 0 \quad \text{a.e. on } [0,T], \\ u(0) = x. \end{cases}$$

(iv) Let $x \in X_A$ and $f \in C([0,T],X_A^\star)$. If $u \in C^1([0,T],X_A^\star))$ or if $u \in C([0,T],X_A)$, then u satisfies (1.6.2) if and only if u satisfies

$$\begin{cases} u \in C([0,T],X_A) \cap C^1([0,T],X_A^\star), \\ i\dfrac{du}{dt} + \overline{A}u + f = 0 \quad \text{for all } t \in [0,T], \\ u(0) = x. \end{cases}$$

(v) Also, if $x \in D(A)$ and $f \in L^1([0,T],X)$, and if $u \in W^{1,1}((0,T),X))$ or if $u \in L^1((0,T),D(A))$, then u satisfies (1.6.2) if and only if u satisfies

$$\begin{cases} u \in L^1((0,T),D(A)) \cap W^{1,1}((0,T),X), \\ i\dfrac{du}{dt} + Au + f = 0 \quad \text{a.e. on } [0,T], \\ u(0) = x. \end{cases}$$

(vi) Suppose $x \in D(A)$ and $f \in C([0,T],X)$. If $u \in C^1([0,T],X))$ or $u \in C([0,T],D(A))$, then u satisfies (1.6.2) if and only if u satisfies

$$\begin{cases} u \in C([0,T],D(A)) \cap C^1([0,T],X), \\ i\dfrac{du}{dt} + Au + f = 0 \quad \text{for all } t \in [0,T], \\ u(0) = x. \end{cases}$$

1.7. Some Compactness Tools

It is well known that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is not compact. In order to pass to the limit in certain problems, we will use some specific tools that take into account the lack of compactness. The first one is due to W. Strauss [323] (see also Berestycki and Lions [25]).

PROPOSITION 1.7.1. *Let $(u_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$ be a bounded sequence of spherically symmetric functions. If $N \geq 2$ or if $u_n(x)$ is a nonincreasing function of $|x|$ for every $n \geq 0$, then there exist a subsequence $(u_{n_k})_{k\geq 0}$ and $u \in H^1(\mathbb{R}^N)$ such that $u_{n_k} \to u$ as $k \to \infty$ in $L^p(\mathbb{R}^N)$ for every $2 < p < 2N/(N-2)$ ($2 < p \leq \infty$ if $N = 1$).*

Proposition 1.7.1 is an immediate consequence of the following two lemmas.

LEMMA 1.7.2. *Let $(u_n)_{n\geq 0}$ be a bounded sequence in $H^1(\mathbb{R}^N)$. Suppose $u_n(x) \to 0$ as $|x| \to \infty$, uniformly in $n \geq 0$. It follows that there exist a subsequence $(u_{n_k})_{k\geq 0}$ and $u \in H^1(\mathbb{R}^N)$ such that $u_{n_k} \to u$ as $k \to \infty$ in $L^p(\mathbb{R}^N)$ for every $2 < p < \frac{2N}{N-2}$ ($2 < p \leq \infty$ if $N = 1$).*

PROOF. It follows from Remark 1.3.1(iii) that there exist $u \in H^1(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\geq 0}$ such that $u_{n_k} \rightharpoonup u$ as $n \to \infty$ in $H^1(\mathbb{R}^N)$. Fix $\varepsilon > 0$ and let $R > 0$ to be chosen later. Given p as in the statement, we have

$$\begin{aligned} \|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} &= \|u_{n_k} - u\|_{L^p(B_R)} + \|u_{n_k} - u\|_{L^p(\{|x|\geq R\})} \\ &\leq \|u_{n_k} - u\|_{L^p(B_R)} + \|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Let $\delta > 0$. We first fix R large enough so that (by uniform convergence)

$$\|u_{n_k} - u\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)} \leq \frac{\varepsilon}{2}.$$

Next, since $(u_{n_k}|_{B_R})_{k\geq 0}$ is bounded in $H^1(B_R)$, it follows from Rellich's compactness theorem that $u_{n_k}|_{B_R} \to u|_{B_R}$ in $L^p(B_R)$. Therefore for k large enough we have

$$\|u_{n_k} - u\|_{L^p(B_R)} \leq \frac{\varepsilon}{2},$$

and so $\|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} \leq \varepsilon$. This proves the result. □

LEMMA 1.7.3. *If $u \in H^1(\mathbb{R}^N)$ is a radially symmetric function, then*

$$\sup_{x\in\mathbb{R}^N} |x|^{\frac{N-1}{2}} |u(x)| \leq C\|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}. \tag{1.7.1}$$

If, in addition, $u(x)$ is a nonincreasing function of $|x|$, then

$$\sup_{x\in\mathbb{R}^N} |x|^{\frac{N}{2}} |u(x)| \leq C\|u\|_{L^2}. \tag{1.7.2}$$

PROOF. Suppose first $u \in C_c^\infty(\mathbb{R}^N)$. We have

$$r^{N-1}u(r)^2 = -\int_r^\infty \frac{d}{ds}(s^{N-1}u(s)^2)ds \leq 2\int_r^\infty s^{N-1}u(s)u'(s)ds,$$

and (1.7.1) follows from the Cauchy-Schwarz inequality. If $u(x)$ is a nonincreasing function of $|x|$, then

$$\|u\|_{L^2}^2 \geq \int_{\{|x|<R\}} |u(x)^2| dx \geq |\{|x| < R\}||u(r)|^2,$$

proving (1.7.2). The general case then follows by a density argument. □

The other type of compactness argument we will use does not require radial symmetry. It is due to P.-L. Lions [235, 236] and is known as the *concentration-compactness* method. That method is designed to pass to the limit in variational problems with lack of compactness. It can be formulated in many ways, but we describe only the form which we will use (Proposition 1.7.6 below). We begin with a first lemma concerning the *concentration function*.

LEMMA 1.7.4. *Let $u \in L^2(\mathbb{R}^N)$ with $\|u\|_{L^2} = a > 0$ and let the concentration function $\rho(u, \cdot)$ be defined by*

$$\rho(u,t) = \sup_{y\in\mathbb{R}^N} \int_{\{|x-y|<t\}} |u(x)|^2\, dx \quad \text{for } t > 0. \tag{1.7.3}$$

(i) *$\rho(u,t)$ is a nondecreasing function of t. $\rho(u,0) = 0$, $0 < \rho(u,t) \leq a$ for $t > 0$, and $\lim_{t\to\infty} \rho(u,t) = a$.*

(ii) *There exists* $y(u,t) \in \mathbb{R}^N$ *such that*

$$\rho(u,t) = \int_{\{|x-y(u,t)|<t\}} |u(x)|^2\,dx\,.$$

(iii) *If* $u \in L^r(\mathbb{R}^N)$ *for some* $r > 2$, *then*

$$|\rho(u,t) - \rho(u,s)| \le C\|u\|_{L^r}^2 |t^N - s^N|^{\frac{r-2}{r}} \quad \textit{for all } s,t>0 \textit{ with } C = C(N,r)\,.$$

PROOF. Property (i) is immediate. Next, given $t > 0$, there is a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\rho(u)(t) = \lim_{n\to\infty} \int_{\{|x-y_n|<t\}} |u(x)|^2\,dx > 0\,.$$

We claim that the sequence $(y_n)_{n\ge 0}$ is bounded. Otherwise, there exists a subsequence $(y_{n_j})_{j\ge 0}$ such that $\{|x-y_{n_j}| < t\} \cap \{|x - y_{n_\ell}| < t\} = \varnothing$ for $j \ne \ell$, and so

$$\int_{\mathbb{R}^N} |u|^2 \ge \sum_{j\ge 0} \int_{\{|x-y_{n_j}|<t\}} |u(x)|^2\,dx = +\infty\,,$$

which is absurd. Therefore, $(y_n)_{n\ge 0}$ has a convergent subsequence, and its limit $y(u,t)$ satisfies (ii). Finally, consider $0 < s \le t < \infty$. We have

$$\begin{aligned}
|\rho(u,t) - \rho(u,s)| &= \int_{\{|x-y(u,t)|<t\}} |u|^2 - \int_{\{|x-y(u,s)|<s\}} |u|^2 \\
&= \int_{\{s<|x-y(u,t)|<t\}} |u|^2 + \int_{\{|x-y(u,t)|<s\}} |u|^2 - \int_{\{|x-y(u,s)|<s\}} |u|^2 \\
&\le \int_{\{s<|x-y(u,t)|<t\}} |u|^2,
\end{aligned}$$

by (ii) and the definition of $\rho(u,s)$. Therefore, by Hölder's inequality,

$$|\rho(u,t) - \rho(u,s)| \le \|u\|_{L^r}^2 |\{s < |x-y(u,t)| < t\}|^{\frac{r-2}{r}}\,,$$

and (iii) follows. □

We next study the limit of a sequence of concentration functions.

LEMMA 1.7.5. *Let* $(u_n)_{n\ge 0} \subset H^1(\mathbb{R}^N)$ *be such that*

$$\|u_n\|_{L^2} = a > 0\,, \tag{1.7.4}$$

$$\sup_{n\ge 0} \|\nabla u_n\|_{L^2} < \infty\,, \tag{1.7.5}$$

and let $\rho(u_n,t)$ *be defined by* (1.7.3). *Set*

$$\mu = \lim_{t\to\infty} \liminf_{n\to\infty} \rho(u_n,t)\,. \tag{1.7.6}$$

Then there exist a subsequence $(u_{n_k})_{k\geq 0}$, a nondecreasing function $\gamma(t)$, and a sequence $t_k \to \infty$ with the following properties:

(i) *$\rho(u_{n_k}, \cdot) \to \gamma(\cdot) \in [0, a]$ as $k \to \infty$ uniformly on bounded sets of $[0, \infty)$.*

(ii) *$\mu = \lim_{t\to\infty} \gamma(t) = \lim_{k\to\infty} \rho(u_{n_k}, t_k) = \lim_{k\to\infty} \rho(u_{n_k}, t_k/2)$.*

PROOF. We deduce from (1.7.6) that there exist $t_k \to \infty$ such that

$$\mu = \lim_{k\to\infty} \rho(u_{n_k}, t_k)\,. \tag{1.7.7}$$

Note that $\rho(u_n, t) \leq \|u_n\|_{L^2}^2 \leq a$. Since $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for some $r > 2$, it follows from (1.7.4), (1.7.5), and property (iii) of Lemma 1.7.4 that $\rho(u_n, \cdot)$ is uniformly Hölder continuous. Therefore, (i) follows from Ascoli's theorem (after renaming the sequence n_k). Note that we do not loose property (1.7.7) by passing to a subsequence. Since $\rho(u_n, \cdot)$ is nondecreasing, it follows from (1.7.7) that

$$\limsup_{k\to\infty} \rho\Big(u_{n_k}, \frac{t_k}{2}\Big) \leq \limsup_{k\to\infty} \rho(u_{n_k}, t_k) = \mu\,. \tag{1.7.8}$$

Next, for every $t > 0$,

$$\liminf_{k\to\infty} \rho(u_{n_k}, t) \geq \liminf_{n\to\infty} \rho(u_n, t)\,,$$

so that, by letting $t \to \infty$ and using (1.7.6) and (i),

$$\lim_{t\to\infty} \gamma(t) \geq \mu\,. \tag{1.7.9}$$

Finally, given $t > 0$, we have $t_k/2 > t$ for k large, so that

$$\rho\Big(u_{n_k}, \frac{t_k}{2}\Big) \geq \rho(u_{n_k}, t) \underset{k\to\infty}{\longrightarrow} \gamma(t)\,.$$

Letting $t \to \infty$, we obtain

$$\liminf_{k\to\infty} \rho\Big(u_{n_k}, \frac{t_k}{2}\Big) \geq \mu\,. \tag{1.7.10}$$

Part (ii) follows from (1.7.8), (1.7.9), and (1.7.10). □

PROPOSITION 1.7.6. *Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ satisfy (1.7.4)–(1.7.5), let $\rho(u_n, t)$ be defined by (1.7.3), and let μ be defined by (1.7.6). There exists a subsequence $(u_{n_k})_{k\geq 0}$ that satisfies the following properties.*

(i) *If $\mu = a$, then there exists a sequence $(y_k)_{k\geq 0} \subset \mathbb{R}^N$ and $u \in H^1(\mathbb{R}^N)$ such that $u_{n_k}(\cdot - y_k) \to u$ as $k \to \infty$ in $L^p(\mathbb{R}^N)$ for all $2 \leq p < \frac{2N}{N-2}$ ($2 \leq p \leq \infty$ if $N = 1$).*

(ii) *If $\mu = 0$, then $\|u_{n_k}\|_{L^p} \to 0$ as $k \to \infty$ for all $2 < p < \frac{2N}{N-2}$ ($2 < p \leq \infty$ if $N = 1$).*

(iii) *There exist* $(v_k)_{k\geq 0}, (w_k)_{k\geq 0} \subset H^1(\mathbb{R}^N)$ *such that*

$$\operatorname{supp} v_k \cap \operatorname{supp} w_k = \varnothing\,, \tag{1.7.11}$$

$$|v_k| + |w_k| \leq |u_{n_k}|\,, \tag{1.7.12}$$

$$\|v_k\|_{H^1} + \|w_k\|_{H^1} \leq C\|u_{n_k}\|_{H^1}\,, \tag{1.7.13}$$

$$\|v_k\|_{L^2}^2 \underset{k\to\infty}{\longrightarrow} \mu\,, \quad \|w_k\|_{L^2}^2 \underset{k\to\infty}{\longrightarrow} a-\mu\,, \tag{1.7.14}$$

$$\liminf_{k\to\infty} \Big\{ \int |\nabla u_{n_k}|^2 - \int |\nabla v_k|^2 - \int |\nabla w_k|^2 \Big\} \geq 0\,, \tag{1.7.15}$$

$$\Big| \int |u_{n_k}|^p - \int |v_k|^p - \int |w_k|^p \Big| \underset{k\to\infty}{\longrightarrow} 0 \tag{1.7.16}$$

for all $2 \leq p < \frac{2N}{N-2}$ $(2 \leq p < \infty$ *if* $N = 1)$.

For the proof, we will use the following Sobolev-type inequality.

LEMMA 1.7.7. *There exists a constant* K *such that*

$$\int_{\mathbb{R}^N} |u|^{\frac{2N+4}{N}} \leq K\rho(u,t)^{\frac{2}{N}} \Big(\int_{\mathbb{R}^N} |\nabla u|^2 + t^{-2} \int_{\mathbb{R}^N} |u|^2 \Big) \tag{1.7.17}$$

for all $u \in H^1(\mathbb{R}^N)$ *and all* $t > 0$, *where* ρ *is defined by* (1.7.3).

PROOF. Let $(Q_j)_{j\geq 0}$ be a sequence of open, unit cubes of $\mathbb{R}^N$ such that $Q_j \cap Q_k = \varnothing$ if $j \neq k$ and $\overline{\bigcup_{j\geq 0} Q_j} = \mathbb{R}^N$. It follows that

$$\int_{\mathbb{R}^N} |u|^{\alpha+2} = \sum_{j=0}^{\infty} \int_{Q_j} |u|^{\alpha+2} \quad \text{and} \quad \|u\|_{H^1}^2 = \sum_{j=0}^{\infty} \int_{Q_j} (|\nabla u|^2 + |u|^2)\,.$$

We now proceed in two steps.

STEP 1. There exists a constant C independent of j such that

$$\int_{Q_j} |u|^{\frac{2N+4}{N}} \leq C\Big(\int_{Q_j} |u|^2 \Big)^{\frac{2}{N}} \Big(\int_{Q_j} |\nabla u|^2 + |u|^2 \Big) \quad \text{for all } u \in H^1(Q_j)\,. \tag{1.7.18}$$

Indeed, suppose first $N \geq 3$. It follows from Sobolev's embedding that

$$\|u\|_{L^{\frac{2N}{N-2}}(Q_j)} \leq C\|u\|_{H^1(Q_j)}\,,$$

and (1.7.18) follows by using Hölder's inequality. Suppose now $N = 2$. It follows from Sobolev's embedding that

$$\|u\|_{L^2(Q_j)} \leq C(\|\nabla u\|_{L^1(Q_j)} + \|u\|_{L^1(Q_j)})\,.$$

Changing u to $|u|^2$ and using the estimate $|\nabla |u|^2| \leq 2|u||\nabla u|$ together with Hölder's inequality, we obtain (1.7.18). Suppose now $N = 1$. Sobolev's embedding yields

$$\|u\|_{L^\infty(Q_j)} \leq C(\|\nabla u\|_{L^1(Q_j))} + \|u\|_{L^1(Q_j)})\,.$$

Therefore, changing u to $|u|^2$, we deduce as above that

$$\|u\|^2_{L^\infty(Q_j)} \le C\|u\|_{L^2(Q_j)}(\|\nabla u\|_{L^2(Q_j)}) + \|u\|_{L^2(Q_j)}),$$

and (1.7.18) follows from Hölder's inequality

$$\|u\|^6_{L^6(Q_j)} \le \|u\|^2_{L^2(Q_j)}\|u\|^4_{L^\infty(Q_j)}.$$

Finally, the fact that in the above calculations the constant is independent of j follows from translation invariance.

STEP 2. Proof of (1.7.17). Summing on j the inequality (1.7.18), we obtain

$$\int_{\mathbb{R}^N} |u|^{\frac{2N+4}{N}} \le C\Big(\sup_{j\in\mathbb{N}} \int_{Q_j} |u|^2\Big)^{\frac{2}{N}} \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2,$$

and (1.7.17) for $t = 1$ follows easily. The general case of (1.7.17) is obtained by changing $u(x)$ to $u(tx)$. □

PROOF OF PROPOSITION 1.7.6. We use the functions $\gamma(\cdot)$ and $y(\cdot,\cdot)$ and the sequences $(u_{n_k})_{k\ge0}$ and $(t_k)_{k\ge0}$ constructed in Lemmas 1.7.4 and 1.7.5. Fix T sufficiently large so that $\gamma(T) > a/2$ and let $y_k = y(u_{n_k}, T)$. By possibly extracting a subsequence, we may assume that there exists $u \in H^1(\mathbb{R}^N)$ so that

$$u_{n_k}(\cdot - y_k) \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^N) \text{ as } k \to \infty. \tag{1.7.19}$$

We now proceed in three steps.

STEP 1. Proof of (i). Suppose $\mu = a$. We claim that if u is given by (1.7.19), then

$$\|u\|^2_{L^2} = a, \tag{1.7.20}$$

from which (i) follows. We now prove the claim (1.7.20). Note that, since the embedding $H^1(B_R) \hookrightarrow L^2(B_R)$ is compact,

$$\int_{\{|x|<R\}} |u(x)|^2\,dx = \lim_{k\to\infty} \int_{\{|x-y_k|<R\}} |u_{n_k}(x)|^2\,dx \quad \text{for every } R > 0. \tag{1.7.21}$$

On the other hand, it follows from what precedes that $\rho(u_{n_k}, T) > a/2$ for k large. Fix $\varepsilon < a/2$ and let τ be large enough so that $\rho(u_{n_k}, \tau) > a - \varepsilon$ for k large. Since

$$\int_{\{|x-y_k|<T\}} |u_{n_k}|^2 + \int_{\{|x-y(u_{n_k},\tau)|<\tau\}} |u_{n_k}|^2 > \frac{a}{2} + a - \varepsilon > a \quad \text{for } k \text{ large},$$

we see that $\{|x-y_k| < T\} \cap \{|x - y(u_{n_k}, \tau)| < \tau\} \ne \varnothing$. In particular, if $R = T + 2\tau$, then $\{|x - y(u_{n_k}, \tau)| < \tau\} \subset \{|x - y_k| < R\}$, and so

$$\int_{\{|x-y_k|<R\}} |u_{n_k}|^2 \ge \int_{\{|x-y(u_{n_k},\tau)|<\tau\}} |u_{n_k}|^2 > a - \varepsilon \quad \text{for } k \text{ large}.$$

Applying (1.7.21), we deduce that

$$\|u\|_{L^2}^2 \geq \int_{\{|x|<R\}} |u(x)|^2\,dx \geq a - \varepsilon\,.$$

(1.7.20) follows by letting $\varepsilon \downarrow 0$.

STEP 2. Proof of (ii). Suppose $\mu = 0$. It follows from Lemma 1.7.5(ii) that $\rho(u_{n_k}, 1) \to 0$ as $k \to \infty$, and (ii) follows from (1.7.17).

STEP 3. Proof of (iii). We fix $\theta, \varphi \in C^\infty([0,\infty))$ such that $0 \leq \theta, \varphi \leq 1$ and

$$\theta(t) \equiv 1 \quad \text{for } 0 \leq t \leq \frac{1}{2}\,, \qquad \theta(t) \equiv 0 \quad \text{for } t \geq \frac{3}{4}\,,$$
$$\varphi(t) \equiv 0 \quad \text{for } 0 \leq t \leq \frac{3}{4}\,, \qquad \varphi(t) \equiv 1 \quad \text{for } t \geq 1\,,$$

and we set

$$v_k = \theta_k u_{n_k}\,, \quad w_k = \varphi_k u_{n_k}\,,$$

where

$$\theta_k(x) = \theta\Big(\frac{|x - y(u_{n_k}, t_k/2)|}{t_k}\Big), \quad \varphi_k(x) = \varphi\Big(\frac{|x - y(u_{n_k}, t_k/2)|}{t_k}\Big)\,.$$

Properties (1.7.11)–(1.7.13) are then immediate. Next,

$$\begin{aligned}
\rho\Big(u_{n_k}, \frac{t_k}{2}\Big) = \int_{\{|x-y(u_{n_k},t_k/2)|\leq t_k/2\}} |u_{n_k}|^2 &\leq \int_{\mathbb{R}^N} |v_k|^2 \\
&\leq \int_{\{|x-y(u_{n_k},t_k/2)|\leq t_k\}} |u_{n_k}|^2 \\
&\leq \int_{\{|x-y(u_{n_k},t_k)|\leq t_k\}} |u_{n_k}|^2 \leq \rho(u_{n_k}, t_k),
\end{aligned}$$

so that

$$\|v_k\|_{L^2}^2 \underset{k\to\infty}{\longrightarrow} \mu \tag{1.7.22}$$

by Lemma 1.7.4(ii). Set now $z_k = u_{n_k} - v_k - w_k$, so that in particular $|z_k| \leq |u_{n_k}|$. We have

$$\begin{aligned}
\int_{\mathbb{R}^N} |z_k|^2 &\leq \int_{\{t_k/2\leq|x-y(u_{n_k},t_k/2)|\leq t_k\}} |u_{n_k}|^2 \\
&= \int_{\{|x-y(u_{n_k},t_k/2)|\leq t_k\}} |u_{n_k}|^2 - \int_{\{|x-y(u_{n_k},t_k/2)|\leq t_k/2\}} |u_{n_k}|^2 \\
&\leq \int_{\{|x-y(u_{n_k},t_k)|\leq t_k\}} |u_{n_k}|^2 - \int_{\{|x-y(u_{n_k},t_k/2)|\leq t_k/2\}} |u_{n_k}|^2 \\
&= \rho(u_{n_k}, t_k) - \rho\Big(u_{n_k}, \frac{t_k}{2}\Big)\,,
\end{aligned}$$

so that

$$\|z_k\|_{L^2}^2 \underset{k\to\infty}{\longrightarrow} 0 \tag{1.7.23}$$

by Lemma 1.7.4(ii). We deduce from (1.7.11), (1.7.22), and (1.7.23) that

$$\|w_k\|_{L^2}^2 \underset{k\to\infty}{\longrightarrow} a - \mu\,,$$

which proves (1.7.14). Next, one easily verifies that

$$\big| |u_{n_k}|^p - |v_k|^p - |w_k|^p \big| \le C|u_{n_k}|^{p-1}|z_k|\,,$$

and (1.7.15) follows. (Note that z_k is bounded in H^1 and converges to 0 in L^2, hence in L^p.) Finally, it follows from an easy calculation that

$$\begin{aligned}
&|\nabla u_{n_k}|^2 - |\nabla v_k|^2 - |\nabla w_k|^2 \\
&\quad = |\nabla u_{n_k}|^2 (1 - \theta_k^2 - \varphi_k^2) \\
&\qquad - |u_{n_k}|^2 \big(|\nabla \theta_k|^2 + |\nabla \varphi_k|^2 \big) - \operatorname{Re}(\overline{u_{n_k}} \nabla u_{n_k}) \cdot \nabla \big(\theta_k^2 + \varphi_k^2 \big) \\
&\quad \ge -\frac{C}{t_k^2} |u_{n_k}|^2 - \frac{C}{t_k} |u_{n_k}| |\nabla u_{n_k}|\,,
\end{aligned}$$

from which (1.7.16) follows. □

CHAPTER 2

The Linear Schrödinger Equation

This chapter is devoted to the study of some fundamental properties of the (linear) Schrödinger equation. We study in particular the dispersive properties and the smoothing effect of the equation in $\mathbb{R}^N$.

2.1. Basic Properties

Let Ω be an open subset of $\mathbb{R}^N$ (Ω does not need to be smooth or bounded). We define the operator A on $L^2(\Omega)$ by

$$\begin{cases} D(A) = \{u \in H_0^1(\Omega),\ \Delta u \in L^2(\Omega)\}, \\ Au = \Delta u \quad \text{for } u \in D(A). \end{cases}$$

Evidently, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ if Ω is smooth enough. It is well known that A is self-adjoint and ≤ 0, and so we may apply the results of Section 1.6. Observe that the space X_A is nothing other than $H_0^1(\Omega)$. Indeed $\|\cdot\|_A = \|\cdot\|_{H_0^1}$, and $\mathcal{D}(\Omega) \subset D(A)$, so that $H_0^1(\Omega) \subset X_A$. Since also $D(A)$ is a subset of $H_0^1(\Omega)$, we see that $X_A \subset H_0^1(\Omega)$, and so $X_A = H_0^1(\Omega)$ with equality of the norms. It follows that $X_A^\star = H^{-1}(\Omega)$. On the other hand, note that $D(A) \neq H_0^2(\Omega)$, and so $D(A)^\star \neq H^{-2}(\Omega)$. The operator $\overline{A} \in \mathcal{L}(L^2(\Omega), (D(A))^\star)$ is simply defined by

$$(\overline{A}u, v)_{(D(A))^\star, D(A)} = (u, \Delta v)_{L^2} \quad \text{for } u \in L^2(\Omega) \text{ and } v \in D(A)\,.$$

Let us denote by $(\mathcal{T}(t))_{t\in\mathbb{R}}$ the group of isometries generated by iA in any of the spaces $D(A)$, $H_0^1(\Omega)$, $L^2(\Omega)$, $H^{-1}(\Omega)$, $(D(A))^\star$. We have the following result.

PROPOSITION 2.1.1. *Given $\varphi \in L^2(\Omega)$, $u(t) = \mathcal{T}(t)\varphi$ is the unique solution of the problem*

$$\begin{cases} u \in C(\mathbb{R}, L^2(\Omega)) \cap C^1(\mathbb{R}, (D(A))^\star), \\ iu_t + \Delta u = 0 \text{ in } (D(A))^\star \quad \text{for every } t \in \mathbb{R}, \\ u(0) = \varphi. \end{cases}$$

Moreover, $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for every $t \in \mathbb{R}$. If $\varphi \in H_0^1(\Omega)$, then for every $t \in \mathbb{R}$, $u \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, H^{-1}(\Omega))$ and $\|\nabla u(t)\|_{L^2} = \|\nabla\varphi\|_{L^2}$.

REMARK 2.1.2. It follows from Section 1.6 that $\mathcal{T}(t)^\star = \mathcal{T}(-t)$ for every $t \in \mathbb{R}$. On the other hand, with the notation of Proposition 2.1.1, let $v(t) = \overline{u}(-t)$. We have

$$\begin{cases} iv_t + \Delta v = 0, \\ v(0) = \overline{\varphi}, \end{cases}$$

and so $\mathcal{T}(-t)\varphi = \overline{\mathcal{T}(t)\overline{\varphi}}$ for every $\varphi \in L^2(\Omega)$.

2.2. Fundamental Properties in $\mathbb{R}^N$

In this section we consider the case $\Omega = \mathbb{R}^N$. In this case, $\mathcal{T}(t)$ can be expressed explicitly in Fourier variables. Indeed, let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and let $u \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^N))$ be defined by

$$\widehat{u(t)}(\xi) = e^{-4\pi^2 i|\xi|^2 t}\widehat{\varphi}(\xi) \quad \text{for all } t \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N.$$

We have $i\hat{u}_t - 4\pi^2|\xi|^2\hat{u} = 0$ in $\mathbb{R} \times \mathbb{R}^N$, and so $iu_t + \Delta u = 0$ in $\mathbb{R} \times \mathbb{R}^N$. Since $u(0) = \varphi$, we deduce that $u(t) = \mathcal{T}(t)\varphi$. Thus we see that

$$\mathcal{F}(\mathcal{T}(t)\varphi)(\xi) = e^{-4\pi^2 i|\xi|^2 t}\widehat{\varphi}(\xi) \tag{2.2.1}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$, $t \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$.

REMARK 2.2.1. Formula (2.2.1) has the following simple applications:

(i) We deduce from formula (2.2.1) that $\mathcal{T}(\cdot)\varphi \in C(\mathbb{R}, \mathcal{S}(\mathbb{R}^N))$ for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$. By duality, $\mathcal{T}(t)$ can be extended to $\mathcal{S}'(\mathbb{R}^N)$, and

$$\mathcal{T}(\cdot)\varphi \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^N))$$

for every $\varphi \in \mathcal{S}'(\mathbb{R}^N)$. It follows in particular that if $\varphi_n \to \varphi$ in $\mathcal{S}'(\mathbb{R}^N)$ and if $t_n \to t$, then

$$\mathcal{T}(t_n)\varphi_n \to \mathcal{T}(t)\varphi \quad \text{in } \mathcal{S}'(\mathbb{R}^N).$$

Indeed, given $\theta \in \mathcal{S}(\mathbb{R}^N)$, we have $\mathcal{T}(-t_n)\theta \to \mathcal{T}(-t)\theta$ in $\mathcal{S}(\mathbb{R}^N)$, so that

$$\begin{aligned}\langle \mathcal{T}(t_n)\varphi_n, \theta\rangle_{\mathcal{S}',\mathcal{S}} &= \langle \varphi_n, \mathcal{T}(-t_n)\theta\rangle_{\mathcal{S}',\mathcal{S}} \\ &\underset{n\to\infty}{\longrightarrow} \langle \varphi, \mathcal{T}(-t)\theta\rangle_{\mathcal{S}',\mathcal{S}} = \langle \mathcal{T}(t)\varphi, \theta\rangle_{\mathcal{S}',\mathcal{S}}\end{aligned}$$

by theorem 2.17 of [304].

(ii) It follows from (2.2.1) that $|\mathcal{F}(\mathcal{T}(t)\varphi)(\xi)| = |\varphi(\xi)|$ for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. In particular, we see that for all $s \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$\|\mathcal{T}(t)\varphi\|_{H^s} = \|\varphi\|_{H^s}.$$

Since $\mathcal{S}(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$ for all $s \in \mathbb{R}$, we deduce that for any $s \in \mathbb{R}$, $(\mathcal{T}(t))_{t\in\mathbb{R}}$ can be extended to a group of isometries in $H^s(\mathbb{R}^N)$, which we still denote by $(\mathcal{T}(t))_{t\in\mathbb{R}}$. The generator of $(\mathcal{T}(t))_{t\in\mathbb{R}}$ in $H^s(\mathbb{R}^N)$ is the operator A_s defined by $D(A_s) = H^{s+2}(\mathbb{R}^N)$, and $A_s u = i\Delta u$ for $u \in D(A_s)$. It follows easily that if $\varphi \in H^s(\mathbb{R}^N)$, then $u(t) = \mathcal{T}(t)\varphi$ satisfies $u \in \bigcap_{j\geq 0} C^j(\mathbb{R}, H^{s-2j}(\mathbb{R}^N))$.

(iii) Let I be a bounded, open interval of $\mathbb{R}$ with $0 \in I$. Let $s \in \mathbb{R}$, $\varphi \in H^s(\mathbb{R}^N)$, $f \in L^1(I, H^s(\mathbb{R}^N))$, and $u \in C(I, H^s(\mathbb{R}^N))$. We deduce from (ii) above and the results of Section 1.6 that u satisfies

$$u(t) = \mathcal{T}(t)\varphi + i\int_0^t \mathcal{T}(t-s)f(s)ds$$

for $t \in I$ if and only if $u \in W^{1,1}(I, H^{s-2}(\mathbb{R}^N))$ and

$$\begin{cases} iu_t + \Delta u + f = 0 & \text{for } t \in I, \\ u(0) = \varphi. \end{cases}$$

If, in addition, $f \in C(I, H^{s-2}(\mathbb{R}^N))$, then $u \in C^1(I, H^{s-2}(\mathbb{R}^N))$.

REMARK 2.2.2. Here are some comments on the scaling properties of $\mathcal{T}(t)$. Let $\varphi \in L^2(\mathbb{R}^N)$, $\gamma > 0$, and set $\psi(x) = \varphi(\gamma x)$ so that $\psi \in L^2(\mathbb{R}^N)$. It follows that for all $t \in \mathbb{R}$,

$$[\mathcal{T}(t)\psi](x) = [\mathcal{T}(\gamma^2 t)\varphi](\gamma x) \quad \text{a.e.} \tag{2.2.2}$$

Indeed, let $u(t) = \mathcal{T}(t)\varphi$ and set $v(t,x) = u(\gamma^2 t, \gamma x)$. Since $iu_t + \Delta u = 0$, it follows that $iv_t + \Delta v = 0$; and since $v(0,x) = \varphi(\gamma x) = \psi(x)$, we see that $v(t) = \mathcal{T}(t)\psi$. Similarly, let $f \in L^1(\mathbb{R}, L^2(\mathbb{R}^N))$, $\gamma > 0$, and set $g(t,x) = \gamma^2 f(\gamma^2 t, \gamma x)$. If

$$u(t) = i\int_0^t \mathcal{T}(t-s)f(s)ds, \quad v(t) = i\int_0^t \mathcal{T}(t-s)g(s)ds,$$

then for all $t \in \mathbb{R}$,

$$v(t,x) = u(\gamma^2 t, \gamma x) \quad \text{a.e.} \tag{2.2.3}$$

Indeed, both v and w defined by $w(t,x) = u(\gamma^2 t, \gamma x)$ are solutions of the equation $iz_t + \Delta z + f = 0$ with the initial condition $z(0) = 0$, so that $v = w$. These calculations are justified when φ and f are sufficiently smooth ($\varphi \in H^2(\mathbb{R}^N)$ and $f \in L^1(\mathbb{R}, H^2(\mathbb{R}^N))$, say). Then (2.2.2) and (2.2.3) follow in the general case by a density argument.

The following well-known result is the fundamental estimate for $\mathcal{T}(t)$.

PROPOSITION 2.2.3. *If $p \in [2,\infty]$ and $t \neq 0$, then $\mathcal{T}(t)$ maps $L^{p'}(\mathbb{R}^N)$ continuously to $L^p(\mathbb{R}^N)$ and*

$$\|\mathcal{T}(t)\varphi\|_{L^p(\mathbb{R}^N)} \le (4\pi|t|)^{-N(\frac{1}{2}-\frac{1}{p})}\|\varphi\|_{L^{p'}(\mathbb{R}^N)} \quad \textit{for all } \varphi \in L^{p'}(\mathbb{R}^N). \tag{2.2.4}$$

The proof of Proposition 2.2.3 relies on the following lemma.

LEMMA 2.2.4. *Given $t \neq 0$, define the function K_t by*

$$K_t(x) = \left(\frac{1}{4\pi i t}\right)^{\frac{N}{2}} e^{\frac{i|x|^2}{4t}} \quad \textit{for } x \in \mathbb{R}^N.$$

It follows that $\mathcal{T}(t)\varphi = K_t \star \varphi$; i.e.,

$$\mathcal{T}(t)\varphi(x) = (4\pi i t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{\frac{i|x-y|^2}{4t}}\varphi(y)dy \tag{2.2.5}$$

for all $t \neq 0$ and all $\varphi \in \mathcal{S}(\mathbb{R}^N)$.

PROOF. Since $\widehat{K}_t(\xi) = e^{-i4\pi^2|\xi|^2 t}$, the result follows from (2.2.1). □

PROOF OF PROPOSITION 2.2.3. Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$. It follows from Lemma 2.2.4 that

$$\|\mathcal{T}(t)\varphi\|_{L^\infty} \le (4\pi|t|)^{-\frac{N}{2}}\|\varphi\|_{L^1}.$$

By density, $\mathcal{T}(t) \in \mathcal{L}(L^1(\mathbb{R}^N), L^\infty(\mathbb{R}^N))$ and

$$\|\mathcal{T}(t)\|_{\mathcal{L}(L^1(\mathbb{R}^N),L^\infty(\mathbb{R}^N))} \le (4\pi|t|)^{-\frac{N}{2}}.$$

The general case is obtained by interpolation between the cases $p = 2$ and $p = \infty$ (use the Riesz-Thorin interpolation theorem). □

REMARK 2.2.5. It follows from formula (2.2.5) that

$$\mathcal{T}(t)\varphi(x) = \left(\frac{1}{4\pi it}\right)^{\frac{N}{2}} e^{\frac{i|x|^2}{4t}} \int e^{-\frac{ix\cdot y}{2t}} e^{\frac{i|y|^2}{4t}} \varphi(y)dy.$$

In other words, up to a rescaling and to multiplication by a function of modulus 1, $\mathcal{T}(t)$ is nothing but the Fourier transform. More precisely, if we define a multiplier M_t by $M_t(x) = e^{i|x|^2/4t}$ and dilation operator D_t by $D_t w(x) = (4\pi it)^{-\frac{N}{2}} w(x/4\pi it)$, then

$$\mathcal{T}(t)\varphi = M_t D_t \mathcal{F}(M_t\varphi).$$

In particular, estimate (2.2.4) is optimal in the sense that if $\mathcal{T}(t) \in \mathcal{L}(L^q, L^p)$, then necessarily $2 \le p \le \infty$ and $q = p'$.

COROLLARY 2.2.6. *If $t \neq 0$, then*

$$\|\mathcal{T}(t)\varphi\|_{H^{s,p}} \le (4\pi|t|)^{-N(\frac{1}{2}-\frac{1}{p})}\|\varphi\|_{H^{s,p'}} \quad \textit{for all } \varphi \in \mathcal{S}'(\mathbb{R}^N),$$

where $s \in \mathbb{R}$ and $2 \le p \le \infty$. The same estimate holds with the norms of $H^{s,p}(\mathbb{R}^N)$ and $H^{s,p'}(\mathbb{R}^N)$ replaced by the norms of $\dot H^{s,p}(\mathbb{R}^N)$ and $\dot H^{s,p'}(\mathbb{R}^N)$. Moreover

$$\|\mathcal{T}(t)\varphi\|_{B^s_{p,q}} \le (4\pi|t|)^{-N(\frac{1}{2}-\frac{1}{p})}\|\varphi\|_{B^s_{p',q}} \quad \textit{for all } \varphi \in \mathcal{S}'(\mathbb{R}^N),$$

where $s \in \mathbb{R}$ and $2 \le p \le \infty$ and $1 \le q \le \infty$. The same estimate holds with the norms of $B^s_{p,q}(\mathbb{R}^N)$ and $B^s_{p',q}(\mathbb{R}^N)$ replaced by the norms of $\dot B^s_{p,q}(\mathbb{R}^N)$ and $\dot B^s_{p',q}(\mathbb{R}^N)$.

PROOF. Fix $t \neq 0$ and let $u = \mathcal{T}(t)\varphi$. Given $w \in \mathcal{S}(\mathbb{R}^N)$, it follows from (2.2.1) that

$$\begin{aligned} \mathcal{F}^{-1}(w\widehat{u(t)}) &= \mathcal{F}^{-1}(we^{-4\pi^2 i|\xi|^2 t}\widehat{\varphi}) \\ &= \mathcal{F}^{-1}(e^{-4\pi^2 i|\xi|^2 t}\mathcal{F}\mathcal{F}^{-1}(w\widehat{\varphi})) = \mathcal{T}(t)(\mathcal{F}^{-1}(w\widehat{\varphi})). \end{aligned} \tag{2.2.6}$$

In particular, it follows from (2.2.4) that

$$\|\mathcal{F}^{-1}(w\widehat{u})\|_{L^p} \le (4\pi|t|)^{-N(\frac{1}{2}-\frac{1}{p})}\|\mathcal{F}^{-1}(w\widehat{\varphi})\|_{L^{p'}} \quad \text{for any } 2 \le p \le \infty.$$

The result follows immediately from the above estimate and the definitions of the various Sobolev and Besov norms (see Section 1.4). □

2.3. Strichartz's Estimates

Estimate (2.2.4) is remarkable but it is not quite handy for solving the nonlinear problems, since the L^p spaces are not stable by $\mathcal{T}(t)$. However, we will derive from (2.2.4) space-time estimates that are essential for solving the nonlinear Schrödinger equations. The first estimates of that kind were obtained by Strichartz [327] as a Fourier restriction theorem. Strichartz's estimates were generalized by Ginibre and Velo [136], who gave a remarkable, elementary proof. Strichartz's estimate for the nonhomogeneous problem was generalized by Yajima [364] and by Cazenave and Weissler [68]. Finally, the endpoint estimates were established by Keel and Tao [210]. We begin by introducing the notion of admissible pair.

DEFINITION 2.3.1. We say that a pair (q, r) is admissible if

$$\frac{2}{q} = N\Big(\frac{1}{2} - \frac{1}{r}\Big) \tag{2.3.1}$$

and

$$2 \le r \le \frac{2N}{N-2} \quad (2 \le r \le \infty \text{ if } N = 1,\ 2 \le r < \infty \text{ if } N = 2). \tag{2.3.2}$$

REMARK 2.3.2. If (q, r) is an admissible pair, then $2 \le q \le \infty$. Note that the pair $(\infty, 2)$ is always admissible. The pair $(2, \frac{2N}{N-2})$ is admissible if $N \ge 3$.

THEOREM 2.3.3. (Strichartz's estimates) *The following properties hold:*

(i) *For every $\varphi \in L^2(\mathbb{R}^N)$, the function $t \mapsto \mathcal{T}(t)\varphi$ belongs to*

$$L^q(\mathbb{R}, L^r(\mathbb{R}^N)) \cap C(\mathbb{R}, L^2(\mathbb{R}^N))$$

for every admissible pair (q, r). Furthermore, there exists a constant C such that

$$\|\mathcal{T}(\cdot)\varphi\|_{L^q(\mathbb{R}, L^r)} \le C\|\varphi\|_{L^2} \quad \text{for every } \varphi \in L^2(\mathbb{R}^N). \tag{2.3.3}$$

(ii) *Let I be an interval of $\mathbb{R}$ (bounded or not), $J = \overline{I}$, and $t_0 \in J$. If (γ, ρ) is an admissible pair and $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$, then for every admissible pair (q, r), the function*

$$t \mapsto \Phi_f(t) = \int_{t_0}^{t} \mathcal{T}(t - s) f(s)\,ds \quad \text{for } t \in I \tag{2.3.4}$$

belongs to $L^q(I, L^r(\mathbb{R}^N)) \cap C(J, L^2(\mathbb{R}^N))$. Furthermore, there exists a constant C independent of I such that

$$\|\Phi_f\|_{L^q(I, L^r)} \le C\|f\|_{L^{\gamma'}(I, L^{\rho'})} \quad \text{for every } f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N)). \tag{2.3.5}$$

REMARK 2.3.4. Theorem 2.3.3 deserves a few comments. One is easily convinced that property (i) describes a quite remarkable smoothing effect. Indeed for all $t \in \mathbb{R}$, $\mathcal{T}(t)L^2 = L^2$. In particular, given $t \neq 0$ and $p > 2$, there exists a dense subset E_p of L^2 such that $\mathcal{T}(t)\varphi \notin L^p$ for every $\varphi \in E_p$. However, it follows from property (i) that for every $\varphi \in L^2$, $\mathcal{T}(t)\varphi \in L^p$ for a.a. $t \in \mathbb{R}$ and all $2 \le p \le \frac{2N}{N-2}$ if $N \ge 3$. Note that by the preceding observation, the restriction "for a.a. $t \in \mathbb{R}$" cannot be reduced to "for all $t \neq 0$" in general.

Concerning property (ii), note that the definition of Φ_f makes sense. Indeed, $L^{\rho'} \hookrightarrow H^{-1}$, and so $f \in L^1(I', H^{-1})$ for every bounded interval $I' \subset I$. In particular, $\Phi_f \in C(I', H^{-1})$. Evidently, properties (i) and (ii) give an estimate of the solution of the nonhomogeneous Schrödinger equation in terms of f and φ,

$$\begin{cases} iu_t + \Delta u + f = 0 \\ u(0) = \varphi. \end{cases}$$

REMARK 2.3.5. The estimates of Theorem 2.3.3 are called endpoint estimates in the case $r = \frac{2N}{N-2}$ or $\rho = \frac{2N}{N-2}$. Note also that an estimate similar to (2.3.3) but with the space and time integration reversed holds. More precisely,

$$\left(\int_{\mathbb{R}^N} \left(\int_{-\infty}^{+\infty} |u(t,x)|^2 \, dt \right)^{\frac{N}{N-2}} dx \right)^{\frac{N-2}{2N}} \le C \|\varphi\|_{L^2} \quad \text{for every } \varphi \in L^2(\mathbb{R}^N);$$

that is,

$$\|u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N, L^2(\mathbb{R}))} \le C \|\varphi\|_{L^2}$$

(see Ruiz and Vega [306]).

PROOF OF THEOREM 2.3.3. We only give the proof away from the endpoints, i.e., for $r, \rho \neq \frac{2N}{N-2}$. The proof in the case $r = \frac{2N}{N-2}$ or $\rho = \frac{2N}{N-2}$ is more delicate and the reader is referred to Keel and Tao [210]. Note that we will make an essential use of the endpoint estimates only in the proof of Propositions 4.2.5, 4.2.7, and 4.2.13.

We divide the proof into five steps, and we first establish property (ii). For convenience, we assume that $I = [0, T)$ for some $T \in (0, \infty)$ and that $t_0 = 0$, the proof being the same in the general case. It is convenient to define, in the same way as Φ, the operators Ψ and Θ_t (where $t \in (0, T)$ is a parameter) by

$$\Psi_f(s) = \int_s^T \mathcal{T}(s-t) f(t) dt \quad \forall s \in [0, T)$$

and

$$\Theta_{t,f}(s) = \int_0^t \mathcal{T}(s-\sigma) f(\sigma) d\sigma \quad \forall s \in [0, T).$$

It is clear that both Ψ and Θ_t are continuous $L^1_{\text{loc}}([0,T), H^{-1}) \to C([0,T), H^{-1})$.

STEP 1. For every admissible pair (q, r), the mappings Φ, Ψ, and Θ_t are continuous $L^{q'}((0,T), L^{r'}(\mathbb{R}^N)) \to L^q((0,T), L^r(\mathbb{R}^N))$. We only prove the estimate for Φ, the other ones being obtained similarly. By density, we need only consider the case $f \in C_c([0,T),L^{r'})$. In this case, Proposition 2.2.3 shows that $\Phi_f \in C([0,T),L^r)$, and that

$$\|\Phi_f(t)\|_{L^r} \le \int_0^t |t-s|^{-N(\frac{1}{2}-\frac{1}{r})} \|f(s)\|_{L^{r'}}\, ds \le \int_0^T |t-s|^{\frac{-2}{q}} \|f(s)\|_{L^{r'}}\, ds\,.$$

It follows from the Riesz potential inequalities (cf. Stein [319], theorem 1, p. 119) that

$$\|\Phi_f\|_{L^q((0,T),L^r)} \le C\|f\|_{L^{q'}((0,T),L^{r'})}$$

where C depends only on q.

STEP 2. For every admissible pair (q,r), the mappings Φ, Ψ, and Θ_t are continuous $L^{q'}((0,T),L^{r'}(\mathbb{R}^N)) \to C([0,T],L^2(\mathbb{R}^N))$. We only prove the estimate for Φ, the other ones being obtained similarly. By density, we need only consider the case $f \in C_c([0,T),L^{r'})$. By using the embedding $L^{r'} \hookrightarrow H^{-1}$ and applying the operator $(I-\varepsilon\Delta)^{-1}$, one may thus assume that $f \in C_c([0,T),L^{r'}) \cap C_c([0,T),L^2)$. It follows that $\Phi_f \in C([0,T),L^2)$, and so

$$\begin{aligned}
\|\Phi_f(t)\|_{L^2}^2 &= \Big(\int_0^t \mathcal{T}(t-s)f(s)ds, \int_0^t \mathcal{T}(t-\sigma)f(\sigma)d\sigma\Big)_{L^2} \\
&= \int_0^t\int_0^t (\mathcal{T}(t-s)f(s),\mathcal{T}(t-\sigma)f(\sigma))_{L^2}\, d\sigma\, ds \\
&= \int_0^t\int_0^t (f(s),\mathcal{T}(s-\sigma)f(\sigma))_{L^2}\, d\sigma\, ds = \int_0^t (f(s),\Theta_{t,f}(s))_{L^2}\, ds\,,
\end{aligned}$$

where we used the property $\mathcal{T}(t)^* = \mathcal{T}(-t)$. Applying Hölder's inequality in space, then in time, and applying Step 1, we deduce that

$$\|\Phi_f(t)\|_{L^2}^2 \le \|f\|_{L^{q'}((0,T),L^{r'})}\|\Theta_{t,f}\|_{L^q((0,T),L^r)} \le C\|f\|_{L^{q'}((0,T),L^{r'})}^2\,.$$

This proves the result, since t is arbitrary.

STEP 3. For every admissible pair (q,r), Φ is continuous $L^1((0,T),L^2(\mathbb{R}^N)) \to L^q((0,T),L^r(\mathbb{R}^N))$. Let $f \in L^1((0,T),L^2)$ and consider $\varphi \in C_c([0,T),\mathcal{D}(\mathbb{R}^N))$. We have

$$\begin{aligned}
\int_0^T (\Phi_f(t),\varphi(t))_{L^2}\, dt &= \int_0^T\int_0^t (\mathcal{T}(t-s)f(s),\varphi(t))_{L^2}\, ds\, dt \\
&= \int_0^T\int_s^T (f(s),\mathcal{T}(s-t)\varphi(t))_{L^2}\, dt\, ds \\
&= \int_0^T (f(s),\Psi_\varphi(s))_{L^2}\, ds\,,
\end{aligned}$$

and so, by the Cauchy-Schwartz inequality and Step 2,

$$\begin{aligned}
\Big|\int_0^T (\Phi_f(t),\varphi(t))_{L^2}\, dt\Big| &\le \|f\|_{L^1((0,T),L^2)}\|\Psi_\varphi\|_{L^\infty((0,T),L^2)} \\
&\le C\|f\|_{L^1((0,T),L^2)}\|\varphi\|_{L^{q'}((0,T),L^{r'})}\,.
\end{aligned} \tag{2.3.6}$$

On the other hand, one shows easily by choosing appropriate test functions that

$$\|g\|_{L^q((0,T),L^r)} = \sup\left\{ \int_0^T (g(t),\varphi(t))_{L^2}\,dt;\, \varphi \in C_c^\infty((0,T)\times\mathbb{R}^N),\ \|\varphi\|_{L^{q'}((0,T),L^{r'})} = 1\right\}$$

for all $g \in L^1((0,T),L^2(\mathbb{R}^N))$. The result follows from (2.3.6) and the above relation applied with $g = \Phi_f$.

STEP 4. Proof of (ii). Let (γ,ρ) be an admissible pair. We deduce from Steps 1 and 2 that Φ is continuous $L^{\gamma'}((0,T),L^{\rho'}(\mathbb{R}^N)) \to L^\infty((0,T),L^2(\mathbb{R}^N))$ and $L^{\gamma'}((0,T),L^{\rho'}(\mathbb{R}^N)) \to L^\gamma((0,T),L^\rho(\mathbb{R}^N))$. Consider an admissible pair (q,r) for which $2 \le q \le \rho$, and let $\theta \in [0,1]$ be such that

$$\frac{1}{r} = \frac{\theta}{\rho} + \frac{1-\theta}{2} \quad\text{and}\quad \frac{1}{q} = \frac{\theta}{\gamma} + \frac{1-\theta}{\infty}.$$

By applying Hölder's inequality in space, then in time, we obtain

$$\|\Phi_f\|_{L^q((0,T),L^r)} \le \|\Phi_f\|^\theta_{L^\gamma((0,T),L^\rho)} \|\Phi_f\|^{1-\theta}_{L^\infty((0,T),L^2)} \le C\|f\|_{L^{\gamma'}((0,T),L^{\rho'})}\,.$$

Thus Φ is continuous $L^{\gamma'}((0,T),L^{\rho'}(\mathbb{R}^N)) \to L^q((0,T),L^r(\mathbb{R}^N))$.

Let now (q,r) be an admissible pair for which $\rho < r$, and let $\mu \in [0,1]$ be such that

$$\frac{1}{\gamma'} = \frac{\mu}{1} + \frac{1-\mu}{q'} \quad\text{and}\quad \frac{1}{\rho'} = \frac{\mu}{2} + \frac{1-\mu}{r'}.$$

By Steps 1 and 3, Φ is continuous $L^{q'}((0,T),L^{r'}(\mathbb{R}^N)) \to L^q((0,T),L^r(\mathbb{R}^N))$ and $L^1((0,T),L^2(\mathbb{R}^N)) \to L^q((0,T),L^r(\mathbb{R}^N))$. By interpolation (see Bergh and Löfström [28], theorem 5.1.2, p. 107), Φ is continuous

$$L^\sigma((0,T),L^\delta(\mathbb{R}^N)) \to L^q((0,T),L^r(\mathbb{R}^N))$$

for every pair (σ,δ) such that for some $\theta \in [0,1]$,

$$\frac{1}{\sigma} = \frac{\theta}{1} + \frac{1-\theta}{q'} \quad\text{and}\quad \frac{1}{\delta} = \frac{\theta}{2} + \frac{1-\theta}{r'}.$$

The result follows by choosing $\theta = \mu$.

STEP 5. Proof of (i). The proof is parallel to the proof of (ii), and we describe only the main steps. Let

$$\Lambda_f(t) = \int_{-\infty}^{+\infty} \mathcal{T}(t-s)f(s)ds \quad\text{and}\quad \Gamma_f = \int_{-\infty}^{+\infty} \mathcal{T}(-t)f(t)dt\,.$$

One shows (see Step 1) that

$$\|\Lambda_f\|_{L^q((0,T),L^r)} \le C\|f\|_{L^{q'}((0,T),L^{r'})}$$

for every admissible pair (q,r). Deduce (see Step 2) that

$$\|\Gamma_f\|_{L^2} \le C\|f\|_{L^{q'}((0,T),L^{r'})}\,,$$

from which one obtains that

$$\left|\int_{-\infty}^{+\infty} (\mathcal{T}(t)\varphi, \psi(t))_{L^2}\, dt\right| = \left(\varphi, \int_{-\infty}^{+\infty} \mathcal{T}(-t)\psi(t)dt\right)_{L^2} \le C\|\varphi\|_{L^2}\|\psi\|_{L^{q'}((0,T),L^{r'})}$$

for every $\varphi \in L^2(\mathbb{R}^N)$ and $\psi \in C_c([0,T),\mathcal{D}(\mathbb{R}^N))$. Assertion (i) follows (see Step 3). This completes the proof. □

COROLLARY 2.3.6. *Let $I = (T,\infty)$ for some $T \ge -\infty$ (respectively, $I = (-\infty, T)$ for some $T \le \infty$) and let $J = \overline{I}$. Let (γ, ρ) be an admissible pair, and let $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$. It follows that the function*

$$t \mapsto \Phi_f(t) = \int_t^{\infty} \mathcal{T}(t-s)f(s)ds \quad \left(\textit{respectively, } \Phi_f(t) = \int_t^{-\infty} \mathcal{T}(t-s)f(s)ds\right)$$

for every $t \in J$, makes sense as the uniform limit in $L^2(\mathbb{R}^N)$, as $m \to +\infty$ (respectively, as $m \to -\infty$), of the functions

$$\Phi_f^m(t) = \int_t^m \mathcal{T}(t-s)f(s)ds \quad \textit{for every } t \in J.$$

In addition for every admissible pair (q,r), $\Phi_f \in L^q(I, L^r(\mathbb{R}^N)) \cap C(J, L^2(\mathbb{R}^N))$. Furthermore, there exists a constant C such that

$$\|\Phi_f\|_{L^q(I,L^r)} \le C\|f\|_{L^{\gamma'}(I,L^{\rho'})} \quad \textit{for every } f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N)).$$

PROOF. We consider, for example, the case $I = (T,\infty)$. Let j, m be two integers, $T < j < m$. For every $t \in J$,

$$\|\Phi_f^m(t) - \Phi_f^j(t)\|_{L^2} = \|\mathcal{T}(m-t)(\Phi_f^m(t) - \Phi_f^j(t))\|_{L^2} = \left\|\int_j^m \mathcal{T}(m-s)f(s)ds\right\|_{L^2}.$$

By (2.3.5), there exists a constant $C(\gamma)$ such that

$$\|\Phi_f^m(t) - \Phi_f^m(t)\|_{L^2} \le C\|f\|_{L^{\gamma'}((j,\infty),L^{\rho'})}.$$

Thus Φ^m is a Cauchy sequence in $L^\infty(I, L^2(\mathbb{R}^N))$, and so $\Phi_f \in C(J, L^2(\mathbb{R}^N))$ and

$$\|\Phi_f\|_{L^\infty(I,L^2)} \le C\|f\|_{L^{\gamma'}(I,L^{\rho'})}. \tag{2.3.7}$$

Finally, given any admissible pair (q,r), it follows from (2.3.5) that there exists a constant C such that

$$\|\Phi_f^m\|_{L^q(I,L^r)} \le C\|f\|_{L^{\gamma'}(I,L^{\rho'})}. \tag{2.3.8}$$

For $j \in \mathbb{N}$, $j \ge T$, define $f_j \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$ by

$$f_j(t) = \begin{cases} f(t) & \text{if } t \le j \\ 0 & \text{if } t > j. \end{cases}$$

Since $f_j \to f$ in $L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$ as $j \to \infty$, we deduce from (2.3.7) that

$$\Phi_{f_j} \to \Phi_f \quad \text{in } L^2(\mathbb{R}^N) \quad \text{uniformly in } t \in J. \tag{2.3.9}$$

Note that for $m \geq j$, $\Phi^m_{f_j}$ is independent of m. It follows from (2.3.8) that $\Phi_{f_j} \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$. Furthermore, letting $T \leq j \leq k$, we deduce from (2.3.8) that

$$\|\Phi_{f_j} - \Phi_{f_k}\|_{L^q(I,L^r)} \leq C\|f_j - f_k\|_{L^{\gamma'}(I,L^{\rho'})} \leq C\|f\|_{L^{\gamma'}((j,k),L^{\rho'})}.$$

In particular, Φ_{f_j} is a Cauchy sequence in $L^q(I, L^r(\mathbb{R}^N))$, which possesses a limit ψ such that, by (2.3.8),

$$\|\psi\|_{L^q(I,L^r)} \leq C\|f\|_{L^{\gamma'}(I,L^{\rho'})}. \tag{2.3.10}$$

Therefore, there exists a subsequence, which we still denote by f_j, such that $\Phi_{f_j}(t) \to \psi(t)$ in $L^r(\mathbb{R}^N)$ for a.a. $t \in I$. Applying (2.3.9), we see that $\psi(t) = \Phi_f(t)$ a.e. on I, and the result follows from (2.3.10). □

COROLLARY 2.3.7. *If $\varphi \in H^1(\mathbb{R}^N)$ and $r \in (2, \frac{2N}{N-2})$ ($r \in (2, \infty]$ if $N = 1$), then $\|\mathcal{T}(t)\varphi\|_{L^r} \to 0$ as $t \to \pm\infty$.*

PROOF. Let q be such that (q, r) is an admissible pair. It follows from Gagliardo-Nirenberg's inequality that there exists C such that for every $t, s \in \mathbb{R}$,

$$\|u(t) - u(s)\|_{L^r} \leq C\|u(t) - u(s)\|_{H^1}^{\frac{2}{q}} \|u(t) - u(s)\|_{L^2}^{\frac{q-2}{q}}.$$

Since $\varphi \in H^1(\mathbb{R}^N)$, $u(t)$ is bounded in $H^1(\mathbb{R}^N)$, and so

$$\|u(t) - u(s)\|_{L^r} \leq C\|u(t) - u(s)\|_{L^2}^{\frac{q-2}{q}}.$$

Furthermore, by Proposition 2.1.1, u_t is bounded in $H^{-1}(\mathbb{R}^N)$, and so u is globally Lipschitz continuous $\mathbb{R} \to H^{-1}(\mathbb{R}^N)$ (see Theorem 1.3.10). Therefore (see Remark 1.3.8(iii)), there exists C such that

$$\|u(t) - u(s)\|_{L^2} \leq C|t - s|^{\frac{1}{2}}$$

and so

$$\|u(t) - u(s)\|_{L^r} \leq C|t - s|^{\frac{q-2}{2q}}.$$

In particular, $u : \mathbb{R} \to L^r(\mathbb{R}^N)$ is uniformly continuous. The result now follows from the property $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ (Theorem 2.3.3), since $q < \infty$. □

REMARK 2.3.8. The estimates of Theorem 2.3.3 (and Corollary 2.3.6) can be generalized to various spaces involving derivatives. For example, for every $m \geq 0$, we may replace φ by $D^\alpha\varphi$ in (2.3.3) with $|\alpha| = m$. Since $D^\alpha\mathcal{T}(t) = \mathcal{T}(t)D^\alpha$, we deduce that

$$\|\mathcal{T}(\cdot)\varphi\|_{L^q(\mathbb{R},W^{m,r})} \leq C\|\varphi\|_{H^m}.$$

Similarly, applying (2.3.5) to $D^\alpha f$, we see that

$$\|\Phi_f\|_{L^q(I,W^{m,r})} \leq C\|f\|_{L^{\gamma'}(I,W^{m,\rho'})}.$$

Since

$$\mathcal{T}(t)[\mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{s}{2}}\widehat{\varphi})] = \mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{s}{2}}\mathcal{F}(\mathcal{T}(t)\varphi)]$$

by (2.2.1), we obtain as well

$$\|\mathcal{T}(\cdot)\varphi\|_{L^q(\mathbb{R},H^{s,r})} \leq C\|\varphi\|_{H^s}, \quad \|\Phi_f\|_{L^q(I,H^{s,r})} \leq C\|f\|_{L^{\gamma'}(I,H^{s,\rho'})}.$$

Using the Littlewood-Paley decomposition, it is easy to establish similar estimates in Besov spaces. (See Corollary 2.3.9 below.) Such estimates are useful to study the nonlinear Schrödinger equation in the fractional order Sobolev space $H^s(\mathbb{R}^N)$; see, for example, Cazenave and Weissler [70], Kato [206], Pecher [295], and Section 4.9 below.

COROLLARY 2.3.9. *Given any $s \in \mathbb{R}$, the following properties hold:*

(i) *If (q, r) is an admissible pair, then there exists a constant C such that*

$$\|\mathcal{T}(\cdot)\varphi\|_{L^q(\mathbb{R},B^s_{r,2})} \le C\|\varphi\|_{B^s_{2,2}}$$

for every $\varphi \in H^s(\mathbb{R}^N)$.

(ii) *If (q, r) is an admissible pair, then there exists a constant C such that*

$$\|\mathcal{T}(\cdot)\varphi\|_{L^q(\mathbb{R},\dot{B}^s_{r,2})} \le C\|\varphi\|_{\dot{B}^s_{2,2}}$$

for every $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ such that $\|\varphi\|_{\dot{B}^s_{2,2}} < \infty$.

(iii) *Let I be an interval of $\mathbb{R}$ (bounded or not), let $J = \overline{I}$, and let $t_0 \in J$. If (γ, ρ) and (q, r) are admissible pairs, then there exists a constant C independent of I such that*

$$\|\Phi_f\|_{L^q(I,B^s_{r,2})} \le C\|f\|_{L^{\gamma'}(I,B^s_{\rho',2})}$$

for every $f \in L^{\gamma'}(I, B^s_{\rho',2}(\mathbb{R}^N))$, where Φ_f is defined by (2.3.4).

(iv) *With the notation of (iii) above, it follows that*

$$\|\Phi_f\|_{L^q(I,\dot{B}^s_{r,2})} \le C\|f\|_{L^{\gamma'}(I,\dot{B}^s_{\rho',2})}$$

for some constant C independent of I.

PROOF. We only prove the homogeneous estimates (ii) and (iv), the proofs of (i) and (iii) being similar. Also, we assume $q < \infty$, the necessary modifications to treat the case $q = \infty$ are obvious. Let η and ψ_j satisfy (1.4.1)–(1.4.2).

STEP 1. Proof of (ii). We set $u(t) = \mathcal{T}(t)\varphi$ and we observe that (see (2.2.6))

$$\mathcal{F}^{-1}\big(|\xi|^s\psi_j\widehat{u(t)}\big) = \mathcal{T}(t)\big(\mathcal{F}^{-1}(|\xi|^s\psi_j\widehat{\varphi})\big),$$

and so

$$\|u\|^2_{L^q(\mathbb{R},\dot{B}^s_{r,2})} = \bigg(\int\Big(\sum_j 2^{2sj}\|\mathcal{T}(t)\big(\mathcal{F}^{-1}(|\xi|^s\psi_j\widehat{\varphi})\big)\|^2_{L^r}\Big)^{\frac{q}{2}}dt\bigg)^{\frac{2}{q}}.$$

Setting $a_j(t) = 2^{2sj}\|\mathcal{T}(t)(\mathcal{F}^{-1}(|\xi|^s\psi_j\widehat{\varphi}))\|^2_{L^r}$ and $p = q/2 \ge 1$, we obtain

$$\begin{aligned}\|u\|^2_{L^q(\mathbb{R},\dot{B}^s_{r,2})} &= \bigg(\int\Big(\sum_j a_j(t)\Big)^p dt\bigg)^{\frac{1}{p}}\\ &= \Big\|\sum_j a_j(\cdot)\Big\|_{L^p(\mathbb{R})}\\ &\le \sum_j \|a_j(\cdot)\|_{L^p(\mathbb{R})}\\ &= \sum_j 2^{2sj}\|\mathcal{T}(\cdot)\big(\mathcal{F}^{-1}(|\xi|^s\psi_j\widehat{\varphi})\big)\|^2_{L^q(\mathbb{R},L^r(\mathbb{R}^N))}.\end{aligned}$$

Applying (2.3.3), we deduce that

$$\|u\|_{L^q(\mathbb{R},\dot{B}^s_{r,2})} \le C\Big(\sum_j 2^{2sj}\big\|\mathcal{F}^{-1}\big(|\xi|^s\psi_j\widehat{\varphi}\big)\big\|^2_{L^2}\Big)^{\frac{1}{2}} = C\|\varphi\|_{\dot{B}^s_{2,2}},$$

which is the desired estimate.

STEP 2. Proof of (iv). We set $u(t) = \Phi_f(t)$ and we observe that (see (2.2.6))

$$2^{sj}\mathcal{F}^{-1}\big(|\xi|^s\psi_j\widehat{u(t)}\big) = \int_{t_0}^t \mathcal{T}(t-s)\big(2^{sj}\mathcal{F}^{-1}(|\xi|^s\psi_j\widehat{f(s)})\big)ds = \Phi_{v_j}(t),$$

where

$$v_j(t) = 2^{sj}\mathcal{F}^{-1}\big(|\xi|^s\psi_j\widehat{f(t)}\big).$$

Therefore,

$$\|u\|^2_{L^q(I,\dot{B}^s_{r,2})} = \Big(\int_I\Big(\sum_j \|\Phi_{v_j}(t)\|^2_{L^r}\Big)^{\frac{q}{2}}dt\Big)^{\frac{2}{q}}.$$

Arguing as in the proof of (ii) above, we obtain

$$\|u\|^2_{L^q(I,\dot{B}^s_{r,2})} \le \sum_j \|\Phi_{v_j}(\cdot)\|^2_{L^q(I,L^r)}.$$

Applying now (2.3.5), we deduce that

$$\|u\|^2_{L^q(I,\dot{B}^s_{r,2})} \le C\sum_j \|v_j\|^2_{L^{\gamma'}(I,L^{\rho'})} = C\sum_j\Big(\int_I b_j(t)dt\Big)^p,$$

where $b_j(t) = \|v_j(t)\|^{\gamma'}_{L^{\rho'}}$ and $p = 2/\gamma' \ge 1$. It follows that

$$\begin{aligned}\|u\|^{\frac{2}{p}}_{L^q(I,\dot{B}^s_{r,2})} &\le C\Big\|\int_I b_j(t)dt\Big\|_{\ell^p(\mathbb{Z})}\\ &\le C\int_I \|b_j(t)\|_{\ell^p(\mathbb{Z})}\,dt\\ &= C\int_I\Big(\sum_j \|v_j(t)\|^2_{L^{\rho'}}\Big)^{\frac{\gamma'}{2}}dt = C\|f\|^{\gamma'}_{L^{\gamma'}(I,\dot{B}^s_{\rho',2})},\end{aligned}$$

which completes the proof. □

2.4. Strichartz's Estimates for Nonadmissible Pairs

It is natural to wonder if (2.3.3) or (2.3.5) hold for nonadmissible pairs (q,r) and (γ,ρ). Concerning (2.3.3), the answer is no. One sees easily that the condition (2.3.1) is necessary. Indeed, assume (2.3.3) holds for some pair (q,r) with $q,r \ge 1$. Fix $\theta \in L^2(\mathbb{R}^N)$, $\theta \ne 0$ and, given $\gamma > 0$, let $\varphi(x) = \theta(\gamma x)$. Setting $w(t) = \mathcal{T}(t)\theta$ and $u(t) = \mathcal{T}(t)\varphi$, it follows from (2.2.2) that $u(t,x) = w(\gamma^2 t, \gamma x)$, so that (2.3.3) implies that

$$\gamma^{-\frac{2}{q}-\frac{N}{r}}\|w\|_{L^q(\mathbb{R},L^r)} \le C\gamma^{-\frac{N}{2}}\|\theta\|_{L^2}.$$

Since this holds for arbitrary $\gamma > 0$, we obtain (2.3.1). In particular, we see that $r \ge 2$ (otherwise $q < 0$). If $N = 2$ and $(q, r) = (2, \infty)$, then the estimate (2.3.3) is false, even if one replaces L^∞ by BMO. (See Montgomery-Smith [252]. Note that (2.3.3) with $(q, r) = (2, \infty)$ holds for radial functions; see Tao [334].) The case $N \ge 3$, $r > 2N/(N-2)$ is more easily eliminated (see Keel and Tao [210]).

Assume now that (2.3.5) holds for some pairs (q, r) and (γ, ρ). Changing f to $\sigma^2 f(\sigma^2 t, \sigma x)$ and applying (2.2.3), we obtain by arguing as above the necessary condition

$$\frac{2}{q} - N\Big(\frac{1}{2} - \frac{1}{r}\Big) + \frac{2}{\gamma} - N\Big(\frac{1}{2} - \frac{1}{\rho}\Big) = 0. \tag{2.4.1}$$

This is clearly satisfied if (q, r) and (γ, ρ) are admissible, but (2.4.1) allows many more choices. We present here a simple case where (2.3.5) holds for nonadmissible pairs. This case corresponds to $\rho = r$ in (2.4.1).

PROPOSITION 2.4.1. *Let I be an interval of $\mathbb{R}$ (bounded or not), set $J = \overline{I}$, let $t_0 \in J$, and consider Φ defined by* (2.3.4). *Assume $2 < r < 2N/(N-2)$ ($2 < r \le \infty$ if $N = 1$) and let $1 < a, \widetilde{a} < \infty$ satisfy*

$$\frac{1}{\widetilde{a}} + \frac{1}{a} = N\Big(\frac{1}{2} - \frac{1}{r}\Big). \tag{2.4.2}$$

It follows that $\Phi_f \in L^a(I, L^r(\mathbb{R}^N))$ for every $f \in L^{\widetilde{a}'}(I, L^{r'}(\mathbb{R}^N))$. Moreover, there exists a constant C independent of I such that

$$\|\Phi_f\|_{L^a(I,L^r)} \le C\|f\|_{L^{\widetilde{a}'}(I,L^{r'})} \quad \textit{for every } f \in L^{\widetilde{a}'}(I, L^{r'}(\mathbb{R}^N)). \tag{2.4.3}$$

PROOF. By density, we need only prove (2.4.3) for $f \in C_c(I, \mathcal{S}(\mathbb{R}^N))$. It follows from (2.2.4) that

$$\|\Phi_f(t)\|_{L^r} \le \int_{t_0}^t (4\pi|t-s|)^{-N(\frac{1}{2}-\frac{1}{r})} \|f(s)\|_{L^{r'}},$$

and so (2.4.3) is an immediate consequence of the Riesz potential inequalities (Stein [319], theorem 1, p. 119). ☐

REMARK 2.4.2. It seems that no necessary and sufficient condition is known for the validity of (2.3.5). The best available results are obtained in Vilela [353]. (See Montgomery-Smith [252]. Note that (2.3.3) with $(q, r) = (2, \infty)$ holds for radial functions; see Tao [334].) The case $N \ge 3$, $r > 2N/(N-2)$ is more easily eliminated (see Keel and Tao [210]).

2.5. Space Decay and Smoothing Effect in $\mathbb{R}^N$

We still assume in this section that $\Omega = \mathbb{R}^N$. We have seen in Proposition 2.2.3 and Theorem 2.3.3 that $\mathfrak{T}(t)$ has a smoothing effect in some L^p spaces. On the other hand, one easily verifies with the formula of Lemma 2.2.4 that for every $\varphi \in L^1(\mathbb{R}^N)$ supported in a compact subset Ω of $\mathbb{R}^N$, the function $(t, x) \mapsto \mathfrak{T}(t)\varphi(x)$ is analytic in $(0, +\infty) \times \mathbb{R}^N$. In other words, $\mathfrak{T}(t)$, being essentially the Fourier transform (see Remark 2.2.5), maps functions having a nice decay as $|x| \to \infty$ to smooth functions.

In this section we establish precise estimates describing this smoothing effect, which enable us to prove similar results in the nonlinear case. Let us first introduce some notation. For $j \in \{1, \dots, N\}$, let P_j be the partial differential operator on $\mathbb{R}^{N+1}$ defined by

$$P_j u(t,x) = (x_j + 2it\partial_j)u(t,x) = x_j u(t,x) + \frac{\partial_j u}{\partial x_j}(t,x)\,. \tag{2.5.1}$$

For a multi-index α, we define the partial differential operator P_α on $\mathbb{R}^{N+1}$ by

$$P_\alpha = \prod_{i=1}^{N} P_i^{\alpha_i}\,. \tag{2.5.2}$$

Furthermore for $x \in \mathbb{R}^N$, we set

$$x^\alpha = \prod_{i=1}^{N} x_i^{\alpha_i}\,. \tag{2.5.3}$$

Consider a smooth function $u : \mathbb{R}^{N+1} \to \mathbb{C}$. An easy calculation shows that

$$P_j u(t,x) = 2ite^{i\frac{|x|^2}{4t}} \frac{\partial}{\partial_{x_j}} \left(e^{-i\frac{|x|^2}{4t}} u\right),$$

from which we deduce by an obvious recurrence argument that

$$P_\alpha u(t,x) = (2it)^{|\alpha|} e^{i\frac{|x|^2}{4t}} D^\alpha \left(e^{-i\frac{|x|^2}{4t}} u\right). \tag{2.5.4}$$

On the other hand, a formal calculation shows that

$$[P_\alpha, i\partial_t + \Delta] = 0\,, \tag{2.5.5}$$

where $[\cdot,\cdot]$ is the commutator bracket. In other words, if u is a smooth solution of the linear Schrödinger equation, then so is $P_\alpha u$. In particular, if we consider $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and if we set $u(t) = \mathfrak{T}(t)\varphi$, then $u_\alpha = P_\alpha u$ is a solution of Schrödinger's equation. It follows that

$$u_\alpha(t) = \mathfrak{T}(t) u_\alpha(0) = \mathfrak{T}(t) x^\alpha \varphi; \tag{2.5.6}$$

and so $\|u_\alpha\|_{L^2} = \|x^\alpha \varphi\|_{L^2}$. By (2.5.4), this implies that

$$(2|t|)^{|\alpha|} \left\| D^\alpha (e^{-i\frac{|x|^2}{4t}} u(t)) \right\|_{L^2} = \|x^\alpha \varphi\|_{L^2}\,. \tag{2.5.7}$$

By density, we immediately obtain the following result.

PROPOSITION 2.5.1. *Let α be a multi-index. Let $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ be such that $x^\alpha \varphi \in L^2(\mathbb{R}^N)$. If $u(t) = \mathfrak{T}(t)\varphi \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^N))$, then*

$$D^\alpha e^{-i\frac{|x|^2}{4t}} u(t) \in C(\mathbb{R} \setminus \{0\}, L^2(\mathbb{R}^N))$$

and formula (2.5.7) *holds for every $t \neq 0$.*

COROLLARY 2.5.2. *Let $\varphi \in L^2(\mathbb{R}^N)$, and assume that $(1+|x|^m)\varphi \in L^2(\mathbb{R}^N)$ for some nonnegative integer m. It follows that $e^{-i\frac{|x|^2}{4t}}u(t) \in C(\mathbb{R}\setminus\{0\}, H^m(\mathbb{R}^N))$, and if k is the integer part of $m/2$, then*

$$u \in \bigcap_{0\le j\le k} C^j(\mathbb{R}\setminus\{0\}, H^{m-2j}_{\text{loc}}(\mathbb{R}^N)).$$

In particular, if $(1+|x|^m)\varphi \in L^2(\mathbb{R}^N)$ for every nonnegative integer m, then $u \in C^\infty(\mathbb{R}\setminus\{0\}\times\mathbb{R}^N)$.

PROOF. The H^m regularity of $e^{-i\frac{|x|^2}{4t}}u(t)$ follows from Proposition 2.5.1. Since $e^{i\frac{|x|^2}{4t}} \in C^\infty(\mathbb{R}\setminus\{0\}\times\mathbb{R}^N)$, we see that $u \in C(\mathbb{R}\setminus\{0\}, H^m_{\text{loc}}(\mathbb{R}^N))$. The regularity of the time derivatives follows from the equation. □

REMARK 2.5.3. Formula (2.5.6) means that $P_\alpha\mathcal{T}(t)\varphi = \mathcal{T}(t)(x^\alpha\varphi)$ or alternatively, by setting $\varphi = \mathcal{T}(-t)\psi$, $\mathcal{T}(-t)P_\alpha\psi = x^\alpha\mathcal{T}(-t)\psi$. In particular, $(x+2it\nabla)\mathcal{T}(t) = \mathcal{T}(t)x$, or equivalently $\mathcal{T}(-t)(x+2it\nabla) = x\mathcal{T}(-t)$.

COROLLARY 2.5.4. *Let $\varphi \in L^2(\mathbb{R}^N)$ be such that $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$, and let $u(t) = \mathcal{T}(t)\varphi$.*

(i) *The function $t \mapsto (x+2it\nabla)u(t,x)$ belongs to $L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ for every admissible pair (q,r).*
(ii) *$u \in C(\mathbb{R}/\{0\}, L^r(\mathbb{R}^N))$ for every $r \in [2, \frac{2N}{N-2}]$ ($r \in [2,\infty)$ if $N=2$, $r \in [2,\infty]$ if $N=1$), and there exists C, depending only on r and N, such that*

$$\|u(t)\|_{L^r} \le C(\|\varphi\|_{L^2} + \|x\varphi\|_{L^2})|t|^{-N(\frac{1}{2}-\frac{1}{r})} \quad \textit{for every } t \ne 0.$$

PROOF. By (2.5.1) and (2.5.6), $(x+2it\nabla)u(t,x) = \mathcal{T}(t)\psi$, where $\psi(x) = x\varphi(x)$, and so (i) follows from Theorem 2.3.3. Next, let $v(t,x) = e^{-i\frac{|x|^2}{4t}}u(t,x)$. By (2.5.6) and (2.5.7), $\nabla v \in C(\mathbb{R}/\{0\}, L^2(\mathbb{R}^N))$ and

$$\|\nabla v(t)\|_{L^2} \le C|t|^{-1}\|x\varphi\|_{L^2}.$$

The result follows from Gagliardo-Nirenberg's inequality, since $|u| = |v|$. □

2.6. Homogeneous Data in $\mathbb{R}^N$

In this section we study the action of the Schrödinger group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on homogeneous functions. The resulting estimates are useful for constructing self-similar and asymptotically self-similar solutions of certain nonlinear Schrödinger equations. They are also useful to describe the possible decay rates of $\|\mathcal{T}(t)\varphi\|_{L^r}$.

For simplicity, we only consider functions of the form $|x|^{-p}$, so that the proofs depend only on explicit calculations with the gamma function and analytic continuation arguments. We refer to [73, 74, 286, 298, 302] for more general results.

We observe that if $p \in \mathbb{C}$ and $0 < \operatorname{Re} p < N$, then $\psi(x) = |x|^{-p}$ does not belong to any space $L^q(\mathbb{R}^N)$. However, $\psi \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\psi \in \mathcal{S}'(\mathbb{R}^N)$. Since the Schrödinger group operates on $\mathcal{S}'(\mathbb{R}^N)$ by Remark 2.2.1(i), it follows that $\mathcal{T}(t)\psi$ is well defined as a tempered distribution for all $t > 0$. In fact, much more can be said.

THEOREM 2.6.1. *Let $\psi(x) = |x|^{-p}$ with $p \in \mathbb{C}$ and $0 < \operatorname{Re} p < N$. It follows that $\mathcal{T}(t)\psi \in L^r(\mathbb{R}^N)$ for all $t > 0$ and all r such that*

$$r > \max\left\{\frac{N}{\operatorname{Re} p}, \frac{N}{N - \operatorname{Re} p}\right\}. \tag{2.6.1}$$

For such r,

$$\|\mathcal{T}(t)\psi\|_{L^r} = t^{\frac{N}{2r} - \frac{\operatorname{Re} p}{2}} \|\mathcal{T}(1)\psi\|_{L^r} \quad \text{for } t > 0. \tag{2.6.2}$$

Moreover, if $u(t,x) = \mathcal{T}(t)\psi(x)$, then $u \in C^\infty((0,\infty) \times \mathbb{R}^N)$.

Before proceeding to the proof, we make some simple observations. Given $\lambda > 0$, let D_λ be the dilation operator

$$D_\lambda w(x) = \lambda^p w(\lambda x).$$

D_λ is defined on $\mathcal{S}(\mathbb{R}^N)$ and is extended by duality to $\mathcal{S}'(\mathbb{R}^N)$ by

$$\langle D_\lambda w, \psi\rangle_{\mathcal{S}',\mathcal{S}} = \lambda^{2p-N} \langle w, D_{\frac{1}{\lambda}} \psi\rangle_{\mathcal{S}',\mathcal{S}}$$

for all $w \in \mathcal{S}'(\mathbb{R}^N)$ and all $\psi \in \mathcal{S}(\mathbb{R}^N)$. It is immediate that

$$\|D_\lambda w\|_{L^r} = \lambda^{\operatorname{Re} p - \frac{N}{r}} \|w\|_{L^r} \tag{2.6.3}$$

whenever $w \in L^r(\mathbb{R}^N)$. Moreover, it is easy to check, first by applying (2.2.2) for $w \in \mathcal{S}(\mathbb{R}^N)$, then by duality for $w \in \mathcal{S}'(\mathbb{R}^N)$, that $\mathcal{T}(t)w = D_\lambda \mathcal{T}(\lambda^2 t) D_{\frac{1}{\lambda}} w$. In particular, letting $\lambda = t^{-\frac{1}{2}}$,

$$\mathcal{T}(t)w = D_{\frac{1}{\sqrt{t}}} \mathcal{T}(1) D_{\sqrt{t}} w \tag{2.6.4}$$

for all $t > 0$. Now if $\psi = |x|^{-p}$, then $D_\lambda \psi = \psi$ for all $\lambda > 0$. Therefore, it follows from (2.6.4) that

$$\mathcal{T}(t)\psi = D_{\frac{1}{\sqrt{t}}} \mathcal{T}(1)\psi. \tag{2.6.5}$$

In view of (2.6.5) and (2.6.3), all the conclusions of Theorem 2.6.1 follow if we show that

$$\mathcal{T}(1)\psi \in C^\infty(\mathbb{R}^N) \tag{2.6.6}$$

and that

$$\mathcal{T}(1)\psi \in L^r(\mathbb{R}^N) \tag{2.6.7}$$

for all r satisfying (2.6.1).

We next establish some notation and recall some well-known facts. The gamma function satisfies the following relation

$$c^{-z}\Gamma(z) = \int_0^\infty e^{-ct} t^{z-1}\, dt, \tag{2.6.8}$$

valid for $c > 0$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Also, if $\mathcal{O}$ denotes the domain of the standard branch of the logarithm; i.e.,

$$\mathcal{O} = \{z \in \mathbb{C} : z \text{ is not a negative real number or } 0\},$$

then for a fixed complex number p, the function $f(z) = z^p = e^{p \log z}$ is analytic in $\mathcal{O}$. Note that if $r > 0$, then $(rz)^p = r^p z^p$ for all $z \in \mathcal{O}$. Also, $|r^p| = r^{\operatorname{Re} p}$ if $r > 0$.

Another function that plays a central role in the analysis is given by

$$H(y;a,b)=\int_0^1 e^{iyr}r^{a-1}(1-r)^{b-1}\,dr\,, \tag{2.6.9}$$

where $a,b\in\mathbb{C}$ with $\operatorname{Re}a>0$ and $\operatorname{Re}b>0$, and $y\in\mathbb{R}$ (or $\mathbb{C}$). Note that $H(y;a,b)$ is separately analytic as a function of y,a, and b in the domains just specified.

LEMMA 2.6.2. *Let $\psi(x)=|x|^{-p}$ with $0<\operatorname{Re}p<N$. For $t>0$ and $x\in\mathbb{R}^N$,*

$$[\mathfrak{T}(t)\psi](x)=(4it)^{-\frac{p}{2}}\Gamma(p/2)^{-1}H\Big(\frac{|x|^2}{4t};\frac{p}{2},\frac{N-p}{2}\Big)\,, \tag{2.6.10}$$

where the function H is defined by (2.6.9).

PROOF. The basic idea is to express $|x|^{-p}$ using the gamma function, then change variables so that the Gauss kernel

$$G_s(x)=(4\pi s)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4s}}$$

appears in the integral. It will then be possible to apply the operator $e^{z\Delta}$. By formula (2.6.8), if $x\neq 0$

$$\begin{aligned}|x|^{-p}&=\Gamma(p/2)^{-1}\int_0^\infty e^{-|x|^2t}t^{\frac{p}{2}-1}\,dt\\&=4^{-\frac{p}{2}}\Gamma(p/2)^{-1}\int_0^\infty e^{-\frac{|x|^2}{4s}}s^{-\frac{p}{2}-1}\,ds\\&=4^{-\frac{p}{2}}(4\pi)^{\frac{N}{2}}\Gamma(p/2)^{-1}\int_0^\infty G_s(x)s^{\frac{N}{2}-\frac{p}{2}-1}\,ds\,.\end{aligned}$$

This integral, in addition to being absolutely convergent for each $x\neq 0$, is an absolutely convergent Bochner integral in $L^1(\mathbb{R}^N)+C_0(\mathbb{R}^N)$. In other words,

$$\psi=4^{-\frac{p}{2}}(4\pi)^{\frac{N}{2}}\Gamma(p/2)^{-1}\int_0^\infty G_s(\cdot)s^{\frac{N}{2}-\frac{p}{2}-1}\,ds\,.$$

Next, we apply the heat semigroup, $e^{t\Delta}$ for $t>0$, which gives (since $e^{t\Delta}G_s=G_{t+s}$)

$$e^{t\Delta}\psi=4^{-\frac{p}{2}}(4\pi)^{\frac{N}{2}}\Gamma(p/2)^{-1}\int_0^\infty G_{s+t}(\cdot)s^{\frac{N}{2}-\frac{p}{2}-1}\,ds\,.$$

This integral now is absolutely convergent in $C_0(\mathbb{R}^N)$, where pointwise evaluation is a bounded linear functional. Making the change of variables $r=\frac{t}{s+t}$, we see that for all $x\in\mathbb{R}^N$

$$\begin{aligned}\Gamma(p/2)(e^{t\Delta}\psi)(x)&=4^{-\frac{p}{2}}(4\pi)^{\frac{N}{2}}\int_0^1 G_{\frac{t}{r}}(x)\left(\frac{t-tr}{r}\right)^{\frac{N-p-2}{2}}\frac{t}{r^2}\,dr\\&=(4t)^{-\frac{p}{2}}(4\pi t)^{\frac{N}{2}}\int_0^1 G_{\frac{t}{r}}(x)r^{\frac{p-N-2}{2}}(1-r)^{\frac{N-p-2}{2}}\,dr \qquad (2.6.11)\\&=(4t)^{-\frac{p}{2}}\int_0^1 e^{-\frac{r|x|^2}{4t}}r^{\frac{p}{2}-1}(1-r)^{\frac{N-p-2}{2}}\,dr\,.\end{aligned}$$

We next claim that formula (2.6.11) is valid not only for $t>0$, but for all $t\in\mathbb{C}$ with $\operatorname{Re}t>0$. Indeed, if $\eta\in\mathcal{S}(\mathbb{R}^N)$, then $\langle e^{t\Delta}\psi,\eta\rangle$ is an analytic function of t

on the open half plane $\mathrm{Re}\,t > 0$, and continuous on the closed half plane $\mathrm{Re}\,t \geq 0$. Next, if we integrate the right side of (2.6.11) against $\eta(x)$ over $\mathbb{R}^N$, the result is also an analytic function of t on the right half plane $\mathrm{Re}\,t > 0$, continuous at least on the closed half plane with $t = 0$ removed. By the identity theorem, these two functions are equal on the open half plane. By continuity, they are equal also for $t = i\tau$, $\tau \in \mathbb{R}$, $\tau \neq 0$. Since η is an arbitrary Schwartz function, (2.6.11), as an identity between two tempered distributions, has been proved for all complex $t \neq 0$ with $\mathrm{Re}\,t \geq 0$. This establishes the proposition. □

COROLLARY 2.6.3. *Let $\psi(x) = |x|^{-p}$ with $0 < \mathrm{Re}\,p < N$. It follows that $\mathfrak{T}(1)\psi \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.*

LEMMA 2.6.4. *If $y > 0$, $\mathrm{Re}\,a > 0$, and $\mathrm{Re}\,b > 0$, and if n and m are nonnegative integers such that*

$$n + 2 > \mathrm{Re}\,a \quad \textit{and} \quad m + 2 > \mathrm{Re}\,b,$$

then

$$\begin{aligned} H(y; a, b) &= y^{-a} \sum_{k=0}^{m} C_k(a,b) e^{\frac{(a+k)\pi i}{2}} y^{-k} \\ &\quad + C_{m+1}(a,b) y^{-a-m-1} \frac{m+1}{\Gamma(m+2-b)} \\ &\quad \times \int_0^\infty \int_0^1 (1-s)^m \left(-i - \frac{st}{y}\right)^{-a-m-1} ds\, e^{-t} t^{m+1-b}\, dt \\ &\quad + e^{iy} y^{-b} \sum_{k=0}^{n} C_k(b,a) e^{-\frac{(b+k)\pi i}{2}} y^{-k} \\ &\quad + C_{n+1}(b,a) e^{iy} y^{-b-n-1} \frac{n+1}{\Gamma(n+2-a)} \\ &\quad \times \int_0^\infty \int_0^1 (1-s)^n \left(i - \frac{st}{y}\right)^{-b-n-1} ds\, e^{-t} t^{n+1-a}\, dt, \end{aligned} \tag{2.6.12}$$

where

$$C_k(a,b) = \frac{\Gamma(a+k)}{k!} \frac{\Gamma(k+1-b)}{\Gamma(1-b)}. \tag{2.6.13}$$

PROOF. For the moment, we assume that

$$0 < \mathrm{Re}\,a < 1, \qquad 0 < \mathrm{Re}\,b < 1.$$

Using formula (2.6.8) twice, first with $c = r$, $z = 1 - a$, and then with $c = 1 - r$, $z = 1 - b$, we rewrite formula (2.6.9) as follows:

$$
\begin{aligned}
&\Gamma(1-a)\Gamma(1-b)H(y;a,b) \\
&= \int_0^\infty \int_0^\infty \int_0^1 e^{iyr} e^{-rs} s^{-a} e^{-(1-r)t} t^{-b}\, dr\, ds\, dt \\
&= \int_0^\infty \int_0^\infty \int_0^1 e^{(iy-s+t)r}\, dr\, s^{-a} e^{-t} t^{-b}\, ds\, dt \\
&= \int_0^\infty \int_0^\infty \frac{e^{iy-s+t}-1}{iy-s+t} s^{-a} e^{-t} t^{-b}\, ds\, dt \\
&= \int_0^\infty \int_0^\infty \frac{s^{-a}}{-iy-t+s} e^{-t} t^{-b}\, ds\, dt \\
&\quad + e^{iy} \int_0^\infty \int_0^\infty \frac{t^{-b}}{iy-s+t} e^{-s} s^{-a}\, dt\, ds \\
&= \int_0^\infty \int_0^\infty \frac{s^{-a}}{-iy-t+s}\, ds\, e^{-t} t^{-b}\, dt \\
&\quad + e^{iy} \int_0^\infty \int_0^\infty \frac{s^{-b}}{iy-t+s}\, ds\, e^{-t} t^{-a}\, dt .
\end{aligned}
\tag{2.6.14}
$$

We therefore consider the integral

$$
\rho(w) = \int_0^\infty \frac{s^{-a}}{w+s}\, ds ,
$$

where $w \in \mathcal{O}$ and $0 < \operatorname{Re} a < 1$. It is known (by changing variables in the beta function) that $\rho(1) = \Gamma(1-a)\Gamma(a)$. Next, if w is a positive real number, we set $s = wt$, and so

$$
\rho(w) = \int_0^\infty \frac{(wt)^{-a}}{w+wt} w\, dt = w^{-a}\Gamma(1-a)\Gamma(a) , \qquad w > 0 . \tag{2.6.15}
$$

Since $\rho(w)$ and $w^{-a} = e^{-a \log w}$ are both holomorphic in $\mathcal{O}$, (2.6.15) is true for all $w \in \mathcal{O}$. Substituting (2.6.15) back into (2.6.14) with $w = \pm iy - t$, we see that

$$
\begin{aligned}
H(y;a,b) &= \frac{\Gamma(a)}{\Gamma(1-b)} \int_0^\infty (-iy-t)^{-a} e^{-t} t^{-b}\, dt \\
&\quad + \frac{\Gamma(b)}{\Gamma(1-a)} e^{iy} \int_0^\infty (iy-t)^{-b} e^{-t} t^{-a}\, dt .
\end{aligned}
\tag{2.6.16}
$$

The next step is to replace $(-iy-t)^{-a}$ and $(iy-t)^{-b}$ in (2.6.16) by their finite Taylor formulas around $t = 0$ with integral remainder terms. If $f(t) = (-iy-t)^{-a}$ and $g(t) = (iy-t)^{-b}$, then

$$
\begin{aligned}
f^{(k)}(t) &= a(a+1)\cdots(a+k-1)(-iy-t)^{-a-k} , \\
g^{(k)}(t) &= b(b+1)\cdots(b+k-1)(-iy-t)^{-b-k} .
\end{aligned}
$$

Since

$$
f(t) = \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(0) t^k + \frac{1}{m!} \int_0^t (t-s)^m f^{(m+1)}(s)\, ds ,
$$

and similarly for $g(t)$, we see that

$$\begin{aligned}H(y;a,b) &= \frac{\Gamma(a)}{\Gamma(1-b)}\sum_{k=0}^{m}\frac{f^{(k)}(0)}{k!}\int_0^\infty e^{-t}t^{-b+k}\,dt\\
&+\frac{\Gamma(a)}{\Gamma(1-b)}\int_0^\infty \frac{1}{m!}\int_0^t (t-s)^m f^{(m+1)}(s)ds\, e^{-t}t^{-b}\,dt\\
&+\frac{\Gamma(b)}{\Gamma(1-a)}e^{iy}\sum_{k=0}^{n}\frac{g^{(k)}(0)}{k!}\int_0^\infty e^{-t}t^{-a+k}\,dt\\
&+\frac{\Gamma(b)}{\Gamma(1-a)}e^{iy}\int_0^\infty \frac{1}{n!}\int_0^t (t-s)^n g^{(n+1)}(s)ds\, e^{-t}t^{-a}\,dt\,;\end{aligned}$$

and so

$$\begin{aligned}&H(y;a,b)\\
&= \frac{\Gamma(a)}{\Gamma(1-b)}\sum_{k=0}^{m}\frac{a(a+1)\cdots(a+k-1)}{k!}(-iy)^{-a-k}\Gamma(k+1-b)\\
&+\frac{\Gamma(a)}{\Gamma(1-b)}\int_0^\infty\frac{1}{m!}\\
&\times\int_0^t (t-s)^m a(a+1)\cdots(a+m)(-iy-s)^{-a-m-1}\,ds\, e^{-t}t^{-b}\,dt\\
&+\frac{\Gamma(b)}{\Gamma(1-a)}e^{iy}\sum_{k=0}^{n}\frac{b(b+1)\cdots(b+k-1)}{k!}(iy)^{-b-k}\Gamma(k+1-a)\\
&+\frac{\Gamma(b)}{\Gamma(1-a)}e^{iy}\int_0^\infty\frac{1}{n!}\\
&\times\int_0^t (t-s)^n b(b+1)\cdots(b+n)(iy-s)^{-b-n-1}\,ds\, e^{-t}t^{-a}\,dt\,.\end{aligned}$$

Furthermore, since $y>0$,

$$\begin{aligned}&\int_0^\infty\int_0^t (t-s)^m(-iy-s)^{-a-m-1}\,ds\,e^{-t}t^{-b}\,dt =\\
&\qquad y^{-a-m-1}\int_0^\infty\int_0^1 (1-s)^m\left(-i-\frac{st}{y}\right)^{-a-m-1} ds\, e^{-t}t^{m+1-b}\,dt\,;\end{aligned}$$

and so we obtain the formulation (2.6.12)–(2.6.13).

Formula (2.6.12) has been proved only for $y>0$, $0<\operatorname{Re} a<1$, and $0<\operatorname{Re} b<1$. On the other hand, the right-hand side is an analytic function in a for $0<\operatorname{Re} a<n+2$, with $y>0$ and b such that $0<\operatorname{Re} b<m+2$ fixed, and also an analytic function in b for $0<\operatorname{Re} b<m+2$, with $y>0$ and a such that $0<\operatorname{Re} a<n+2$ fixed. (Recall that $1/\Gamma(z)$ is an entire function.) It follows that (2.6.12) holds for all $y>0$, and all a,b in the region stated in the lemma. □

PROPOSITION 2.6.5. *Let $\psi(x)=|x|^{-p}$ where $0<\operatorname{Re} p<N$. It follows that $\mathfrak{T}(1)\psi\in L^r(\mathbb{R}^N)$ for all r satisfying* (2.6.1). *Moreover, $\mathfrak{T}(1)\psi(x)$ is given by the explicit formula* (2.6.17) *below for $x\neq 0$.*

PROOF. We apply the asymptotic expression from Lemma 2.6.4 to formula (2.6.10) in Lemma 2.6.2 with $a = p/2$, $b = (N-p)/2$, $n > \frac{\mathrm{Re}\,p}{2} - 2$, and $m > \frac{N-\mathrm{Re}\,p}{2} - 2$. We see that if $x \neq 0$, then

$$\begin{aligned}
&[\mathfrak{T}(1)\psi](x)\\
&= |x|^{-p}\sum_{k=0}^{m} A_k(a,b)e^{\frac{k\pi i}{2}}\left(\frac{|x|^2}{4}\right)^{-k}\\
&+ A_{m+1}(a,b)|x|^{-p}\left(\frac{|x|^2}{4}\right)^{-m-1}\frac{(m+1)e^{-\frac{a\pi i}{2}}}{\Gamma(m+2-b)}\\
&\times \int_0^\infty\int_0^1 (1-s)^m\left(-i-\frac{4st}{|x|^2}\right)^{-a-m-1} ds\, e^{-t}t^{m+1-b}\,dt\\
&+ e^{\frac{i|x|^2}{4}}|x|^{-N+p}(4)^{\frac{N}{2}-p}\sum_{k=0}^{n} B_k(b,a)e^{-\frac{(N+2k)\pi i}{4}}\left(\frac{|x|^2}{4}\right)^{-k}\\
&+ e^{\frac{i|x|^2}{4}}|x|^{-N+p}(4)^{\frac{N}{2}-p}B_{n+1}(b,a)\left(\frac{|x|^2}{4}\right)^{-n-1}\frac{(n+1)e^{-\frac{a\pi i}{2}}}{\Gamma(n+2-a)}\\
&\times \int_0^\infty\int_0^1 (1-s)^n\left(i-\frac{4st}{|x|^2}\right)^{-b-n-1} ds\, e^{-t}t^{n+1-a}\,dt,
\end{aligned} \tag{2.6.17}$$

where

$$A_k(a,b) = \frac{C_k(a,b)}{\Gamma(a)} = \frac{\Gamma(a+k)}{\Gamma(a)k!}\frac{\Gamma(k+1-b)}{\Gamma(1-b)}$$

and

$$B_k(b,a) = \frac{C_k(b,a)}{\Gamma(a)} = \frac{\Gamma(b+k)}{\Gamma(a)k!}\frac{\Gamma(k+1-a)}{\Gamma(1-a)}.$$

By Corollary 2.6.3, $\mathfrak{T}(1)\psi \in C^\infty(\mathbb{R}^N)$. Therefore, to determine whether $\mathfrak{T}(1)\psi \in L^r(\mathbb{R}^N)$, it suffices to consider $|x|$ large. Proposition 2.6.5 now follows immediately from formula (2.6.17). □

PROOF OF THEOREM 2.6.1. As observed before, we need only establish properties (2.6.6) and (2.6.7). They follow from Corollary 2.6.3 and Proposition 2.6.5, respectively. □

REMARK 2.6.6. Here are some comments on Theorem 2.6.1.

(i) Note that $A_0(a,b) = B_0(a,b) = 1$. Therefore, the term with slower decay in (2.6.17) is either of order $|x|^{-p}$ or of order $|x|^{N-p}$, depending on p. Thus, if r does not satisfy the condition (2.6.1), then $\mathfrak{T}(1)\psi \notin L^r(\mathbb{R}^N)$. If $\mathrm{Re}\,p < N/2$, then $\mathfrak{T}(1)\psi$ behaves like $|x|^{-p}$ as $|x| \to \infty$. If $\mathrm{Re}\,p > N/2$, then $\mathfrak{T}(1)\psi$ behaves like $ce^{i|x|^2/4}|x|^{-N+p}$ as $|x| \to \infty$. And if $\mathrm{Re}\,p = N/2$, then $\mathfrak{T}(1)\psi$ behaves like $|x|^{-p} + ce^{i|x|^2/4}|x|^{-N+p}$ as $|x| \to \infty$. In particular, $|\mathfrak{T}(1)\psi| \approx |x|^{-\min\{\mathrm{Re}\,p, N-\mathrm{Re}\,p\}}$, so that the decay is at most $|x|^{-N/2}$. This is justified by the fact that $\mathfrak{T}(1)\psi$ cannot be in any L^q space with $q \le 2$ for otherwise we would have $\psi \in L^{q'}$.

(ii) The conclusions of Theorem 2.6.1 hold for the more general homogeneous function $\psi(x) = \omega(x)|x|^{-p}$ with $0 < \mathrm{Re}\,p < N$ and ω homogeneous of degree

0 and "sufficiently smooth." See Cazenave and Weissler [73, 74], Oru [286], Planchon [298], and Ribaud and Youssfi [302].

(iii) In view of (2.6.5), $u(t) = \mathcal{T}(t)\psi$ is a self-similar solution for the group of transformations $u_\lambda(t,x) = \lambda^p u(\lambda^2 t, \lambda x)$. This can also be seen independently by observing that ψ is homogeneous of degree $-p$.

COROLLARY 2.6.7. *Let $\psi(x) = |x|^{-p}$ with $0 < \operatorname{Re} p < N$. Suppose $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies $\varphi - \psi \in L^{r'}(\mathbb{R}^N)$ for some*

$$r > \min\left\{\frac{N}{\operatorname{Re} p}, \frac{N}{N - \operatorname{Re} p}\right\}. \tag{2.6.18}$$

It follows that

$$t^{\frac{\operatorname{Re} p}{2} - \frac{N}{2r}} \|\mathcal{T}(t)(\varphi - \psi)\|_{L^r} \le t^{-\frac{r(N - \operatorname{Re} p) - N}{2r}} \|\varphi - \psi\|_{L^{r'}} \to 0 \quad \text{as } t \to \infty. \tag{2.6.19}$$

In particular,

$$t^{\frac{\operatorname{Re} p}{2} - \frac{N}{2r}} \|\mathcal{T}(t)\varphi\|_{L^r} \to \|\mathcal{T}(1)\psi\|_{L^r} \quad \text{as } t \to \infty. \tag{2.6.20}$$

PROOF. By (2.2.4),

$$\|\mathcal{T}(t)(\varphi - \psi)\|_{L^r} \le t^{-\frac{N(r-2)}{2r}} \|\varphi - \psi\|_{L^{r'}}.$$

Hence (2.6.19) follows by using the assumption (2.6.18). The result (2.6.20) now follows from (2.6.19), (2.6.18), and (2.6.2). □

REMARK 2.6.8. In view of (2.6.2) and (2.6.20), we can determine the possible decay rates as $t \to \infty$ of $\|\mathcal{T}(t)\varphi\|_{L^r}$ for $2 < r \le \infty$.

(i) The decay rate (as $t \to \infty$) given by (2.2.4) is optimal, since

$$\liminf_{t \to \infty} t^{\frac{N(r-2)}{2r}} \|\mathcal{T}(t)\varphi\|_{L^r} > 0 \tag{2.6.21}$$

for every $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, $\varphi \not\equiv 0$, with the convention that $\|\psi\|_{L^r} = +\infty$ if $\psi \notin L^r$. (See Strauss [322] and Kato [205].) Indeed, let $\varphi \in \mathcal{S}(\mathbb{R}^N)$, $\varphi \not\equiv 0$, and set $u(t) = \mathcal{T}(t)\varphi$. It is easy to check that v defined by

$$v(t,x) = t^{-\frac{N}{2}} e^{\frac{i|x|^2}{4t}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)$$

for $t > 0$ is also a solution of Schrödinger's equation in $\mathcal{S}'(\mathbb{R}^N)$. By duality, the same holds for $\varphi \in \mathcal{S}'(\mathbb{R}^N)$. In particular, $v(t) = \mathcal{T}(t)\psi$ for some $\psi \in \mathcal{S}'(\mathbb{R}^N)$. Now, assuming by contradiction that

$$t_n^{\frac{N(r-2)}{2r}} \|\mathcal{T}(t_n)\varphi\|_{L^r} \to 0 \quad \text{for some } t_n \to \infty,$$

we deduce, setting $s_n = 1/t_n$, that

$$\|\mathcal{T}(s_n)\psi\|_{L^r} = t_n^{\frac{N(r-2)}{2r}} \|\mathcal{T}(t_n)\varphi\|_{L^r} \to 0 \quad \text{as } n \to \infty.$$

In particular, $\mathcal{T}(s_n)\psi \to 0$ in $\mathcal{S}'(\mathbb{R}^N)$. By Remark 2.2.1(i), we conclude that $\psi = 0$, thus $\varphi = 0$, which proves the claim. Note also that the maximal decay rate is indeed achieved if $\varphi \in L^{r'}(\mathbb{R}^N)$.

(ii) Let $0 < \nu < \frac{N(r-2)}{2r}$. If $\varphi(x) = |x|^{-2\nu-\frac{N}{r}}$, we deduce from (2.6.2) that $\|\mathcal{T}(t)\varphi\|_{L^r} \approx t^{-\nu}$. Thus all decay rates slower that the maximal decay $t^{-\frac{N(r-2)}{2r}}$ are achieved for some $\varphi \in \mathcal{S}'(\mathbb{R}^N)$.

(iii) $L^2(\mathbb{R}^N)$ being of particular importance for Schrödinger's equation, and even more for its nonlinear versions, one may wonder what are the decay rates achieved for initial values in $L^2(\mathbb{R}^N)$. To see this, let $0 < \nu < \frac{N(r-2)}{2r}$ and $\psi(x) = |x|^{-2\nu-\frac{N}{r}}$. Consider $\varphi \in C^\infty(\mathbb{R}^N)$ with $\varphi(x) = \psi(x)$ for $|x|$ large. Since $\varphi - \psi$ has compact support and the singularity $|x|^{-2\nu-\frac{N}{r}}$, we see that $\varphi - \psi \in L^{r'}(\mathbb{R}^N)$. Thus $\|\mathcal{T}(t)\varphi\|_{L^r} \approx t^{-\nu}$. Next, $\varphi \in L^2(\mathbb{R}^N)$ for $\nu > \frac{N(r-2)}{4r}$. In particular, all possible decay rates between the maximal decay $t^{-\frac{N(r-2)}{2r}}$ and $t^{-\frac{N(r-2)}{4r}}$ are achieved by L^2 solutions. On the other hand, the lower limit $\frac{N(r-2)}{4r}$ is optimal (in fact it is not even achieved), at least for $r \le 2N/(N-2)$ ($r \le \infty$ if $N = 1$, $r < \infty$ if $N = 2$). Indeed, it follows from Strichartz's estimate that for such r's,

$$\|\mathcal{T}(\cdot)\varphi\|_{L^{\frac{4r}{N(r-2)}}((0,\infty),L^r(\mathbb{R}^N))} \le C\|\varphi\|_{L^2}$$

so that

$$\liminf_{t\to\infty} t^{\frac{N(r-2)}{4r}} \|\mathcal{T}(t)\varphi\|_{L^r} = 0\,.$$

2.7. Comments

As we will see in Chapter 4, Theorem 2.3.3 is an essential tool for the study of the nonlinear Schrödinger equation in $\mathbb{R}^N$. Therefore, it is natural to ask if Theorem 2.3.3 can be generalized to a wider class of equations. In fact, a careful analysis of the proof shows that it uses only two properties. The first one is the identity $\mathcal{T}(t)^* = \mathcal{T}(-t)$, which is valid for every skew-adjoint generator. The second one is the estimate (2.2.4), which itself follows from Lemma 2.2.4. Therefore, such an inequality holds whenever $\mathcal{T}(t)$ has a kernel $K(t)$ whose L^∞-norm behaves like $|t|^{-\frac{N}{2}}$ (at least near 0). In particular, we have the following result (see Keel and Tao [210] for more general results).

THEOREM 2.7.1. *Let A be a self-adjoint, ≤ 0 operator on $X = L^2(\Omega)$. Assume that there exists $t_0 > 0$ such that for every $t \in (-t_0, 0) \cup (0, t_0)$, $\mathcal{T}(t) = e^{itA}$ maps $L^1(\Omega)$ to $L^\infty(\Omega)$, with a norm less than $K|t|^{-\frac{N}{2}}$. The following properties hold:*

(i) *For every $\varphi \in L^2(\Omega)$, the function $t \mapsto \mathcal{T}(t)\varphi$ belongs to $L^q_{\mathrm{loc}}(\mathbb{R}, L^r(\Omega))$ for every admissible pair (q,r) with $q > 2$, and there exists a constant C, depending only on K and q, such that*

$$\|\mathcal{T}(\cdot)\varphi\|_{L^q((-T,T),L^r)} \le C\Big(\frac{1+T}{t_0}\Big)^{\frac{1}{q}} \|\varphi\|_{L^2}$$

for every $\varphi \in L^2(\mathbb{R}^N)$ and every $T > 0$.

(ii) *Let* $0 < |T| < \infty$. *If* (γ, ρ) *is an admissible pair with* $\gamma > 2$ *and* $f \in L^{\gamma'}((0,T), L^{\rho'}(\Omega))$, *then for every admissible pair* (q,r) *with* $q > 2$, *the function*

$$t \mapsto \Phi_f(t) = \int_0^t \mathcal{T}(t-s)f(s)ds$$

belongs to $L^q((0,T), L^r(\Omega))$. *Furthermore, there exists a constant* C, *depending only on* K, γ, *and* q, *such that*

$$\|\Phi_f\|_{L^q((0,T),L^r)} \le C\Big(\frac{1+|T|}{t_0}\Big)^{\frac{1}{q}} \|f\|_{L^{\gamma'}((0,T),L^{\rho'})}$$

for every $f \in L^{\gamma'}((0,T), L^{\rho'}(\Omega))$. *In addition,* $\Phi_f \in C([0,T], L^2(\Omega))$.

PROOF. Repeating the proof of Theorem 2.3.3, one shows that estimates (i) and (ii) hold for $T = t_0$. In particular, assuming $q < \infty$,

$$\int_0^{t_0} \|\mathcal{T}(t)\varphi\|_{L^r}^q \, dt \le C\|\varphi\|_{L^2}^q .$$

It follows that for every positive integer k,

$$\int_{kt_0}^{(k+1)t_0} \|\mathcal{T}(t)\varphi\|_{L^r}^q \, dt \le C\|\mathcal{T}(kt_0)\varphi\|_{L^2}^q \le C\|\varphi\|_{L^2}^q .$$

In particular,

$$\int_{-kt_0}^{kt_0} \|\mathcal{T}(t)\varphi\|_{L^r}^q \, dt \le Ck\|\varphi\|_{L^2}^q .$$

Hence (i) is established. One proves (ii) by a similar argument. □

In view of Theorem 2.7.1, it is interesting to know when e^{itA} satisfies estimate (2.2.4) (for possibly small times). In the next remarks, we collect some results in that direction.

REMARK 2.7.2. Estimate (2.2.4) does not hold in a bounded domain $\Omega \subset \mathbb{R}^N$ for any $p > 2$. The reason is that in this case, $L^2(\Omega) \hookrightarrow L^{p'}(\Omega)$, and so if such an estimate held, then $I = \mathcal{T}(t)\mathcal{T}(-t)$ would map

$$L^2(\Omega) \to L^2(\Omega) \hookrightarrow L^{p'}(\Omega) \to L^p(\Omega) .$$

This is absurd, since this would mean that $L^2(\Omega) \hookrightarrow L^p(\Omega)$. However, note that estimate (2.2.4) might hold if, for example, Ω is the complement of a star-shaped domain. Unfortunately, such a result is apparently unknown (see Hayashi [160]). On the other hand, estimate (2.2.4) (hence those of Theorem 2.3.3) hold in certain cones of $\mathbb{R}^N$. For example, they hold if $\Omega = \mathbb{R}^N_+$. To see this, consider $\varphi \in \mathcal{D}(\mathbb{R}^N_+)$, and let $\overline{\varphi}$ be defined by

$$\overline{\varphi}(x_1, \dots, x_n) = \begin{cases} \varphi(x_1, \dots, x_n) & \text{if } x_n > 0 \\ -\varphi(x_1, \dots, -x_n) & \text{if } x_n < 0. \end{cases}$$

It follows that $\overline{\varphi} \in \mathcal{D}(\mathbb{R}^N)$. Let $\overline{u} = \overline{\mathcal{T}(t)}\overline{\varphi}$, where $\overline{\mathcal{T}(t)}$ is the group of isometries generated by $i\Delta$ in $\mathbb{R}^N$. One easily verifies, by uniqueness, that $\overline{u}|_{\mathbb{R}^N_+} = \mathcal{T}(t)\varphi$, where $\mathcal{T}(t)$ is the group of isometries generated by $i\Delta$ in $\mathbb{R}^N_+$. This proves the result, by Proposition 2.2.3 and density. This applies in particular to the case where $\Omega \subset \mathbb{R}$ is a half-line. One can repeat this argument and obtain the estimate (2.2.4) when Ω is a cone of $\mathbb{R}^N$ of a certain type. For example, (2.2.4) holds when $\Omega \subset \mathbb{R}^2$ is defined by $\Omega = \{\rho e^{i\theta} : \rho > 0, 0 < \theta < \pi/2^m\}$ for some nonnegative integer m.

REMARK 2.7.3. The estimates of Theorem 2.3.3 fail in a bounded domain $\Omega \subset \mathbb{R}^N$. In the case where Ω is a cube of $\mathbb{R}^N$ there are, however, substitutes to these estimates that can be used to solve the Cauchy problem for the nonlinear equation. (In fact, this holds in the more general case of periodic boundary conditions.) On these questions, see Bourgain [35, 38].

REMARK 2.7.4. Estimate (2.2.4) holds when one replaces the Laplacian by a more general pseudodifferential operator on $\mathbb{R}^N$ (see Balabane [11, 12], and Balabane and Emami Rad [13, 14]).

REMARK 2.7.5. We note that the results of this chapter have been stated for the equation $iu_t + \Delta u = 0$. It is clear that similar results hold for the equation $iu_t + a\Delta u = 0$, where $a \in \mathbb{R}$, $a \neq 0$, which is equivalent by an obvious scaling.

REMARK 2.7.6. Consider the operator $A = \Delta - V$, where $V : \mathbb{R}^N \to \mathbb{R}$ is a given potential. If the negative part of V is not too large, then A defines a self-adjoint operator on $L^2(\mathbb{R}^N)$ (see for example, Kato [202]). If V is small enough in $L^1 \cap L^\infty$, then it follows from a perturbation method that $\mathcal{T}(t) = e^{itA}$ satisfies (2.2.4) (see Schonbek [308]). More general cases are considered in Journé, Soffer, and Sogge [200].

If $V \in C^\infty(\mathbb{R}^N)$ is nonnegative and if $D^\alpha V \in L^\infty(\mathbb{R}^N)$ for all $|\alpha| \geq 2$ (the model case is $V(x) = |x|^2$), then also $\mathcal{T}(t) = e^{itA}$ satisfies (2.2.4) (see Fujiwara [115, 116], A. Weinstein [355], Zelditch [368], and Oh [277, 278]).

On the other hand, such estimates do not hold in general for several reasons. First of all, A may have eigenvalues. Therefore, if λ is an eigenvalue of A and if φ is a corresponding eigenvector, then $\mathcal{T}(t)\varphi = e^{i\lambda t}\varphi$, and so $\mathcal{T}(t)\varphi$ does not decay as $|t| \to \infty$. But there is a more subtle reason that prevents estimate (2.2.4) from holding. Even if one removes the eigenvectors, that is, if one works in the supplement (in L^2) of the space spanned by the eigenvectors, then a resonance effect can occur, even for short range (i.e., localized) potentials. The reader should consult on that subject the very interesting papers of Rauch [300], Jensen and Kato [197], and Murata [253].

REMARK 2.7.7. Estimates of the form (2.2.4) hold for the Schrödinger equation with an external magnetic potential. See Chapter 9.

REMARK 2.7.8. In addition to the smoothing effects of Sections 2.3 and 2.5, a third kind of smoothing effect was discovered. It says that for every $\varphi \in L^2(\mathbb{R}^N)$, then $u(t) = \mathcal{T}(t)\varphi$ belongs to $H^{1/2}_{\mathrm{loc}}(\mathbb{R}^N)$ for a.a. $t \in \mathbb{R}$. It was discovered independently by Constantin and Saut [94, 95], Sjölin [315], and Vega [351]. See also Ben Artzi and Devinatz [21], Ben Artzi and Klainerman [22], and Kato and Yajima [207] for further developments, as well as Kenig, Ponce, and Vega [211] for a related smoothing effect. A typical result in this direction is the following (see Ben

Artzi and Klainerman [22] for a rather simple proof): There exists a constant C such that for every $\varphi \in L^2(\mathbb{R}^N)$, $u(t) = \mathcal{T}(t)\varphi$ satisfies

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}^N} \frac{1}{1+|x|^2} |Pu(t,x)|^2 \, dx \, dt \leq C \|\varphi\|_{L^2}^2 ,$$

where $P = (I - \Delta)^{1/4}$ is the pseudodifferential operator defined by $\widehat{Pu}(\xi) = (1 + 4\pi^2|\xi|^2)^{1/4}\widehat{u}(\xi)$. One can obtain similar estimates for the nonhomogeneous problem. More precisely, if $f \in L^2([0,T], L^2(\mathbb{R}^N))$, then for every bounded open set $B \subset \mathbb{R}^N$ $u \in L^2([0,T], H^{1/2}(B))$ (see Constantin and Saut [94, 95]). Therefore, there is locally a gain of half a derivative. As a matter of fact, if one is willing to reverse the time and space integrations, then the gain is one derivative. More precisely, if $\{Q_\alpha\}_{\alpha \in \mathbb{Z}^N}$ is a family of disjoint open cubes of size R such that $\mathbb{R}^N = \bigcup_{\alpha \in \mathbb{Z}^N} \overline{Q_\alpha}$, then (see Kenig, Ponce, and Vega [212])

$$\sup_{\alpha \in \mathbb{Z}^N} \Bigl(\int_{Q_\alpha} \int_{-\infty}^{+\infty} |\nabla u(t,x)|^2 dt \, dx \Bigr)^{\frac{1}{2}} \leq CR \sum_{\alpha \in \mathbb{Z}^N} \Bigl(\int_{Q_\alpha} \int_{-\infty}^{+\infty} |f(t,x)|^2 dt \, dx \Bigr)^{\frac{1}{2}} .$$

See also Ruiz and Vega [306] for related estimates. Similar estimates hold for $A = \Delta - V$, under appropriate assumptions on the potential V (see Constantin and Saut [95]). Estimates of the above type are essential for solving the Cauchy problem for quasi-linear Schrödinger equations (i.e., with nonlinearities containing derivatives of u). See the comments and references in Section 9.5.

REMARK 2.7.9. Strichartz's estimates similar to those of Theorem 2.3.3 hold in certain exterior domains $\Omega \subset \mathbb{R}^N$ under the geometric assumption that Ω is nontrapping. See Burq, Gérard, and Tzvetkov [47].

REMARK 2.7.10. Strichartz's estimates similar to those of Theorem 2.3.3 hold for the Schrödinger equation on certain nonflat manifolds. See Burq, Gérard, and Tzvetkov [49].

CHAPTER 3

The Cauchy Problem in a General Domain

In this chapter we consider a class of nonlinear Schrödinger equations in a general domain $\Omega \subset \mathbb{R}^N$. In the case $\Omega = \mathbb{R}^N$, the Strichartz estimates are an essential tool for the study of the Cauchy problem. In the case of a general domain $\Omega \subset \mathbb{R}^N$, Strichartz's estimates do not hold and fewer results are known. Our goal is to obtain, using energy techniques, a rather general existence result of solutions in the energy space, which can be adapted to many situations where local existence in that space is known. There is a wide literature on this subject. The study of the Cauchy problem in the energy space was initiated by Ginibre and Velo [133, 132] for local nonlinearities, and by Ginibre and Velo [134] for nonlocal nonlinearities of Hartree type.

In Section 3.1, we introduce various notions of solutions and in Section 3.2 some typical examples of nonlinearities to which we will apply our results. In Section 3.3, we prove our main local existence result and in Section 3.3, we establish some global existence results via energy estimates. In Sections 3.5 and 3.6, we apply the results of Sections 3.3 and 3.4 to the nonlinear Schrödinger equation in some subdomains of $\mathbb{R}$ and $\mathbb{R}^2$. The results of Section 3.3 will also be applied in Chapter 9 to some generalizations.

3.1. The Notion of Solution

In this section we make precise various notions of solution that we will use throughout the text. Let $\Omega \subset \mathbb{R}^N$ and, given a nonlinearity g, consider the initial value problem

$$\begin{cases} iu_t + \Delta u + g(u) = 0 \\ u|_{\partial\Omega} = 0 \\ u(0) = \varphi \,. \end{cases} \tag{3.1.1}$$

In order to motivate our definitions, we first consider the model case of the pure power nonlinearity $g(u) = \lambda |u|^\alpha u$ with $\lambda \in \mathbb{R}$ and $\alpha \geq 0$; i.e., consider the model equation

$$\begin{cases} iu_t + \Delta u + \lambda |u|^\alpha u = 0 \\ u|_{\partial\Omega} = 0 \\ u(0) = \varphi \,. \end{cases} \tag{3.1.2}$$

We first observe that, multiplying the equation by $\overline{u}$, integrating over Ω, and taking the imaginary part, we obtain formally the conservation of charge

$$\frac{d}{dt}\int_{\Omega} |u(t,x)|^2\,dx = 0\,. \tag{3.1.3}$$

Therefore, the L^2 norm of the solution is constant. Next, multiplying the equation by $\overline{u}_t$, integrating over Ω, and taking the real part, we obtain formally the conservation of energy

$$\frac{d}{dt}E(u(t)) = 0\,, \tag{3.1.4}$$

where the energy E is defined by

$$E(w) = \int_{\Omega} \Big\{ \frac{1}{2}|\nabla w(x)|^2 - \frac{\lambda}{\alpha+2}|w(x)|^{\alpha+2} \Big\} dx\,.$$

Finally, multiplying the equation by $\nabla\overline{u}$, integrating over Ω, and taking the real part, we obtain formally the conservation of momentum

$$\frac{d}{dt}\operatorname{Im}\int_{\Omega} u(t,x)\nabla\overline{u}(t,x)dx = 0\,. \tag{3.1.5}$$

When $N = 1$ and $\alpha = 2$, equation (3.1.2) is completely integrable and there are infinitely many conservation laws. When $\alpha = 4/N$ and $\Omega = \mathbb{R}^N$, there is the pseudoconformal conservation law (see Section 7.2). In general, however, the only known conservation laws for (3.1.2) are (3.1.3), (3.1.4), and (3.1.5). Since (3.1.5) does not involve any positive quantity, only (3.1.3) and (3.1.4) can possibly provide useful estimates of the solutions.

The above conservation laws suggest two possible "energy spaces," namely, $L^2(\Omega)$ associated with (3.1.3), and $H_0^1(\Omega)$ associated with (3.1.4). The point in working in an energy space is that, if there is a "good" local existence result, then the global existence of solutions follows from *a priori* estimates. These in general follow from the conservation of energy under some relevant assumptions on the nonlinearity.

We will study the local Cauchy problem in L^2 in Chapter 4. For the moment, we restrict our attention to solutions in $H_0^1(\Omega)$, and we make the following definition.

DEFINITION 3.1.1. Consider $g \in C(H_0^1(\Omega), H^{-1}(\Omega))$, $\varphi \in H_0^1(\Omega)$ and an interval $I \ni 0$.

(i) *A weak H_0^1-solution u of (3.1.1) on I is a function*

$$u \in L^\infty(I, H_0^1(\Omega)) \cap W^{1,\infty}(I, H^{-1}(\Omega))$$

such that $iu_t + \Delta u + g(u) = 0$ in $H^{-1}(\Omega)$ for a.a. $t \in I$ and $u(0) = \varphi$.

(ii) *A strong H_0^1-solution u of (3.1.1) on I is a function*

$$u \in C(I, H_0^1(\Omega)) \cap C^1(I, H^{-1}(\Omega))$$

such that $iu_t + \Delta u + g(u) = 0$ in $H^{-1}(\Omega)$ for all $t \in I$ and $u(0) = \varphi$.

REMARK 3.1.2. Here are some comments on Definition 3.1.1.

(i) The boundary condition $u|_{\partial\Omega}=0$ is included in the assumption $u(t)\in H_0^1(\Omega)$.

(ii) If $u\in L^\infty(I,H_0^1(\Omega))\cap W^{1,\infty}(I,H^{-1}(\Omega))$, then $u\in C(\overline{I},L^2(\Omega))$ so that the condition $u(0)=\varphi$ makes sense.

(iii) Let $u\in L^\infty(I,H_0^1(\Omega))$. If $g(u)\in L^\infty(I,H^{-1}(\Omega))$ (which is automatically the case if $g: H_0^1(\Omega)\to H^{-1}(\Omega)$ is bounded on bounded sets), then $\Delta u+g(u)\in L^\infty(I,H^{-1}(\Omega))$. Therefore, if u satisfies $iu_t+\Delta u+g(u)=0$ in the sense of distributions, then $u\in W^{1,\infty}(I,H^{-1}(\Omega))$. In addition, if $u\in C(I,H_0^1(\Omega))$ satisfies $iu_t+\Delta u+g(u)=0$ in the sense of distributions, then $u\in C^1(I,H^{-1}(\Omega))$.

(iv) We gave the definitions of weak and strong H_0^1-solutions of the Cauchy problem at $t=0$, i.e., with the initial condition $u(0)=\varphi$. Of course, given any $t_0\in\mathbb{R}$, one can give similar definitions for the Cauchy problem at $t=t_0$, i.e., with the initial condition $u(t_0)=\varphi$.

On applying the results of Section 1.6, we deduce the following property.

PROPOSITION 3.1.3. (DUHAMEL'S FORMULA) *Let I be an interval such that $0\in I$; let $g\in C(H_0^1(\Omega),H^{-1}(\Omega))$ and $\varphi\in H_0^1(\Omega)$. If g is bounded on bounded sets and $u\in L^\infty(I,H_0^1(\Omega))$, then u is a weak H_0^1-solution of* (3.1.1) *on I if and only if*

$$u(t)=\mathcal{T}(t)\varphi+i\int_0^t \mathcal{T}(t-s)g(u(s))ds \quad \textit{for a.a. } t\in I. \tag{3.1.6}$$

A function $u\in C(I,H_0^1(\Omega))$ is a strong H_0^1-solution of (3.1.1) *on I if and only if it satisfies* (3.1.6) *for all $t\in I$.*

We now introduce the notion of uniqueness in H^1.

DEFINITION 3.1.4. Consider $g\in C(H_0^1(\Omega),H^{-1}(\Omega))$. We say that there is uniqueness in H^1 for problem (3.1.1) if, given any $\varphi\in H_0^1(\Omega)$ and any interval $I\ni 0$, it follows that any two weak H_0^1-solutions of (3.1.1) on I coincide.

Finally, we introduce the notion of local well-posedness for problem (3.1.1).

DEFINITION 3.1.5. Consider $g\in C(H_0^1(\Omega),H^{-1}(\Omega))$. We say that the initial-value problem (3.1.1) is locally well posed in $H_0^1(\Omega)$ if the following properties hold:

(i) There is uniqueness in H^1 for the problem (3.1.1).

(ii) For every $\varphi\in H_0^1(\Omega)$, there exists a strong H_0^1-solution of (3.1.1) which is defined on a maximal interval $(-T_{\min},T_{\max})$, with $T_{\max}=T_{\max}(\varphi)\in(0,\infty]$ and $T_{\min}=T_{\min}(\varphi)\in(0,\infty]$.

(iii) There is the blowup alternative: If $T_{\max}<\infty$, then $\lim_{t\uparrow T_{\max}}\|u(t)\|_{H^1}=+\infty$ (respectively, if $T_{\min}<\infty$, then we have $\lim_{t\downarrow -T_{\min}}\|u(t)\|_{H^1}=+\infty$).

(iv) The solution depends continuously on the initial value; i.e., if $\varphi_n \underset{n\to\infty}{\longrightarrow}\varphi$ in $H_0^1(\Omega)$ and if $I\subset(-T_{\min}(\varphi),T_{\max}(\varphi))$ is a closed interval, then the maximal solution u_n of (3.1.1) with the initial condition $u_n(0)=\varphi_n$ is defined on $\overline{I}$ for n large enough and satisfies $u_n \underset{n\to\infty}{\longrightarrow} u$ in $C(\overline{I},H_0^1(\Omega))$.

REMARK 3.1.6. Here are some comments on Definition 3.1.5.

(i) The property that $(-T_{\min}, T_{\max})$ is the maximum interval of existence means that if $I \ni 0$ is an interval such that there exists a strong H^1_0-solution of (3.1.1) on I, then $I \subset (-T_{\min}, T_{\max})$.

(ii) If $T_{\max} < \infty$ (respectively, $T_{\min} < \infty$), then by the blowup alternative $\lim_{t \uparrow T_{\max}} \|u(t)\|_{H^1} = +\infty$ (respectively, $\lim_{t \downarrow -T_{\min}} \|u(t)\|_{H^1} = +\infty$). In this case, the solution u is said to blow up at $T_{\max}$ (respectively, $-T_{\min}$). If $T_{\max} = \infty$ (respectively, $T_{\min} = \infty$), the solution is said to be positively (respectively, negatively) global. Note that in this case the blowup alternative does not say anything about the possible boundedness of $\|u(t)\|_{H^1}$ as $t \to \infty$.

(iii) Note that the continuous dependence property implies that the functions $T_{\max}$ and $T_{\min}$ are lower semicontinuous $H^1_0(\Omega) \to (0, \infty]$.

(iv) There are various notions of well-posedness in the literature. We adopted a quite strong notion of well-posedness by requiring uniqueness, the blowup alternative, and continuous dependence.

3.2. Some Typical Nonlinearities

In this section, we introduce various classical models of nonlinearities.

EXAMPLE 3.2.1. The external potential. Consider a real-valued potential $V : \Omega \to \mathbb{R}$. Assume that

$$V \in L^p(\Omega) \tag{3.2.1}$$

with

$$p \geq 1, \quad p > \frac{N}{2}. \tag{3.2.2}$$

Let g be defined by

$$g(u) = Vu \tag{3.2.3}$$

for all measurable $u : \Omega \to \mathbb{C}$, and G be defined by

$$G(u) = \frac{1}{2} \int_\Omega V(x)|u(x)|^2 \, dx \tag{3.2.4}$$

for all measurable $u : \Omega \to \mathbb{C}$ such that $V|u|^2 \in L^1(\Omega)$. We have the following result.

PROPOSITION 3.2.2. *Let V satisfy* (3.2.1) *and* (3.2.2), *let g and G be defined by* (3.2.3) *and* (3.2.4), *respectively, and set*

$$r = \frac{2p}{p-1}. \tag{3.2.5}$$

The following properties hold:

(i) $G \in C^1(H^1_0(\Omega), \mathbb{R})$, $g \in C(H^1_0(\Omega), H^{-1}(\Omega))$, *and* $G' = g$.

(ii) $2 \leq r < \frac{2N}{N-2}$ ($2 \leq r \leq \infty$ *if* $N = 1$).

(iii) $g \in C(L^r(\Omega), L^{r'}(\Omega))$ *and* $\|g(u)\|_{L^{r'}} \le \|V\|_{L^p}\|u\|_{L^r}$ *for all* $u \in L^r(\Omega)$.

(iv) $\operatorname{Im} g(u)\overline{u} = 0$ *a.e. in* Ω *for all* $u \in H_0^1(\Omega)$.

PROOF. Part (ii) follows from (3.2.2) and (3.2.5). (iii) is a consequence of (ii) and Hölder's inequality. In particular, $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ and $L^{r'}(\Omega) \hookrightarrow H^{-1}(\Omega)$ by Sobolev's embedding theorem, which implies that $g \in C(H_0^1(\Omega), H^{-1}(\Omega))$. Next, it follows from Hölder's inequality that $2\|G(u\| \le \|V\|_{L^p}\|u\|_{L^r}^2$, so that G is well defined on $H_0^1(\Omega)$. Furthermore,

$$G(u+v) - G(u) - (g(u), v)_{H^{-1},H_0^1} = \frac{1}{2}\int_\Omega V|v|^2$$

for all $u, v \in H_0^1(\Omega)$, and one deduces easily that $G \in C^1(H_0^1(\Omega), \mathbb{R})$ and $G' = g$. Hence (i) is established. Finally, (iv) follows from the fact that V is real valued. □

REMARK 3.2.3. Let V be a real-valued potential, $V \in L^p(\Omega) + L^\infty(\Omega)$. If p satisfies (3.2.2), then we may write $V = V_1 + V_2$, where V_1 satisfies (3.2.2) and V_2 satisfies (3.2.2) with p replaced by ∞. In particular, we may apply Proposition 3.2.2 to both V_1 and V_2.

EXAMPLE 3.2.4. The local nonlinearity. Consider a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $f(x, u)$ is measurable in x and continuous in u. Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by

(3.2.6) $$F(x,u) = \int_0^u f(x,s)ds \quad \text{for all } u \ge 0.$$

Assume that

(3.2.7) $$f(x,0) = 0 \quad \text{for a.a. } x \in \Omega,$$

and that for every $K > 0$ there exists $L(K) < \infty$ such that

(3.2.8) $$|f(x,u) - f(x,v)| \le L(K)|u - v|$$

for a.a. $x \in \Omega$ and all u, v such that $|u|, |v| \le K$. Assume further that

(3.2.9) $$\begin{cases} L \in C([0,\infty)) & \text{if } N = 1 \\ L(t) \le Ct^\alpha \text{ with } 0 \le \alpha < \dfrac{4}{N-2} & \text{if } N \ge 2. \end{cases}$$

Extend f to the complex plane by setting

(3.2.10) $$f(x,u) = \frac{u}{|u|} f(x,|u|) \quad \text{for all } u \in \mathbb{C}, u \ne 0.$$

Finally, set

(3.2.11) $$g(u)(x) = f(x,u(x)) \quad \text{a.e. in } \Omega$$

for all measurable $u : \Omega \to \mathbb{C}$, and

(3.2.12) $$G(u) = \int_\Omega F(x,|u(x)|)dx$$

for all measurable $u : \Omega \to \mathbb{C}$ such that $F(\cdot, u(\cdot)) \in L^1(\Omega)$. We have the following result.

PROPOSITION 3.2.5. *Let f satisfy* (3.2.7), (3.2.8), *and* (3.2.9); *let g and G be defined by* (3.2.10), (3.2.11), (3.2.6), *and* (3.2.12), *and set*

$$r = \begin{cases} 2 & \text{if } N = 1 \\ \alpha + 2 & \text{if } N \geq 2. \end{cases} \tag{3.2.13}$$

The following properties hold:

(i) $G \in C^1(H_0^1(\Omega), \mathbb{R})$, $g \in C(H_0^1(\Omega), H^{-1}(\Omega))$ *and* $G' = g$.
(ii) *If* $N \geq 2$, *then* $2 \leq r < \frac{2N}{N-2}$.
(iii) $g \in C(L^r(\Omega), L^{r'}(\Omega))$.
(iv) *For all* $M > 0$, *there exists* $C(M) < \infty$ *such that* $\|g(u) - g(v)\|_{L^{r'}} \leq C(M)\|u - v\|_{L^r}$ *for all* $u, v \in H_0^1(\Omega)$ *with* $\|u\|_{H^1}, \|v\|_{H^1} \leq M$.
(v) $\operatorname{Im} g(u)\overline{u} = 0$ *a.e. in* Ω *for all* $u \in H_0^1(\Omega)$.

PROOF. (v) is an immediate consequence of (3.2.10). Let now $K > 0$, and let $u, v \in \mathbb{C}$ be such that $|u|, |v| \leq K$. Suppose for definiteness that $|u| \geq |v|$. It follows from (3.2.7), (3.2.8), and (3.2.10) that

$$|f(x, v)| \leq L(K)|v| \,. \tag{3.2.14}$$

On the other hand, we deduce from (3.2.10) that

$$|u|\,|v|(f(x,u) - f(x,v)) = u|v|[f(x,|u|) - f(x,|v|)] \\ + [u(|v| - |u| + |u|(u - v))]f(x,|v|) \,,$$

and so

$$|u|\,|v|\,|f(x,u) - f(x,v)| \leq |u|\,|v|\,|f(x,|u|) - f(x,|v|)| + 2|u|\,|u - v|\,|f(x,|v|)| \\ \leq 3|u|\,|v|L(K)|u - v| \,,$$

where we used (3.2.8) and (3.2.14) in the last inequality. Therefore, replacing $L(K)$ by $3L(K)$, we have

$$|f(x,u) - f(x,v)| \leq L(K)|u - v| \tag{3.2.15}$$

for all $u, v \in \mathbb{C}$ such that $|u|, |v| \leq K$.

We first consider the case $N \geq 2$. (ii) follows from (3.2.9) and (3.2.13). In particular, $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ by Sobolev's embedding theorem. Therefore, by (3.2.15), (3.2.9), and Hölder's inequality,

$$\|g(u) - g(v)\|_{L^{r'}} \leq C\big(\|u\|_{L^r}^{\alpha} + \|u\|_{L^r}^{\alpha}\big)\|u - v\|_{L^r} \,.$$

Hence (iii) and (iv) are proven. Next, it follows from (3.2.7), (3.2.8), and (3.2.6) that

$$|f(x,u)| \leq C|u|^{r-1} \,, \quad |F(x,|u|)| \leq C|u|^r \,, \tag{3.2.16}$$

so that G is well defined on $H_0^1(\Omega)$. We now deduce from (3.2.6) and (3.2.16) that

$$|G(u+v) - G(u)| = \left| \int_\Omega \int_{|u|}^{|u+v|} f(x,s)\,ds\,dx \right| \leq C \int_\Omega |v|(|u| + |v|)^{r-1} \,,$$

so that $G \in C(H_0^1(\Omega), \mathbb{R})$. Fix now $u, v \in H_0^1(\Omega)$. Given $0 < t \le 1$, (3.2.16) implies that

$$\frac{1}{t}|F(x,u+tv) - F(x,u) - t\,\mathrm{Re}(f(x,u)\overline{v})| \le C|v|(|u|+|v|)^{r-1} \in L^1(\Omega)\,.$$

Since clearly

$$\frac{1}{t}[F(x,u+tv) - F(x,u) - t\,\mathrm{Re}(f(x,u)\overline{v})] \underset{t\downarrow 0}{\longrightarrow} 0\,,$$

it follows from the dominated convergence theorem that

$$\frac{1}{t}[G(u+tv) - G(u) - (g(u),v)_{H^{-1},H_0^1}] \underset{t\downarrow 0}{\longrightarrow} 0\,.$$

Therefore, G is gâteaux differentiable and $G' = g$. Since $g \in C(H_0^1(\Omega), H^{-1}(\Omega))$, (i) follows.

We finally consider the case $N = 1$. Since $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$, we deduce from (3.2.15) that, after possibly modifying the function L,

$$|f(x,u) - f(x,v)| \le L(M)|u-v|$$

for all $u, v \in H_0^1(\Omega)$ such that $\|u\|_{H^1}, \|v\|_{H^1} \le M$. The rest of the proof is then an obvious modification of the argument used in the case $N \ge 2$. □

REMARK 3.2.6. A typical f to which we may apply Proposition 3.2.5 is $f(u) = |u|^\alpha u$ with $0 \le \alpha < \frac{2N}{N-2}$ ($0 \le \alpha < \infty$ if $N = 1$).

REMARK 3.2.7. Let $g(u) = f(\cdot, u(\cdot))$, where f satisfies (3.2.7) and (3.2.8) with

$$\begin{cases} L(t) \in C([0,\infty)) & \text{if } N = 1 \\ L(t) \le C(1+t^\alpha) \text{ with } 0 \le \alpha < \dfrac{4}{N-2} & \text{if } N \ge 2. \end{cases} \tag{3.2.17}$$

If $N \ge 2$, define the functions f_1 and f_2 by

$$f_1(x,u) = \begin{cases} f(x,u) & \text{if } 0 \le u \le 1 \\ f(x,1) & \text{if } u \ge 1 \end{cases}$$

and

$$f_2(x,u) = \begin{cases} 0 & \text{if } 0 \le u \le 1 \\ f(x,u) - f(x,1) & \text{if } u \ge 1. \end{cases}$$

We have $f = f_1 + f_2$, and f_1 and f_2 both satisfy (3.2.7). Furthermore, one easily verifies that f_1 satisfies (3.2.8) and (3.2.9) with α replaced by 0 and that f_2 satisfies (3.2.8) and (3.2.9) with α as in (3.2.17). In particular, we can write $g = g_1 + g_2$ where both g_1 and g_2 satisfy the assumptions of Proposition 3.2.5.

EXAMPLE 3.2.8. The Hartree nonlinearity in $\mathbb{R}^N$. Let $\Omega = \mathbb{R}^N$ and consider a real-valued potential $W : \mathbb{R}^N \to \mathbb{R}$. Assume that

$$W \in L^p(\mathbb{R}^N) \tag{3.2.18}$$

for some

$$p \ge 1\,, \quad p > \frac{N}{4}\,, \tag{3.2.19}$$

and

$$W \quad \text{is even.} \tag{3.2.20}$$

Let g be defined by

$$g(u) = (W \star |u|^2)u \tag{3.2.21}$$

for all measurable $u : \mathbb{R}^N \to \mathbb{R}$ such that $W \star |u|^2$ is measurable, and let G be defined by

$$G(u) = \frac{1}{4} \int_{\mathbb{R}^N} (W \star |u|^2)(x)|u(x)|^2 \, dx \tag{3.2.22}$$

for all measurable $u : \mathbb{R}^N \to \mathbb{R}$ such that $(W \star |u|^2)(x)|u(x)|^2$ is integrable. We have the following result.

PROPOSITION 3.2.9. *Let W satisfy* (3.2.18), (3.2.19), *and* (3.2.20), *and let g and G be defined by* (3.2.21) *and* (3.2.22), *respectively. Set*

$$r = \frac{4p}{2p-1} \,. \tag{3.2.23}$$

Then the following properties hold:

(i) $G \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$, $g \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$, *and* $G' = g$.
(ii) $2 \le r < \frac{2N}{N-2}$ $(2 \le r < \infty$ *if* $N = 1)$.
(iii) $g \in C(L^r(\mathbb{R}^N), L^{r'}(\mathbb{R}^N))$.
(iv) *For all* $M > 0$, *there exists* $C(M) < \infty$ *such that* $\|g(u) - g(v)\|_{L^{r'}} \le C(M)\|u - v\|_{L^r}$ *for all* $u, v \in H^1(\mathbb{R}^N)$ *with* $\|u\|_{L^r}, \|v\|_{L^r} \le M$.
(v) $\operatorname{Im} g(u)\overline{u} = 0$ *a.e. in* $\mathbb{R}^N$ *for all* $u \in \mathbb{R}^N$.

PROOF. Part (ii) follows from (3.2.19) and (3.2.23). Next, we deduce from (3.2.19), (3.2.23), and Hölder and Young's inequalities that

$$\|(W \star (uv))w\|_{L^{r'}} \le \|W\|_{L^p} \|u\|_{L^r} \|v\|_{L^r} \|w\|_{L^r} \,.$$

Statements (iii) and (iv) follow easily. On the other hand, we deduce in particular from (ii) that $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ and $L^{r'}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$ by Sobolev's embedding theorem, which implies that $g \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$. It also follows from Hölder and Young's inequalities that

$$\int_{\mathbb{R}^N} (W \star (uv))wz \le \|W\|_{L^p} \|u\|_{L^r} \|v\|_{L^r} \|w\|_{L^r} \|z\|_{L^r}, \tag{3.2.24}$$

so that G is well defined $H^1(\mathbb{R}^N) \to \mathbb{R}$. Since W is even, we see that

$$\int_{\mathbb{R}^N} (W \star \varphi)\psi = \int_{\mathbb{R}^N} (W \star \psi)\varphi \,.$$

Therefore,

$$G(u+v) - G(u) - (g(u), v)_{H^{-1},H_0^1} = \int_{\mathbb{R}^N} (W \star |v|^2)\Big(\frac{|u|^2 + |v|^2}{4} + \operatorname{Re}(u\overline{v})\Big) + \int_{\mathbb{R}^N} (W \star \operatorname{Re}(u\overline{v})) \operatorname{Re}(u\overline{v}) .$$

Applying now (3.2.24), we obtain

$$\big|G(u+v) - G(u) - (g(u), v)_{H^{-1},H_0^1}\big| \le C\|W\|_{L^p}(\|u\|_{L^r}^2 + \|v\|_{L^r}^2)\|v\|_{L^r}^2 ,$$

and so $G \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and $G' = g$. This proves (i). Assertion (v) follows from the fact that W is real valued. □

REMARK 3.2.10. Let W be an even, real-valued potential, $W \in L^p(\Omega) + L^\infty(\Omega)$. If p satisfies (3.2.19), then we may write $W = W_1 + W_2$, where W_1 satisfies (3.2.19) and W_2 satisfies (3.2.19) with p replaced by ∞. In particular, we may apply Proposition 3.2.9 to both W_1 and W_2.

EXAMPLE 3.2.11. Let

$$g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u ,$$

where V, f, and W are as follows:

- V is a real-valued potential, $V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $p \ge 1$, $p > N/2$.
- $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}$ and satisfies (3.2.7), (3.2.8), and (3.2.17). f is extended to $\mathbb{R}^N \times \mathbb{C}$ by (3.2.10).
- W is an even, real-valued potential; $W \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $q \ge 1$, $q > N/4$.

Applying Remarks 3.2.3, 3.2.7, and 3.2.10, we see that we may write

$$g = g_1 + \dots + g_6 ,$$

where each of the g_j's satisfies the following conditions:

(i) $g_j = G_j'$ for some $G_j \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$,

(ii) $g_j \in C(L^{r_j}(\mathbb{R}^N), L^{r_j'}(\mathbb{R}^N))$,

(iii) for every $M < \infty$, there exists $C(M) < \infty$ such that $\|g_j(v) - g_j(u)\|_{L^{r_j'}} \le C(M)\|v - u\|_{L^{r_j}}$ for all $u, v \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1} + \|v\|_{H^1} \le M$,

(iv) $\operatorname{Im}(g_j(u)\overline{u}) = 0$ a.e. in $\mathbb{R}^N$ for every $u \in H^1(\mathbb{R}^N)$

for some $r_j \in [2, \frac{2N}{N-2})$ ($r_j, \rho_j \in [2, \infty]$ if $N = 1$).

3.3. Local Existence in the Energy Space

We begin with an abstract result for which we use the notation introduced in Section 1.6.

THEOREM 3.3.1. *Let X be a complex Hilbert space with the real scalar product $(\cdot,\cdot)_X$. Let A be a $\mathbb{C}$-linear, self-adjoint, ≤ 0 operator on X with domain $D(A)$.*

Let X_A be the completion of $D(A)$ for the norm $\|x\|_{X_A}^2 = \|x\|_X^2 - (Ax,x)_X$, $X_A^\star = (X_A)^\star$, and $\overline{A}$ be the extension of A to $(D(A))^\star$. Finally, let $\mathcal{T}(t)$ be the group of isometries generated on $(D(A))^\star$, $X_A^\star$, X, X_A, or $D(A)$ by the skew-adjoint operator iA. Assume that $g : X \to X$ is Lipschitz continuous on bounded sets of X and that there exists $G \in C^1(X_A, \mathbb{R})$ such that $G'(x) = g(x)$ for all $x \in X_A$. Assume further that

$$(g(x), ix)_X = 0 \quad \text{for all } x \in X. \tag{3.3.1}$$

For $x \in X_A$, set

$$E(x) = \frac{1}{2}(\|x\|_{X_A}^2 - \|x\|_X^2) - G(x) = -\frac{1}{2}(Ax,x)_X - G(x) \tag{3.3.2}$$

so that $E \in C^1(X_A, \mathbb{R})$ and $E'(x) = -Ax - g(x) \in X_A^\star$ for every $x \in X_A$. It follows that, for every $x \in X$, there exists a unique solution u of the problem

$$\begin{cases} u \in C(\mathbb{R}, X) \cap C^1(\mathbb{R}, (D(A))^\star), \\ i\dfrac{du}{dt} + \overline{A}u + g(u) = 0 \quad \text{for all } t \in \mathbb{R}, \\ u(0) = x. \end{cases} \tag{3.3.3}$$

In addition, the following properties hold:

(i) $\|u(t)\|_X = \|x\|_X$ *for every $t \in \mathbb{R}$ (conservation of charge).*

(ii) *If $x \in X_A$, then $u \in C(\mathbb{R}, X_A) \cap C^1(\mathbb{R}, X_A^\star)$ and $E(u(t)) = E(x)$ for every $t \in \mathbb{R}$ (conservation of energy).*

(iii) *If $x \in D(A)$, then also $u \in C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, X)$.*

PROOF. We proceed in five steps.

STEP 1. It is well known that for every $x \in X$, there exists a unique, maximal solution $u \in C((T_1, T_2), X)$ of (3.3.3), $T_1 < 0 < T_2$. u is maximal in the sense that if $|T_i| < \infty$ (for $i = 1,2$), then $\|u(t)\|_X \to \infty$ as $t \to T_i$. In addition, if $x \in D(A)$, then $u \in C((T_1,T_2), D(A)) \cap C^1((T_1,T_2), X)$. Furthermore, u depends continuously on x in X, uniformly on compact subsets of the maximal existence interval. This follows essentially from Segal [309] (see Cazenave and Haraux [64, 65], Brezis and Cazenave [44], and Pazy [294]).

STEP 2. Assume $x \in D(A)$, and take the scalar product of the equation with iu. We obtain that

$$(u_t, u)_X = (iu_t, iu)_X = -(Au, iu)_X - (g(u), iu)_X.$$

The first term of the right-hand side vanishes by self-adjointness, and the second by (3.3.1). Therefore,

$$\frac{d}{dt}\|u(t)\|_X^2 = 2(u_t, u)_X = 0.$$

Hence the conservation of charge follows. Multiplying the equation by u_t, we obtain

$$0 = (iu_t, u_t)_X = (-Au, u_t)_X - (g(u), u_t)_X.$$

Therefore,

$$\frac{d}{dt}E(u(t)) = 0. \tag{3.3.4}$$

This establishes the conservation of energy.

STEP 3. By Step 2 and continuous dependence, we obtain conservation of charge when $x \in X$. Therefore $\|u(t)\|_X$ is uniformly bounded on the maximal existence interval, and so the solution exists on $(-\infty, \infty)$. Hence (i) and (iii) follow.

STEP 4. Let $x \in X_A$, and let $x_n \in D(A)$ converge to x in X_A as $n \to \infty$. We denote by u_n the solution of (3.3.3) with initial value x_n. By (i), u_n is bounded in $L^\infty(\mathbb{R}, X)$, and so $G(u_n)$ is uniformly bounded. We deduce from the conservation of energy (3.3.4) that u_n is bounded in $L^\infty(\mathbb{R}, X_A)$, and from the equation that $(u_n)_t$ is bounded in $L^\infty(\mathbb{R}, X_A^\star)$. On the other hand, it follows from continuous dependence that for every $t \in \mathbb{R}$, $u_n(t) \to u(t)$ as $n \to \infty$, strongly in X, hence weakly in X_A. Therefore, $u \in L^\infty(\mathbb{R}, X_A) \cap W^{1,\infty}(\mathbb{R}, X_A^\star)$ and $E(u(t)) \le E(x)$ for every $t \in \mathbb{R}$.

STEP 5. Let $t \in \mathbb{R}$, let $y = u(t)$, and let v be the solution of (3.3.3) with initial value y. We deduce from Step 4 that $E(v(-t)) \le E(y)$. On the other hand, $v(-t) = x$ by uniqueness so that $E(u(t)) = E(x)$ for every $t \in \mathbb{R}$. Hence there is conservation of energy. In particular, the function $t \mapsto \|u(t)\|_{X_A}^2$ is continuous. Since $u : \mathbb{R} \to X_A$ is weakly continuous, we obtain $u \in C(\mathbb{R}, X_A)$, and so $u \in C^1(\mathbb{R}, X_A^\star)$ by the equation. Hence (ii) is proven. □

REMARK 3.3.2. Note that the assumption (3.3.1) is only needed to ensure conservation of charge, which implies that all solutions of (3.3.3) are global. Without that assumption, we would have a local version of Theorem 3.3.1 (without the conservation of charge).

Theorem 3.3.1 is not applicable in general for solving the local Cauchy problem in the energy space for the nonlinear Schrödinger equation (3.1.1) for "large" nonlinearities. Indeed, we must take $X = L^2(\Omega)$, and so we need g to be locally Lipschitz continuous on $L^2(\Omega)$. If g is of the form $g(u)(x) = f(u(x))$ for some function $f : \mathbb{C} \to \mathbb{C}$, then f needs to be globally Lipschitz continuous, and in particular sublinear. Thus, we need to improve Theorem 3.3.1 under weaker assumptions on g.

We now use the notation of Chapter 2. In particular, Ω is an open subset of $\mathbb{R}^N$, A is the Laplacian with Dirichlet boundary conditions, and so $X = L^2(\Omega)$, $X_A = H_0^1(\Omega)$, and $X_A^\star = H^{-1}(\Omega)$. We want to go as far as possible under fairly general assumptions on g. The main results of this section are Theorems 3.3.5 and 3.3.9. In Theorem 3.3.5, we show the existence of local weak H_0^1 solutions. In Theorem 3.3.9, we show the local well-posedness of the Cauchy problem in $H_0^1(\Omega)$, provided we have the "a priori" information that solutions are unique. The reason we proceed that way is that in order to apply Theorem 3.3.9, we will only need to show uniqueness, and the known techniques for proving uniqueness depend heavily on the the type of nonlinearity and on geometric properties of Ω.

We make the following assumption on the nonlinearity g:

$$g = G' \quad \text{for some } G \in C^1\big(H_0^1(\Omega), \mathbb{R}\big)\,. \tag{3.3.5}$$

In particular, $g \in C(H_0^1(\Omega), H^{-1}(\Omega))$. We assume that there exist $r, \rho \in [2, \frac{2N}{N-2})$ ($r, \rho \in [2, \infty]$ if $N = 1$) such that

$$g \in C\big(H_0^1(\Omega), L^{\rho'}(\Omega)\big) \tag{3.3.6}$$

and such that for every $M < \infty$ there exists $C(M) < \infty$ such that

$$\|g(v) - g(u)\|_{L^{\rho'}} \leq C(M)\|v - u\|_{L^r} \tag{3.3.7}$$

for all $u, v \in H_0^1(\Omega)$ such that $\|u\|_{H^1} + \|v\|_{H^1} \leq M$. Finally, we assume that, for every $u \in H_0^1(\Omega)$,

$$\operatorname{Im}(g(u)\overline{u}) = 0 \quad \text{a.e. in } \Omega. \tag{3.3.8}$$

We define the energy E by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - G(u) \quad \text{for every } u \in H_0^1(\Omega), \tag{3.3.9}$$

so that $E \in C^1(H_0^1(\Omega), \mathbb{R})$ and $E'(u) = -\Delta u - g(u)$ for every $u \in H_0^1(\Omega)$.

REMARK 3.3.3. Assumptions (3.3.5)–(3.3.8) deserve some comments. The energy space being here $H_0^1(\Omega)$, it is natural to require that $g : H_0^1(\Omega) \to H^{-1}(\Omega)$, as the Laplacian does. The assumption that g is the gradient of some functional G is stronger. It allows us to define the energy, and the conservation of energy is essential in our proof of local existence. Note that most of the classical examples from theoretical physics possess this property. However, in the case of local nonlinearities in $\Omega = \mathbb{R}^N$, local existence can be proved without conservation of energy (see Kato [203, 204, 205, 206] and Chapter 4). Assumptions (3.3.6) require that g is slightly better than a mapping $H_0^1(\Omega) \to H^{-1}(\Omega)$, and assumption (3.3.7) is a type of local Lipschitz condition. Finally, assumption (3.3.8) implies the conservation of charge. It is essential for our proof, but may be replaced by other hypotheses on g with different proofs (see Kato [203, 204, 205, 206], and Cazenave and Weissler [68]).

REMARK 3.3.4. Note that all the nonlinearities introduced in Section 3.2 satisfy the assumptions (3.3.5)–(3.3.8).

We begin with the following result.

THEOREM 3.3.5. *Let $g = g_1 + \cdots + g_k$, where each of the g_j's satisfies the assumptions* (3.3.5)–(3.3.8) *for some exponents r_j, ρ_j. Set $G = G_1 + \cdots + G_k$ and $E = E_1 + \cdots + E_k$. For every $M > 0$, there exists $T(M) > 0$ with the following property: For every $\varphi \in H_0^1(\Omega)$ such that $\|\varphi\|_{H^1} \leq M$, there exists a weak H_0^1-solution u of* (3.1.1) *on $I = (-T(M), T(M))$. In addition,*

$$\|u\|_{L^\infty((-T(M),T(M)),H^1)} \leq 2M\,. \tag{3.3.10}$$

Furthermore,

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}\,, \tag{3.3.11}$$

$$E(u(t)) \leq E(\varphi)\,, \tag{3.3.12}$$

for all $t \in (-T(M), T(M))$.

Before proceeding to the proof of the proposition, we establish two elementary lemmas.

LEMMA 3.3.6. *Let $I \subset \mathbb{R}$ be an interval. It follows that for every $u \in L^\infty(I, H_0^1(\Omega)) \cap W^{1,\infty}(I, H^{-1}(\Omega))$,*

$$\|u(t) - u(s)\|_{L^2(\Omega)} \le C|t-s|^{\frac{1}{2}} \quad \textit{for all } s, t \in I,$$

where $C = \max\{\|u\|_{L^\infty(I,H^1)}, \|u'\|_{L^\infty(I,H^{-1})}\}$.

PROOF. The result is a consequence of Remark 1.3.11 applied with $X = H^{-1}(\Omega)$ and $p = \infty$, and of the inequality $\|v\|_{L^2}^2 \le \|v\|_{H^{-1}}\|v\|_{H_0^1}$ (see Remark 1.3.8(iii)). □

LEMMA 3.3.7. *If g satisfies (3.3.5)–(3.3.8), then, after possibly modifying the function $C(M)$,*

$$\|g(v) - g(u)\|_{L^{\rho'}} \le C(M)\|v-u\|_{L^2}^a, \tag{3.3.13}$$

$$|G(v) - G(u)| \le C(M)\|v-u\|_{L^2}^b, \tag{3.3.14}$$

for every $u, v \in H_0^1(\Omega)$ such that $\|u\|_{H^1} + \|v\|_{H^1} \le M$, with $a = 1 - N(\frac{1}{2} - \frac{1}{r})$ and $b = 1 - N(\frac{1}{2} - \frac{1}{\rho})$.

PROOF. (3.3.13) follows from (3.3.7) and from Gagliardo-Nirenberg's inequality

$$\|w\|_{L^r} \le C\|w\|_{H_0^1}^{1-a}\|w\|_{L^2}^a.$$

(3.3.14) follows from the identity

$$\begin{aligned} G(v) - G(u) &= \int_0^1 \frac{d}{ds} G(sv + (1-s)u)ds \\ &= \int_0^1 \langle g(sv + (1-s)u), v - u\rangle_{L^{\rho'},L^\rho} ds \end{aligned}$$

and the inequality

$$\|w\|_{L^\rho} \le C\|w\|_{H_0^1}^{1-b}\|w\|_{L^2}^b.$$

□

PROOF OF THEOREM 3.3.5. We give the proof in the case where g satisfies (3.3.5)–(3.3.8). The proof in the case $g = g_1 + \cdots + g_k$ is trivially adapted.

The proof proceeds in three steps. We first approximate g by a family of nicer nonlinearities for which we may apply Theorem 3.3.1 in order to construct approximate solutions. Next, we obtain uniform estimates on the approximate solutions by using the conservation laws. Finally, we use these estimates to pass to the limit in the approximate equation.

Note that the proof of Theorem 3.3.5 requires at some stage a regularization procedure. Indeed, construction of solutions could be made, under appropriate assumptions on g and in the case $\Omega = \mathbb{R}^N$, by a fixed point argument (see Kato [203, 204, 205, 206], Cazenave and Weissler [70]). However, the energy inequality (3.3.12) is obtained, at least formally, by taking the scalar product of the equation with iu_t. Note that for a solution with values in $H_0^1(\Omega)$, u_t is only in $H^{-1}(\Omega)$, and so one cannot multiply the equation by u_t. Hence the necessity of the regularization.

Now, in principle, we have the choice on the type of regularization. For a given type of nonlinearity, a natural regularization appears, but which is of a different nature according to the nonlinearity. For example, for a local nonlinearity (Example 3.2.4), the most appropriate thing to do would be to truncate f for large values of u. For a linear potential (Example 3.2.1), it would be natural to truncate the potential; and for a Hartree type nonlinearity (Example 3.2.8) it would be natural to use the convolution with a sequence of mollifiers. Since we want a proof that applies to these different nonlinearities, and that works as well when $\Omega = \mathbb{R}^N$ or when Ω is bounded, we find it convenient to regularize the nonlinearity by applying $(I - \varepsilon\Delta)^{-1}$.

We obtain estimates on the approximate solutions by using the conservation of energy for the approximate problem. For that purpose, we need g to be the gradient of some functional G (assumption (3.3.5)).

As usual, the difficulty is to pass to the limit in the nonlinearity. The crux is that for the limiting problem there is conservation of charge (Lemma 3.3.8). Note that there is necessarily a little bit of technicality at that point. Indeed, we make a local assumption on g (assumption (3.3.8)), we apply a global regularization, and eventually we recover a local property at the limit. This seems rather unnatural, but there does not seem to be any obvious way of avoiding that difficulty.

From now on, we consider $\varphi \in H_0^1(\Omega)$ and we set $M = \|\varphi\|_{H^1}$.

STEP 1. Construction of approximate solutions. Given a positive integer m, let

$$J_m = \left(I - \frac{1}{m}\Delta\right)^{-1}.$$

In other words for every $f \in H^{-1}(\Omega)$, $J_m f \in H_0^1(\Omega)$ is the unique solution of the equation

$$u - \frac{1}{m}\Delta u = f \quad \text{in } H^{-1}(\Omega).$$

We summarize below the main properties of the self-adjoint operator J_m (see Section 1.5).

$$\|J_m\|_{\mathcal{L}(H^{-1},H_0^1)} \le m, \tag{3.3.15}$$

$$\|J_m\|_{\mathcal{L}(L^p,L^p)} \le 1 \quad \text{for } 1 \le p < \infty. \tag{3.3.16}$$

Moreover, if X is any of the spaces $H_0^1(\Omega), L^2(\Omega)$, or $H^{-1}(\Omega)$, then

$$\|J_m\|_{\mathcal{L}(X,X)} \le 1, \tag{3.3.17}$$

$$J_m u \underset{m\to\infty}{\longrightarrow} u \text{ in } X \text{ for all } u \in X, \tag{3.3.18}$$

$$\text{if } \sup_m \|u_m\|_X < \infty, \text{ then } J_m u_m - u_m \rightharpoonup 0 \text{ in } X \text{ as } m \to \infty. \tag{3.3.19}$$

We define

$$g_m(u) = J_m(g(J_m u)) \quad \text{and} \quad G_m(u) = G(J_m u) \quad \text{for every } u \in H_0^1(\Omega).$$

It is clear from (3.3.15) that the above definitions make sense. It is easy to verify by using (3.3.15) and (3.3.7) that g_m is Lipschitz continuous on bounded sets of $L^2(\Omega)$, and by (3.3.15) and (3.3.5) that $G_m \in C^1(H_0^1(\Omega), \mathbb{R})$ and $G_m' = g_m$. In addition, we deduce easily from (3.3.8) that, for every $u \in L^2(\Omega)$,

$$(g_m(u), iu)_{L^2} = (g(J_m u), iJ_m u)_{L^2} = 0.$$

Therefore, we may apply Theorem 3.3.1. Hence there exists a sequence $(u^m)_{m\in\mathbb{N}}$ of functions of $C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, H^{-1}(\Omega))$ such that

$$\begin{cases} iu_t^m + \Delta u^m + g_m(u^m) = 0 \\ u^m(0) = \varphi. \end{cases} \tag{3.3.20}$$

Furthermore,

$$\|u^m(t)\|_{L^2} = \|\varphi\|_{L^2} \tag{3.3.21}$$

and

$$\frac{1}{2}\int_\Omega |\nabla u^m(t)|^2\,dx - G_m(u^m(t)) = \frac{1}{2}\int_\Omega |\nabla\varphi|^2\,dx - G_m(\varphi) \tag{3.3.22}$$

for all $t \in \mathbb{R}$.

STEP 2. Estimates on the sequence u^m. We denote by $C(M)$ various constants depending only on M. Let

$$\theta_m = \sup\left\{\tau > 0 : \|u^m(t)\|_{H^1} \le 2M \text{ on } (-\tau, \tau)\right\}. \tag{3.3.23}$$

Note that, by (3.3.17) and (3.3.16),

$$g_m \text{ satisfies (3.3.6) and (3.3.7) uniformly in } m \in \mathbb{N}. \tag{3.3.24}$$

Therefore, by (3.3.20),

$$\sup_{m\in\mathbb{N}} \|u_t^m\|_{L^\infty((-\theta_m,\theta_m),H^{-1})} \le C(M). \tag{3.3.25}$$

It follows from (3.3.23), (3.3.25), and Lemma 3.3.6 that

$$\|u^m(t) - u^m(s)\|_{L^2} \le C(M)|t-s|^{\frac{1}{2}} \quad \text{for all } s,t \in (-\theta_m, \theta_m). \tag{3.3.26}$$

Applying (3.3.21), (3.3.22), (3.3.24), (3.3.14), (3.3.23), and (3.3.26), we obtain

$$\begin{aligned} \|u^m(t)\|_{H^1}^2 &\le \|\varphi\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2 + 2\big|G_m(u^m(t)) - G_m(\varphi)\big| \\ &\le \|\varphi\|_{H^1}^2 + C(M)|t|^{\frac{b}{2}} \end{aligned} \tag{3.3.27}$$

for all $t \in (-\theta_m, \theta_m)$. If we define $T(M)$ by

$$C(M)T(M)^{\frac{b}{2}} = 2M^2,$$

then

$$\|u^m\|_{L^\infty((-T,T),H^1)} < 2M$$

for $T = \min\{T(M), \theta_m\}$, by (3.3.27). This implies that $T(M) \le \theta_m$, and so

$$\|u^m\|_{L^\infty((-T(M),T(M)),H^1)} \le 2M, \tag{3.3.28}$$

and by (3.3.25),

$$\|u_t^m\|_{L^\infty((-T(M),T(M)),H^{-1})} \le C(M). \tag{3.3.29}$$

STEP 3. Passage to the limit. By applying (3.3.28), (3.3.29), and Proposition 1.3.14, we deduce that there exist

$$u \in L^\infty((-T(M),T(M)), H_0^1(\Omega)) \cap W^{1,\infty}((-T(M),T(M)), H^{-1}(\Omega))$$

and a subsequence, which we still denote by (u^m), such that, for all $t \in [-T(M), T(M)]$,

$$u^m(t) \rightharpoonup u(t) \quad \text{in } H_0^1(\Omega) \text{ as } m \to \infty. \tag{3.3.30}$$

In addition, by (3.3.28), (3.3.29), Lemma 3.3.6, (3.3.24), and (3.3.14), $g_m(u^m)$ is bounded in $C^{0,\frac{\alpha}{2}}((-T(M), T(M)), L^{\rho'}(\Omega))$. Therefore, we deduce from Proposition 1.1.2 that there exist a subsequence, which we still denote by $(g_m(u^m))$, and $f \in C^{0,\frac{\alpha}{2}}((-T(M), T(M)), L^{\rho'}(\Omega))$ such that, for all $t \in [-T(M), T(M)]$,

$$g_m(u^m(t)) \rightharpoonup f(t) \quad \text{in } L^{\rho'}(\Omega) \text{ as } m \to \infty. \tag{3.3.31}$$

On the other hand, it follows from (3.3.20) that for every $w \in H_0^1(\Omega)$ and for every $\phi \in \mathcal{D}(-T(M), T(M))$,

$$\int_{-T(M)}^{T(M)} \left[-\langle iu^m, w\rangle_{H^{-1},H_0^1} \phi'(t) + \langle \Delta u^m + g_m(u^m), w\rangle_{H^{-1},H_0^1} \phi(t)\right] dt = 0.$$

Applying (3.3.30), (3.3.31), and the dominated convergence theorem, we deduce easily that

$$\int_{-T(M)}^{T(M)} \left[-\langle iu, w\rangle_{H^{-1},H_0^1} \phi'(t) + \langle \Delta u + f, w\rangle_{H^{-1},H_0^1} \phi(t)\right] dt = 0.$$

This implies that u satisfies

$$\begin{cases} iu_t + \Delta u + f = 0, \\ u(0) = \varphi, \end{cases} \tag{3.3.32}$$

where the first equation holds for a.a. $t \in (-T(M), T(M))$. Now the crux of the proof is the following result.

LEMMA 3.3.8. *For all $t \in (-T(M), T(M))$, $\operatorname{Im}(f(t)\overline{u(t)}) = 0$ a.e. on Ω.*

PROOF. It suffices to show that for every bounded subset B of Ω,

$$\langle f(t)|_B, iu(t)|_B\rangle_{L^{\rho'}(B),L^\rho(B)} = 0.$$

To see this, we omit the time dependence and we write

$$\begin{aligned} \langle f, iu\rangle_{L^{\rho'}(B),L^\rho(B)} &= \langle f - J_m g(J_m u^m), iu\rangle + \langle J_m g(J_m u^m) - g(J_m u^m), iu\rangle \\ &\quad + \langle g(J_m u^m), i(u - u^m)\rangle + \langle g(J_m u^m), i(u^m - J_m u^m)\rangle \\ &\quad + \langle g(J_m u^m), iJ_m u^m\rangle \\ &\underset{m\to\infty}{\longrightarrow} a + b + c + d + e. \end{aligned}$$

Note first that $J_m g(J_m u^m) = g_m(u^m) \rightharpoonup f$ in $L^{\rho'}(\Omega)$, hence in $L^{\rho'}(B)$. Therefore, $a = 0$. Next, observe that $g(J_m u^m)$ is bounded in $L^{\rho'}(\Omega)$. It follows from (3.3.19) and (3.3.16) that $J_m g(J_m u^m) - g(J_m u^m) \rightharpoonup 0$ in $H^{-1}(\Omega)$, hence in $L^{\rho'}(B)$. Therefore, $b = 0$. Since $u^m \rightharpoonup u$ in $H_0^1(\Omega)$, we have $u^m \to u$ in $L^\rho(B)$. Since $g(J_m u^m)$ is bounded in $L^{\rho'}(B)$, we deduce that $c = 0$. By (3.3.19), $u^m - J_m u^m$ converges weakly to 0 in $H_0^1(\Omega)$. It follows that $u^m - J_m u^m \to 0$ in $L^\rho(B)$. Since $g(J_m u^m)$

is bounded in $L^{\rho'}(B)$, we obtain $d = 0$. Finally, $e = 0$ by (3.3.8), and the result follows. □

END OF THE PROOF OF THEOREM 3.3.5. Taking the $H^{-1} - H_0^1$ duality product of the first equation in (3.3.32) with iu, we find

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 0 \quad \text{for all } t \in (-T(M), T(M))$$

and so

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}\,. \tag{3.3.33}$$

It follows from (3.3.21), (3.3.33), and Proposition 1.3.14(ii) that

$$u^m \to u \quad \text{in } C([-T(M), T(M)], L^2(\Omega))\,. \tag{3.3.34}$$

Applying (3.3.28), (3.3.34), and Gagliardo-Nirenberg's inequality, we deduce that

$$u^m \to u \quad \text{in } C([-T(M), T(M)], L^p(\Omega)) \tag{3.3.35}$$

for every $2 \le p < \frac{2N}{N-2}$. It follows easily from (3.3.7), (3.3.16), (3.3.18), and (3.3.35) that

$$\begin{aligned} g_m(u^m(t)) - g(u(t)) &= J_m[(g(J_m u_m(t)) - g(J_m u(t))] \\ &\quad + J_m[g(J_m u(t)) - g(u(t))] + J_m g(u(t)) - g(u(t)) \\ &\underset{m\to\infty}{\longrightarrow} 0 \end{aligned}$$

in $L^{\rho'}(\Omega)$ for all $t \in (-T(M), T(M))$. Therefore, $f = g(u)$ and so u satisfies (3.1.1). (3.3.10) follows from (3.3.28) and (3.3.11) from (3.3.33). It remains to prove (3.3.12). This is a consequence of (3.3.22), the weak lower semicontinuity of the H^1-norm, and the fact that $G_m(u^m(t)) \to G(u(t))$ as $m \to \infty$. This completes the proof. □

We now show that the initial-value problem (3.1.1) is locally well posed in $H_0^1(\Omega)$, provided we have the a priori information that weak H_0^1 solutions are unique.

THEOREM 3.3.9. *Let $g = g_1 + \cdots + g_k$ where each of the g_j's satisfies* (3.3.5)–(3.3.8) *for some exponents r_j, ρ_j; and set $G = G_1 + \cdots + G_k$ and $E = E_1 + \cdots + E_k$. Assume, in addition, that there is uniqueness for the problem* (3.1.1). *It follows that* (3.1.1) *is locally well posed in $H_0^1(\Omega)$, and that there is conservation of charge and energy; i.e.,*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2} \quad \textit{and} \quad E(u(t)) = E(\varphi)$$

for all $t \in (-T_{\min}, T_{\max})$, where u is the solution of (3.1.1) *with the initial value $\varphi \in H_0^1(\Omega)$.*

PROOF. The proof proceeds in three steps. We first show that the solution u given by Theorem 3.3.5 belongs to

$$C((-T(M), T(M)), H_0^1(\Omega)) \cap C^1((-T(M), T(M)), H^{-1}(\Omega))\,,$$

and that there is conservation of energy. Next, we consider the maximal solutions and show that $T_{\min}$ and $T_{\max}$ satisfy the blowup alternative. Finally, we establish continuous dependence.

STEP 1. Regularity. Let I be an interval and let

$$u \in L^\infty(I, H_0^1(\Omega)) \cap W^{1,\infty}(I, H^{-1}(\Omega))$$

satisfy $iu_t + \Delta u + g(u) = 0$ for a.a. $t \in I$. We claim that u satisfies both conservation of charge and energy, and that

$$u \in C(I, H_0^1(\Omega)) \cap C^1(I, H^{-1}(\Omega)).$$

To see this, consider

$$M = \sup\{\|u(t)\|_{H^1},\, t \in I\},$$

and let us first show that $\|u(t)\|_{L^2}$ and $E(u(t))$ are constant on every interval $J \subset I$ of length at most $T(M)$, where $T(M)$ is given by Theorem 3.3.5. Indeed, let J be as above and let $\sigma, \tau \in J$. Let $\varphi = u(\sigma)$ and let v be the solution of (3.1.1) given by Theorem 3.3.5. $v(\cdot - \sigma)$ is defined on J and, by uniqueness, $v(\cdot - \sigma) = u(\cdot)$ on J. Applying (3.3.11) and (3.3.12), we deduce in particular that

$$\|u(\tau)\|_{L^2} = \|u(\sigma)\|_{L^2}, \quad E(u(\tau)) \le E(u(\sigma)). \tag{3.3.36}$$

Let now $\varphi = u(\tau)$ and let w be the solution of (3.1.1) given by Theorem 3.3.5. $w(\cdot - \tau)$ is defined on J and, by uniqueness, $w(\cdot - \tau) = u(\cdot)$ on J. From (3.3.12), we deduce in particular that

$$E(u(\sigma)) \le E(u(\tau)).$$

Comparing with (3.3.36), we see that both $\|u(t)\|_{L^2}$ and $E(u(t))$ are constant on J. Since J is arbitrary, we have

$$\|u(t)\|_{L^2} = \|u(s)\|_{L^2}, \quad E(u(t)) = E(u(s)) \quad \text{for all } s, t \in I. \tag{3.3.37}$$

Furthermore, note that by Lemma 3.3.6, $u \in C^{0,1/2}(\overline{I}, L^2(\Omega))$, and so the function $t \mapsto G(u(t))$ is continuous $\overline{I} \to \mathbb{R}$ by Lemma 3.3.7. In view of (3.3.37), this implies that $\|u(t)\|_{H^1}$ is continuous $\overline{I} \to \mathbb{R}$. Therefore, $u \in C(\overline{I}, H_0^1(\Omega))$, and, by the equation, $u \in C^1(\overline{I}, H^{-1}(\Omega))$.

STEP 2. Maximality. Consider $\varphi \in H_0^1(\Omega)$ and let

$$T_{\max}(\varphi) = \sup\{T > 0 : \text{there exists a solution of (3.1.1) on } [0, T]\},$$
$$T_{\min}(\varphi) = \sup\{T > 0 : \text{there exists a solution of (3.1.1) on } [-T, 0]\}.$$

It follows from Step 1 and the uniqueness property that there exists a solution

$$u \in C((-T_{\min}, T_{\max}), H_0^1(\Omega)) \cap C^1((-T_{\min}, T_{\max}), H^{-1}(\Omega))$$

of (3.1.1). Suppose now that $T_{\max} < \infty$, and assume that there exist $M < \infty$ and a sequence $t_j \uparrow T_{\max}$ such that $\|u(t_j)\|_{H^1} \le M$. Let k be such that $t_k + T(M) > T_{\max}(\varphi)$. By Theorem 3.3.5 and Step 1, and starting from $u(t_k)$, one can extend u up to $t_k + T(M)$, which contradicts maximality. Therefore,

$$\|u(t)\|_{H^1} \to \infty \quad \text{as } t \uparrow T_{\max}.$$

One shows by the same argument that if $T_{\min}(\varphi) < \infty$, then

$$\|u(t)\|_{H^1} \to \infty \quad \text{as } t \downarrow -T_{\min}.$$

Therefore, so far we have established the existence of a maximal solution, the blowup alternative, and the conservation of charge and energy.

STEP 3. Continuous dependence. Suppose $\varphi_m \to \varphi$ in $H_0^1(\Omega)$ and let

$$M = 2\sup\{\|u(t)\|_{H^1} : t \in [-T_1, T_2]\}.$$

Since $\|\varphi_m\|_{H^1} \le M$ for m large enough, $[-T(M), T(M)] \subset (-T_{\min}(\varphi), T_{\max}(\varphi))$. Thus u_m is bounded in

$$L^\infty((-T(M),T(M)), H_0^1(\Omega)) \cap W^{1,\infty}((-T(M),T(M)), H^{-1}(\Omega)).$$

Applying the argument of Step 3 of the proof of Theorem 3.3.5, we obtain that $u_m \to u$ in $C([-T(M),T(M)], L^2(\Omega))$. By Lemma 3.3.7 and conservation of energy, this implies that $\|u_m\|_{H^1}$ converges to $\|u\|_{H^1}$ uniformly on $[-T(M),T(M)]$. Applying Proposition 1.3.14(iii), we deduce that $u_m \to u$ in $C([-T(M),T(M)], H_0^1(\Omega))$. Since $T(M)$ depends only on M, we may repeat this argument to cover the interval $[-T_1, T_2]$. This completes the proof. □

REMARK 3.3.10. By Theorem 3.3.9, if g is a finite sum of terms g_j, where each of the g_j's satisfies the assumptions (3.3.5)–(3.3.8) for some exponents r_j, ρ_j, then problem (3.1.1) is well posed in $H_0^1(\Omega)$ provided there is uniqueness. Unfortunately, the techniques that are used to prove uniqueness depend on the problem (see the following sections). However, we give below a general sufficient condition for uniqueness.

COROLLARY 3.3.11. *Let $G \in C^1(H_0^1(\Omega), \mathbb{R})$ and let $g = G'$. Assume that $g(0) \in L^2(\Omega)$ and that there exists $C(M)$ for every M such that*

$$\|g(v) - g(u)\|_{L^2} \le C(M)\|v - u\|_{L^2} \tag{3.3.38}$$

for all $u, v \in H_0^1(\Omega)$ such that $\|u\|_{H^1} + \|v\|_{H^1} \le M$. Assume further that $\operatorname{Im} g(u)\overline{u} = 0$ *a.e. for every $u \in H_0^1(\Omega)$. It follows that the conclusions of Theorem* 3.3.9 *hold.*

PROOF. We need only show uniqueness. Let I be an interval containing 0, let $\varphi \in H_0^1(\Omega)$, and let $u_1, u_2 \in L^\infty(I, H_0^1(\Omega)) \cap W^{1,\infty}(I, H^{-1}(\Omega))$ be two solutions of (3.1.1). It follows from Remark 1.6.1(iii) that

$$u_2(t) - u_1(t) = i\int_0^t \mathcal{T}(t-s)(g(u_2(s)) - g(u_1(s)))ds \quad \text{for all } t \in I.$$

Therefore, there exists a constant C such that

$$\|u_2(t) - u_1(t)\|_{L^2} \le C\int_0^t \|u_2(s) - u_1(s)\|_{L^2}\, ds,$$

and the result follows from Gronwall's lemma. □

REMARK 3.3.12. Theorem 3.3.9 (and also Corollary 3.3.11) is stated for one equation, but the method applies as well for systems of the same form. More precisely, consider an integer $\mu \ge 1$ and set $\mathcal{H}_0^1 = (H_0^1(\Omega))^\mu$, $\mathcal{H}^{-1} = (H^{-1}(\Omega))^\mu$, and $\mathcal{L}^p = (L^p(\Omega))^\mu$. Let $(\alpha_\ell)_{1\le\ell\le\mu}$, $(\beta_\ell)_{1\le\ell\le\mu}$ be two families of real numbers such that $\alpha_\ell \neq 0$ and $\beta_\ell \neq 0$ for every $1 \le \ell \le \mu$. Set $\mathcal{A}U = (\alpha_1\Delta u_1, \dots, \alpha_\mu \Delta u_\mu)$ for $U = (u_1, \dots, u_\mu) \in \mathcal{H}_0^1$, and $\mathcal{B}U = (\beta_1 u_1, \dots, \beta_\mu u_\mu)$ for every $U \in \mathbb{C}^\mu$. Suppose $g = g_1 + \dots + g_k$, where each of the g_j's satisfies the assumptions (3.3.5)–(3.3.8) for some exponents r_j, ρ_j, but with $H_0^1(\Omega), H^{-1}(\Omega), L^p(\Omega)$ replaced by $\mathcal{H}_0^1, \mathcal{H}^{-1}, \mathcal{L}^p$. It follows that the conclusions of Theorem 3.3.5 and, under uniqueness assumption, those of Theorem 3.3.9 hold for the system

$$\begin{cases} i\mathcal{B}U_t + \mathcal{A}U + g(U) = 0 \\ U(0) = \Phi, \end{cases}$$

where Φ is a given initial value in $\mathcal{H}_0^1$.

REMARK 3.3.13. Let g be as in Theorem 3.3.9. Consider $\varphi \in H_0^1(\Omega)$, and let u be the maximal solution of (3.1.1). Let u^m be the approximate solutions constructed in Step 1 of the proof of Theorem 3.3.5. Following the argument of the proof of Theorem 3.3.9, one shows easily that $u^m \to u$ in $C([S,T], H_0^1(\Omega))$ as $m \to \infty$ for every interval $[S,T] \subset (-T_{\min}, T_{\max})$.

3.4. Energy Estimates and Global Existence

Given g as in Theorem 3.3.5, there exists a local weak H_0^1-solution of the problem (3.1.1) for every initial value $\varphi \in H_0^1(\Omega)$. In this section we use the conservation of charge (3.3.11) and the energy inequality (3.3.12) to show that, under appropriate assumptions on the nonlinearity g, there exists a global solution of (3.1.1) for some (or every) initial value $\varphi \in H_0^1(\Omega)$. Our first result is the following.

THEOREM 3.4.1. *Let g be as in Theorem 3.3.5. Assume further that there exist $A > 0$, $C(A) > 0$, and $\varepsilon \in (0,1)$ such that*

$$G(u) \le \frac{1-\varepsilon}{2}\|u\|_{H^1}^2 + C(A) \tag{3.4.1}$$

for all $u \in H_0^1(\Omega)$ such that $\|u\|_{L^2} \le A$. If $\varphi \in H_0^1(\Omega)$ satisfies $\|\varphi\|_{L^2} \le A$, then there exists a (global) weak H_0^1-solution u of (3.1.1) *on $\mathbb{R}$. In addition, $u \in L^\infty(\mathbb{R}, H_0^1(\Omega))$ and u satisfies the conservation of charge* (3.3.11) *and the energy inequality* (3.3.12) *for all $t \in \mathbb{R}$.*

PROOF. Let $I \ni 0$ be an interval of $\mathbb{R}$. Consider a weak H_0^1-solution u of (3.1.1) on I. Assume that u satisfies the conservation of charge (3.3.11) and the energy inequality (3.3.12) for all $t \in I$. Since

$$\|u(t)\|_{H^1}^2 = E(u(t)) - 2G(u(t)) + \|u(t)\|_{L^2}^2,$$

we deduce from (3.3.11) and (3.3.12) that

$$\|u(t)\|_{H^1}^2 \le \|\varphi\|_{H^1}^2 - 2G(\varphi) + 2G(u(t)) \quad \text{for all } t \in I.$$

Assuming $\|\varphi\|_{L^2} \le A$, we deduce from (3.4.1) that

$$\|u(t)\|_{H^1}^2 \le \|\varphi\|_{H^1}^2 - 2G(\varphi) + (1-\varepsilon)\|u(t)\|_{H^1}^2 + 2C(\|\varphi\|_{L^2}),$$

and so

$$\|u(t)\|_{H^1}^2 \le \frac{1}{\varepsilon}\big[\|\varphi\|_{H^1}^2 - 2G(\varphi) + 2C(\|\varphi\|_{L^2})\big] \quad \text{for all } t \in I. \tag{3.4.2}$$

We observe that the right-hand side of (3.4.2) depends on the initial value φ but neither on t nor on the weak H^1_0-solution u.

We now proceed as follows. Let $\varphi \in H^1_0(\Omega)$ satisfy $\|\varphi\|_{L^2} \le A$ and set

$$M = \frac{1}{\sqrt{\varepsilon}}\sqrt{\|\varphi\|_{H^1}^2 - 2G(\varphi) + 2C(\|\varphi\|_{L^2})}\,.$$

Since in particular $\|\varphi\|_{H^1} \le M$, it follows from Theorem 3.3.5 that there exists a weak H^1_0-solution u of (3.1.1) on $[0, T(M)]$ which satisfies (3.3.11) and (3.3.12) for all $t \in [0, T(M)]$. We deduce in particular from (3.4.2) that $\|u(T(M))\|_{H^1} \le M$. Setting $\widetilde{\varphi} = u(T(M))$, we may apply again Theorem 3.3.5 and we see that there exists a weak H^1_0-solution $\widetilde{u}$ of (3.1.1) (with the initial value $\widetilde{\varphi}$) on $[0, T(M)]$ which satisfies (3.3.11) and (3.3.12) for all $t \in [0, T(M)]$. We now "glue" u and $\widetilde{u}$ by defining the function $u(t)$ on $[0, 2T(M)]$ as

$$u(t) = \begin{cases} u(t) & \text{if } 0 \le t \le T(M) \\ \widetilde{u}(t - T(M)) & \text{if } 0 \le T(M) \le t \le 2T(M). \end{cases} \tag{3.4.3}$$

It is clear that u defined by (3.4.3) is a weak H^1_0-solution of (3.1.1) on $[0, 2T(M)]$. Moreover,

$$\|u(t)\|_{L^2} = \|\widetilde{u}(t - T(M))\|_{L^2} = \|\widetilde{\varphi}\|_{L^2} = \|u(T(M))\|_{L^2} = \|\varphi\|_{L^2}$$

and

$$E(u(t)) = E\big(\widetilde{u}(t - T(M))\big) \le E(\widetilde{\varphi}) = E\big(u(T(M))\big) \le E(\varphi)$$

for $T(M) \le t \le 2T(M)$. We deduce that u satisfies (3.3.11) and (3.3.12) for all $t \in [0, 2T(M)]$. In particular, we deduce from (3.4.2) that $\|u(2T(M))\|_{H^1} \le M$. We can then repeat the above argument and construct a weak H^1_0-solution u of (3.1.1) on $[0, \infty)$ which satisfies (3.3.11) and (3.3.12) for all $t \ge 0$. We also can argue similarly for $t \le 0$, so that we obtain a weak H^1_0-solution u of (3.1.1) on $\mathbb{R}$ which satisfies (3.3.11) and (3.3.12) for all $t \in \mathbb{R}$. Finally, we deduce from (3.4.2) that $\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1} < \infty$, which completes the proof. □

The following corollary is an immediate consequence of Theorem 3.4.1.

COROLLARY 3.4.2. *Let g be as in Theorem 3.3.5. Assume further that for every $A > 0$, there exist $C(A) > 0$ and $\varepsilon \in (0,1)$ such that (3.4.1) holds. It follows that for every $\varphi \in H^1_0(\Omega)$, there exists a (global) weak H^1_0-solution u of (3.1.1) on $\mathbb{R}$. In addition, $u \in L^\infty(\mathbb{R}, H^1_0(\Omega))$ and u satisfies the conservation of charge (3.3.11) and the energy inequality (3.3.12) for all $t \in \mathbb{R}$.*

Corollary 3.4.2 provides a sufficient condition on the nonlinearity so that for all initial values $\varphi \in H^1_0(\Omega)$, there exists a global weak H^1_0-solution of (3.1.1). We next show that, under a different type of assumption on g, there exists a global weak H^1_0-solution of (3.1.1) for all sufficiently small initial data $\varphi \in H^1_0(\Omega)$. Our result is the following.

THEOREM 3.4.3. *Let g be as in Theorem 3.3.5. Assume further that $G(0) = 0$ and that there exist $\varepsilon > 0$ and a nonnegative function $\theta \in C([0,\varepsilon), \mathbb{R}^+)$ with $\theta(0) = 0$ such that*

$$G(u) \le \frac{1-\varepsilon}{2}\|u\|_{H^1}^2 + \theta(\|u\|_{L^2}) \tag{3.4.4}$$

for all $u \in H_0^1(\Omega)$ such that $\|u\|_{H^1} \le \varepsilon$. It follows that there exists $\delta > 0$ such that for every $\varphi \in H_0^1(\Omega)$ with $\|\varphi\|_{H^1} \le \delta$, there exists a (global) weak H_0^1-solution u of (3.1.1) on $\mathbb{R}$. In addition, $\|u\|_{L^\infty(\mathbb{R},H^1)} \le \varepsilon$ and u satisfies the conservation of charge (3.3.11) and the energy inequality (3.3.12) for all $t \in \mathbb{R}$.

PROOF. Let $I \ni 0$ be an interval of $\mathbb{R}$. Consider a weak H_0^1-solution u of (3.1.1) on I. Assume that u satisfies the conservation of charge (3.3.11) and the energy inequality (3.3.12) for all $t \in I$. Assume that $\|u(t)\|_{H^1} \le \varepsilon$ on some interval $J \subset I$ with $0 \in J$. It follows from (3.3.11), (3.3.12), and (3.4.4) that (see the proof of (3.4.2))

$$\|u(t)\|_{H^1}^2 \le \frac{1}{\varepsilon}\big[\|\varphi\|_{H^1}^2 - 2G(\varphi) + 2\theta(\|\varphi\|_{L^2})\big] \quad \text{for all } t \in J.$$

Note that the right-hand side of (3.4.5) is a continuous function of φ (in $H_0^1(\Omega)$), which vanishes for $\varphi = 0$, and so there exists $0 < \delta \le \varepsilon/2$ such that

$$\frac{1}{\varepsilon}\big[\|\varphi\|_{H^1}^2 - 2G(\varphi) + 2\theta(\|\varphi\|_{L^2})\big] \le \frac{\varepsilon^2}{4} \quad \text{if } \|\varphi\|_{H^1} \le \delta\,.$$

Therefore, if we assume that $\|\varphi\|_{H^1} \le \delta$, we deduce that

$$\|u(t)\|_{H^1} \le \frac{\varepsilon}{2} \tag{3.4.5}$$

on every interval $J \subset I$, $J \ni 0$ on which $\|u(t)\|_{H^1} \le \varepsilon$.

We now proceed as follows. Let $\varphi \in H_0^1(\Omega)$ satisfy $\|\varphi\|_{H^1} \le \delta$ with $\delta > 0$ as above. Since in particular $\|\varphi\|_{H^1} \le \varepsilon/2$, it follows from Theorem 3.3.5 that there exists a weak H_0^1-solution u of (3.1.1) on $[0, T(\varepsilon/2)]$ which satisfies (3.3.11) and (3.3.12) for all $t \in [0, T(\varepsilon/2)]$ and such that, $\|u\|_{L^\infty((0,T(\varepsilon/2)),H^1)} \le \varepsilon$. We deduce in particular from (3.4.5) that $\|u(T(\varepsilon/2))\|_{H^1} \le \varepsilon/2$. Setting $\widetilde{\varphi} = u(T(\varepsilon/2))$, we again apply Theorem 3.3.5. We see that there exists a weak H_0^1-solution $\widetilde{u}$ of (3.1.1) (with the initial value $\widetilde{\varphi}$) on $[0, T(\varepsilon/2)]$ which satisfies (3.3.11)–(3.3.12) for all $t \in [0, T(\varepsilon/2)]$ and such that $\|\widetilde{u}\|_{L^\infty((0,T(\varepsilon/2)),H^1)} \le \varepsilon$. We now "glue" u and $\widetilde{u}$ by defining the function $u(t)$ for $0 \le t \le 2T(\varepsilon/2)$ by (3.4.3). It follows that u defined by (3.4.3) is a weak H_0^1-solution of (3.1.1) on $[0, 2T(\varepsilon/2)]$ which satisfies (3.3.11) and (3.3.12) for all $t \in [0, 2T(\varepsilon/2)]$. (See the proof of Theorem 3.4.1.) Moreover, $\|u\|_{L^\infty((0,2T(\varepsilon/2)),H^1)} \le \varepsilon$. In particular, we deduce from (3.4.2) that $\|u\|_{L^\infty((0,2T(\varepsilon/2)),H^1)} \le \varepsilon/2$. We can then repeat the above argument and construct a weak H_0^1-solution u of (3.1.1) on $[0,\infty)$, then on $\mathbb{R}$, which satisfies the conclusions of the theorem. □

REMARK 3.4.4. The assumption (3.4.4) is very similar in form to the assumption (3.4.1). The major difference is that (3.4.4) is assumed only for small $\|u\|_{H^1}$. If g is a local nonlinearity, then (3.4.4) corresponds to a condition on g near 0, while (3.4.1) corresponds to a condition on g for large u.

3.5. The Nonlinear Schrödinger Equation in One Dimension

In this section we assume that the dimension $N = 1$. Without loss of generality, we may also assume that Ω is connected. Therefore, Ω is either $\mathbb{R}$, or a half line, or a bounded interval. The case Ω is either $\mathbb{R}$ or half line falls into the scope of Theorem 4.3.1 or Remark 4.3.2. Therefore, the local Cauchy problem in $H_0^1(\Omega)$ is well posed, for example, for the type of nonlinearities considered in Corollary 4.3.3. On the other hand, if Ω is a bounded interval, we know (Remark 2.7.2) that estimate (2.2.4) does not hold. However, one can obtain a fairly general result for local nonlinearities by using the embedding $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$.

THEOREM 3.5.1. *If $g(u) = f(\cdot, u(\cdot))$ as in Example 3.2.4 (with $N = 1$), then the initial-value problem (3.1.1) is locally well posed in $H_0^1(\Omega)$. Moreover, there is conservation of charge and energy; i.e.,*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2} \quad \text{and} \quad E(u(t)) = E(\varphi)$$

for all $t \in (-T_{\min}, T_{\max})$, where u is the solution of (3.1.1) with the initial value $\varphi \in H_0^1(\Omega)$.

PROOF. It follows from Proposition 3.2.5 that g satisfies (3.3.38), and the result follows from Corollary 3.3.11. □

COROLLARY 3.5.2. *Let g be as in Theorem 3.5.1. If*

$$F(x,u) \le C(1 + |u|^\delta)|u|^2 \quad \text{for some } \delta < 4,$$

then for every $\varphi \in H_0^1(\Omega)$, the maximal strong H_0^1-solution of (3.1.1) is global and uniformly bounded in H^1. If $\delta = 4$, the same conclusion holds provided $\|\varphi\|_{L^2}$ is small enough.

PROOF. We have $G(u) \le C\|u\|_{L^2}^2 + C\|u\|_{L^{\delta+2}}^{\delta+2}$. Using Gagliardo-Nirenberg's inequality, we deduce that

$$G(u) \le C\|u\|_{L^2}^2 + C\|u\|_{H^1}^{\frac{\delta}{2}} \|u\|_{L^2}^{2+\frac{\delta}{2}}.$$

If $\delta < 4$, then it follows from the inequality $ab \le \varepsilon a^r + C(\varepsilon) b^{r'}$ that

$$G(u) \le \frac{1}{2}\|u\|_{H^1}^2 + C(\|u\|_{L^2}).$$

The result then follows from Corollary 3.4.2. If $\delta = 4$, then

$$G(u) \le C\|u\|_{L^2}^4 \|u\|_{H^1}^2 + C(\|u\|_{L^2}),$$

and the result follows from Theorem 3.4.1. □

COROLLARY 3.5.3. *Let g be as in Theorem 3.5.1. It follows that there exists $\delta > 0$ such that, for every $\varphi \in H_0^1(\Omega)$ with $\|\varphi\|_{H^1} \le \delta$, the maximal strong H_0^1-solution of (3.1.1) is global and uniformly bounded in H^1.*

PROOF. There exists a constant K such that if $\|u\|_{H^1} \le 1$, then $\|u\|_{L^\infty} \le K$. Therefore,

$$G(u) \le C(K)\|u\|_{L^2}^2,$$

and the result follows from Theorem 3.4.3. □

REMARK 3.5.4. In particular, one may apply Theorem 3.5.1 to the case $g(u) = Vu + \lambda |u|^\alpha u$, where V is a real-valued potential $V \in L^\infty(\Omega)$, $\lambda \in \mathbb{R}$, and $0 \le \alpha < \infty$. Solutions with initial value of small H^1 norm are global by Corollary 3.5.3. If $\lambda \le 0$ or if $\alpha < 4$, then, for every initial value in $H^1_0(\Omega)$, the corresponding solution of (3.1.1) is global, as follows from Corollary 3.5.2.

REMARK 3.5.5. Like Theorem 3.3.9, Theorem 3.5.1 is stated for one equation, but the method applies as well for systems of the same form (see Remark 3.3.12).

3.6. The Nonlinear Schrödinger Equation in Two Dimensions

In this section we assume that the dimension $N = 2$. Note that $H^1_0(\Omega) \not\hookrightarrow L^\infty(\Omega)$, and so one may not apply the method of Section 3.5. However, one still can do something by using the fact that $H^1_0(\Omega)$ is "almost" embedded in $L^\infty(\Omega)$, or more precisely by using Trudinger's inequality (Remark 1.3.6).

We have the following result, due to Vladimirov (see Vladimirov [354], Ogawa [273], and Ogawa and Ozawa [274]).

THEOREM 3.6.1. *Let Ω be an open subset of $\mathbb{R}^2$ and let g be as in Remark 3.2.7 with $\alpha \le 2$. It follows that the initial-value problem (3.1.1) is locally well posed in $H^1_0(\Omega)$. Moreover, there is conservation of charge and energy; i.e.,*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2} \quad \text{and} \quad E(u(t)) = E(\varphi)$$

for all $t \in (-T_{\min}, T_{\max})$, where u is the solution of (3.1.1) with the initial value $\varphi \in H^1_0(\Omega)$.

PROOF. By Theorem 3.3.9 and Proposition 3.2.5, we need only show uniqueness. Furthermore, since this is a local property, we need only establish it for possibly small intervals (see Step 2 of the proof of Theorem 4.6.1 below). Let I be an interval containing 0, and let $u, v \in L^\infty(I, H^1_0) \cap W^{1,\infty}(I, H^{-1})$ be two solutions of (3.1.1). Setting $w = v - u$, we have

$$iw_t + \Delta w + g(v) - g(u) = 0 .$$

On multiplying the above equation in the $H^{-1} - H^1_0$ duality by iw, it follows that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2} = \operatorname{Im} \int_\Omega (g(v(t)) - g(u(t)))\overline{w(t)}\, dx .$$

Therefore, if we define the function $h \in L^\infty(I, H^1_0(\Omega))$ by

$$h(t) = |u(t)| + |v(t)| \quad \text{for all } t \in I,$$

then

$$\left|\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2}\right| \le C \int_\Omega (1 + h(s)^2)|w(s)|^2\, dx .$$

Integrating the above inequality between 0 and $t \in I$, we obtain

$$\|w(t)\|^2_{L^2} \le 2C \left| \int_0^t \left(\|w(s)\|^2_{L^2} + \int_\Omega h(s)^2 |w(s)|^2\, dx \right) ds \right| . \tag{3.6.1}$$

Consider any number $p \in (2, \infty)$. We deduce from Hölder's inequality that

$$(3.6.2)\qquad \begin{aligned} \int_\Omega h^2|w|^2\,dx &= \int_\Omega (h^p|w|^2)^{\frac{2}{p}}|w|^{\frac{2p-4}{p}}\,dx \le \Big(\int_\Omega h^p|w|^2\,dx\Big)^{\frac{2}{p}}\|w\|_{L^2}^{\frac{2p-4}{p}} \\ &\le \Big(\int_\Omega h^{2p}\,dx\Big)^{\frac{1}{p}}\|w\|_{L^4}^{\frac{4}{p}}\|w\|_{L^2}^{\frac{2p-4}{p}}. \end{aligned}$$

Note first that $w(t)$ is bounded in $H_0^1(\Omega)$, hence in $L^4(\Omega)$. Furthermore, $h(t)$ is also bounded in $H_0^1(\Omega)$. Therefore (Remark 1.3.6), there exist two positive constants K, μ such that

$$(3.6.3)\qquad \int_\Omega (e^{\mu h(t)^2} - 1)dx \le K.$$

It follows from (3.6.2), (3.6.3), and the elementary inequality

$$x^{2p} \le \Big(\frac{p}{\mu}\Big)^p (e^{\mu x^2} - 1)$$

that

$$\int_\Omega h^2|w|^2\,dx \le CpK^{\frac{1}{p}}\|w\|_{L^2}^{\frac{2p-4}{p}} \quad \text{for some constant } C.$$

Since $K^{\frac{1}{p}} \le 1 + K$, we deduce that

$$\int_\Omega h^2|w|^2\,dx \le Cp\|w\|_{L^2}^{\frac{2p-4}{p}}.$$

Let now $\phi(t) = \|w\|_{L^2}^2$. Applying the above inequality and (3.6.1), we obtain

$$\phi(t) \le C\Big|\int_0^t (\phi(s) + p\phi(s)^{\frac{p-2}{p}})ds\Big|.$$

Note that ϕ is bounded, so that $\phi(t) \le p\phi(t)^{\frac{p-2}{p}}$ for p large enough. Therefore,

$$(3.6.4)\qquad \phi(t) \le Cp\Big|\int_0^t \phi(s)^{\frac{p-2}{p}}\,ds\Big| \quad \text{for all } t \in I.$$

Let now

$$\Phi_p(t) = \int_0^t \phi(s)^{\frac{p-2}{p}}\,ds.$$

It follows from (3.6.4) that $\Phi_p'(t) \le Cp|\Phi_p(t)^{(p-2)/p}|$ for all $t \in I$. Integrating this inequality yields $|\Phi_p(t)| \le (2C|t|)^{p/2}$. Therefore, if $2C|T| < 1$, we obtain

$$\liminf_{p\to\infty} \Phi_p(T) = 0,$$

which implies that

$$\int_0^T \phi(s)ds = 0.$$

Thus $w \equiv 0$ on $[-T, T]$. This gives the result. □

COROLLARY 3.6.2. *Let Ω be an open subset of $\mathbb{R}^2$ and let g be as in Theorem 3.6.1. If $\|\varphi\|_{L^2}$ is small enough, then the maximal strong H_0^1-solution of (3.1.1) given by Theorem 3.6.1 is global and uniformly bounded in H^1. If*

$$F(x,u) \le C(1+|u|^\delta)|u|^2 \quad \textit{for some } \delta < 2,$$

then the same conclusion holds for every $\varphi \in H_0^1(\Omega)$.

PROOF. The assumption on g immediately yields $F(x,u) \le C(1+|u|^2)|u|^2$, so that $G(u) \le C\|u\|_{L^2}^2 + C\|u\|_{L^4}^4$. Using Gagliardo-Nirenberg's inequality, we deduce that $G(u) \le C\|u\|_{L^2}^2 + C\|u\|_{H^1}^2\|u\|_{L^2}^2$. Global existence for small data in L^2 then follows from Theorem 3.4.1. Assuming now $F(x,u) \le C(1+|u|^\delta)|u|^2$ for some $\delta < 2$, we obtain by arguing as above that $G(u) \le C\|u\|_{L^2}^2 + C\|u\|_{H^1}^\delta\|u\|_{L^2}^2$. Applying the inequality $ab \le \varepsilon a^r + C(\varepsilon)b^{r'}$ we deduce that $G(u) \le \frac{1}{2}\|u\|_{H^1}^2 + C(\|u\|_{L^2})$. The result then follows from Corollary 3.4.2. □

REMARK 3.6.3. A global existence result for H^2 solutions (i.e., solutions with values in $H^2(\Omega) \cap H_0^1(\Omega)$) was obtained by Brezis and Gallouët [45].

REMARK 3.6.4. In particular, one may apply Theorem 3.6.1 to the case $g(u) = Vu + \lambda|u|^\alpha u$, where V is a real-valued potential $V \in L^\infty(\Omega)$, $\lambda \in \mathbb{R}$, and $0 \le \alpha \le 2$. In addition, global existence for initial values with small L^2 norm follows from Corollary 3.6.2. Furthermore, if $\lambda \le 0$ or $\alpha < 2$, then for every initial value in $H_0^1(\Omega)$, the corresponding solution of (3.1.1) is global. This follows again from Corollary 3.6.2.

REMARK 3.6.5. Like Theorem 3.3.9, Theorem 3.6.1 is stated for one equation, but the method applies as well for systems of the same form (see Remark 3.3.12).

3.7. Comments

Theorem 3.3.9 admits a generalization in the setting of Theorem 3.3.1. More precisely, with the notation of Theorem 3.3.1, consider a $\mathbb{C}$-linear, self-adjoint ≤ 0 operator A on $X = L^2(\Omega)$. Assume that

$$X_A \hookrightarrow L^p(\Omega) \quad \text{for all } 2 \le p < \frac{2N}{N-2}. \tag{3.7.1}$$

Assume further that for every

$$\frac{2N}{N+2} \le p \le \frac{2N}{N-2},$$

$(I - \varepsilon A)^{-1}$ is continuous $L^p(\Omega) \to L^p(\Omega)$ for all $\varepsilon > 0$, and

$$\sup\left\{\|(I-\varepsilon A)^{-1}\|_{\mathcal{L}(L^p,L^p)} : \varepsilon > 0\right\} < \infty. \tag{3.7.2}$$

Consider a function $g \in C(X_A, X_A^\star)$ such that

$$g = G' \quad \text{for some } G \in C^1(X_A, \mathbb{R}), \tag{3.7.3}$$

and assume that there exist $r, \rho \in [2, \frac{2N}{N-2})$ ($r, \rho \in [2,\infty]$ if $N = 1$) such that

$$g \in C(X_A, L^{\rho'}(\Omega)) \tag{3.7.4}$$

and such that for every $M > 0$, there exists $C(M) < \infty$ such that

$$\|g(v) - g(u)\|_{L^{\rho'}} \le C(M)\|v - u\|_{L^r} \tag{3.7.5}$$

for every $u, v \in X_A$ such that $\|u\|_{X_A} + \|v\|_{X_A} \le M$. (Or, more generally, assume $g = g_1 + \cdots + g_k$, where each g_j satisfies the above assumptions for some r_j, ρ_j.) Finally, assume that for every $u \in X_A$,

$$\operatorname{Im}(g(u)\overline{u}) = 0 \quad \text{a.e. on } \Omega\,, \tag{3.7.6}$$

and let E be defined by (3.3.2).

We consider the problem

$$\begin{cases} iu_t + Au + g(u) = 0 \\ u(0) = x \end{cases} \tag{3.7.7}$$

for a given $x \in X_A$. We have the following result.

THEOREM 3.7.1. *Let A and g be as above. Assume, in addition, that there is uniqueness for the problem* (3.7.7). *It follows that the initial value problem* (3.7.7) *is locally well posed in X_A. Moreover, there is conservation of charge and energy; i.e.,*

$$\|u(t)\|_{L^2} = \|x\|_{L^2} \quad \text{and} \quad E(u(t)) = E(x)$$

for all $t \in (-T_{\min}, T_{\max})$, where u is the solution of (3.7.7) *with the initial value $x \in X_A$. (Here, the notions of uniqueness and local well-posedness are as in Section* 3.1).

PROOF. The proof is an adaptation of the proof of Theorem 3.3.9. We only point out the modifications that are not absolutely trivial. Lemma 3.3.6 is easily adapted with the duality inequality $\|u\|_X^2 \le \|u\|_{X_A}\|u\|_{X_A^\star}$. The proof of Lemma 3.3.7 is adapted as follows. Consider $2 \le p < q < \frac{2N}{N-2}$. By Hölder's inequality and (3.7.1), there exists $a \in (0,1)$ such that

$$\|u\|_{L^p} \le \|u\|_{L^q}^a \|u\|_{L^2}^{1-a} \le \|u\|_A^a \|u\|_{L^2}^{1-a}\,,$$

and the rest of the proof is unchanged. To adapt the proof of Theorem 3.3.5, we need inequalities of the type (3.3.15)–(3.3.16). They follow easily from the self-adjointness of A, except for (3.3.16), which follows from (3.7.2). The rest of the proof, including Lemma 3.3.8, is unchanged except that one has to apply Proposition 1.1.2 instead of Proposition 1.3.14. □

REMARK 3.7.2. Corollary 3.3.11 is easily adapted to the above situation.

REMARK 3.7.3. Like Theorem 3.3.9 (see Remark 3.3.12), Theorem 3.7.1 is stated for one equation, but the method applies as well for systems of the same form. More precisely, considering an integer $\mu \ge 1$, one may assume that A is a self-adjoint operator on $(L^2(\Omega))^\mu$ and replace everywhere $L^p(\Omega)$ by $(L^p(\Omega))^\mu$. It follows that the conclusions of Theorem 3.7.1 remain valid.

REMARK 3.7.4. Using Strichartz-type estimates (see Remark 2.7.3), it is possible to solve the local (or global) initial value problem for certain nonlinear Schrödinger equations in a cube of $\mathbb{R}^N$ with periodic boundary conditions. See Bourgain [34, 35, 37, 38] and Kenig, Ponce, and Vega [214].

REMARK 3.7.5. For $g(u) = -|u|^2 u$ in the unit disc of $\mathbb{R}^2$, it was shown that the initial value problem is ill-posed in $H^s(\Omega)$ for $s < 1/3$. More precisely, given $T > 0$ and a bounded subset B of $H^s(\Omega)$, the map $\varphi \in B \cap H_0^1(\Omega) \mapsto u \in C([0,T], H^s(\Omega))$ is not uniformly continuous. See Burq, Gérard, and Tzvetkov [48, 50]. Note that the case of a disc is therefore different from the case of a square; see Bourgain [38].

REMARK 3.7.6. Nonlinear Schrödinger equations are sometimes considered in exterior domains. When $N = 1$, or when $N = 2$ and under some growth condition on the nonlinearity, local (or global) existence follows from the results of Sections 3.5 and 3.6. In some other cases, one can still obtain global solutions for small initial data and study their asymptotic behavior. See, for example, Chen [78], Esteban and Strauss [111], Hayashi [164, 165], M. Tsutsumi [339], Y. Tsutsumi [340], and Yao [365]. Recently, by using a family of Strichartz estimates (see Remark 2.7.9), it was shown that in the exterior of a nontrapping obstacle in $\mathbb{R}^N$, there is local well-posedness in $H_0^1(\Omega)$ if $\alpha < 2/(N-2)$ and in $L^2(\Omega)$ if $\alpha < 2/N$ when the nonlinearity is, for example, $g(u) = \lambda |u|^\alpha u$. See Burq, Gérard, and Tzvetkov [47].

REMARK 3.7.7. Using a family of Strichartz estimates (see Remark 2.7.10), it is possible to solve the local (or global) Cauchy problem for certain nonlinear Schrödinger equations on nonflat manifolds. See Burq, Gérard, and Tzvetkov [46, 49].

CHAPTER 4

The Local Cauchy Problem

4.1. Outline

In this chapter we study the local Cauchy problem in the case $\Omega = \mathbb{R}^N$. Therefore, we consider the problem

$$\begin{cases} iu_t + \Delta u + g(u) = 0, \\ u(0) = \varphi. \end{cases} \tag{4.1.1}$$

We note that if $I \ni 0$ is an interval and $g \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$ is bounded on bounded sets, then $u \in L^\infty(I, H_0^1(\Omega))$ is a solution of equation (4.1.1) on I if and only if u satisfies the integral equation

$$u(t) = \mathcal{T}(t)\varphi + i\int_0^t \mathcal{T}(t-s)g(u(s))ds \quad \text{for a.a. } t \in I \tag{4.1.2}$$

(see Proposition 3.1.3). A special case of (4.1.1) is the pure power nonlinearity $g(u) = \lambda|u|^\alpha u$ with $\lambda \in \mathbb{C}$ and $\alpha \geq 0$. (4.1.1) then takes the form

$$\begin{cases} iu_t + \Delta u + \lambda|u|^\alpha u = 0, \\ u(0) = \varphi. \end{cases} \tag{4.1.3}$$

We observe that if $\alpha \leq 4/(N-2)$ ($\alpha < \infty$ if $N = 1, 2$), then $u \in L^\infty(I, H_0^1(\Omega))$ is a solution of equation (4.1.3) on I if and only if u satisfies the integral equation

$$u(t) = \mathcal{T}(t)\varphi + i\lambda\int_0^t \mathcal{T}(t-s)|u|^\alpha u(s)ds \quad \text{for a.a. } t \in I. \tag{4.1.4}$$

Of course, the results of Chapter 3 apply in particular to the problem (4.1.1). The essential particularity of the case $\Omega = \mathbb{R}^N$ is that we may use Strichartz's estimates. They are the main tool for obtaining uniqueness results. They also can be used for showing existence results in various spaces by fixed-point arguments or other methods.

In Section 4.2 we establish various uniqueness properties based on Strichartz's estimates. In Section 4.3 we apply the results of Chapter 3 combined with those of Section 4.2. In Section 4.4 we apply a fixed-point argument of Kato to derive existence results. They apply in particular to nonlinearities for which there is neither conservation of charge nor conservation of energy, so that the results of Chapter 3 do not apply. Section 4.5 is devoted to a critical case in $H^1(\mathbb{R}^N)$.

The next sections are devoted to existence results in spaces different from the energy space $H^1(\mathbb{R}^N)$: $L^2(\mathbb{R}^N)$ (Sections 4.6 and 4.7), $H^2(\mathbb{R}^N)$ (Section 4.8), $H^s(\mathbb{R}^N)$ for $s < N/2$ (Section 4.9), and $H^m(\mathbb{R}^N)$ for $m > N/2$ (Section 4.10).

Section 4.11 is devoted to a nonautonomous Schrödinger equation that is derived from (4.1.1) by the pseudoconformal transformation. We will use that equation in Section 7.5.

Finally, we observe that the results of this chapter are stated for one equation, but similar results obviously hold for systems of the same form. See Remark 3.3.12 for an appropriate setting.

4.2. Strichartz's Estimates and Uniqueness

As we have seen in Section 3.3, uniqueness is a key property for the local well-posedness of the initial-value problem (4.1.1). In the present case where $\Omega = \mathbb{R}^N$, Strichartz's estimates are a powerful tool to establish uniqueness. We note that most of the results of this section are due to Kato [206]. We begin with the model case of the pure power nonlinearity; i.e., we consider the problem (4.1.3). We note that if $\alpha \le 4/(N-2)$ ($\alpha < \infty$ if $N = 1,2$), then $g(u) = \lambda |u|^\alpha u$ satisfies $g \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$ so that we may consider weak H^1-solutions of (4.1.3). It turns out that they are unique, as the following result shows.

PROPOSITION 4.2.1. *Assume $\lambda \in \mathbb{C}$ and $0 < \alpha < 4/(N-2)$ ($0 < \alpha < \infty$ if $N = 1,2$). If $\varphi \in H^1(\mathbb{R}^N)$ and u_1, u_2 are two weak H^1-solutions of* (4.1.3) *on some interval $I \ni 0$, then $u_1 = u_2$.*

PROOF. We may assume without loss of generality that I is a bounded interval. It follows from (4.1.4) that

$$(u_1 - u_2)(t) = i\lambda \int_0^t \mathcal{T}(t-s)\big(|u_1|^\alpha u_1 - |u_2|^\alpha u_2\big)(s)ds. \tag{4.2.1}$$

Since

$$\big||u_1|^\alpha u_1 - |u_2|^\alpha u_2\big| \le C\big(|u_1|^\alpha + |u_2|^\alpha\big)|u_1 - u_2|,$$

we deduce from Hölder's inequality that, setting $r = \alpha + 2$,

$$\big\||u_1|^\alpha u_1 - |u_2|^\alpha u_2\big\|_{L^{r'}} \le C\big(\|u_1\|_{L^r}^\alpha + \|u_2\|_{L^r}^\alpha\big)\|u_1 - u_2\|_{L^r}.$$

Let now $q = 4r/N(r-2)$ so that (q,r) is an admissible pair. Applying Hölder's inequality in time, we deduce that if J is an interval such that $0 \in J \subset I$, then

$$\begin{aligned}\big\||u_1|^\alpha u_1 - |u_2|^\alpha u_2\big\|_{L^{q'}(J,L^{r'})} \le \\ C\big(\|u_1\|_{L^\infty(J,L^r)}^\alpha + \|u_2\|_{L^\infty(J,L^r)}^\alpha\big)\|u_1 - u_2\|_{L^{q'}(J,L^r)}.\end{aligned} \tag{4.2.2}$$

It follows from (4.2.1), (4.2.2), and Strichartz's estimate that

$$\|u_1 - u_2\|_{L^q(J,L^r)} \le C\big(\|u_1\|_{L^\infty(J,L^r)}^\alpha + \|u_2\|_{L^\infty(J,L^r)}^\alpha\big)\|u_1 - u_2\|_{L^{q'}(J,L^r)}. \tag{4.2.3}$$

Since $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ and $|I| < \infty$, (4.2.3) yields

$$\|u_1 - u_2\|_{L^q(J,L^r)} \le C\|u_1 - u_2\|_{L^{q'}(J,L^r)}$$

for some constant C independent of J. The result now follows by applying Lemma 4.2.2 below with $k = 1$, $\phi_1(t) = \|u_1(t) - u_2(t)\|_{L^r}$, $a_1 = q'$, and $b_1 = q$. □

LEMMA 4.2.2. *Let $I \ni 0$ be an interval. Let $1 \le a_j < b_j \le \infty$ and $\phi_j \in L^{b_j}(I)$, for $1 \le j \le k$. If there exists a constant $C \ge 0$ such that*

$$\sum_{j=1}^{k} \|\phi_j\|_{L^{b_j}(J)} \le C \sum_{j=1}^{k} \|\phi_j\|_{L^{a_j}(J)} \tag{4.2.4}$$

for every interval J such that $0 \in J \subset I$, then $\phi_1 = \cdots = \phi_k = 0$ a.e. on I.

PROOF. We first consider the case $I = [0, T]$ for some $0 < T < \infty$. Suppose that $\phi_1 = \cdots = \phi_k = 0$ a.e. on some interval $(0, \tau)$ with $0 \le \tau < T$. Letting $J = [0, t]$ with $\tau < t < T$, it follows from (4.2.4) and Hölder's inequality that

$$\sum_{j=1}^{k} \|\phi_j\|_{L^{b_j}(\tau,t)} \le C \sum_{j=1}^{k} \|\phi_j\|_{L^{a_j}(\tau,t)} \le C \sum_{j=1}^{k} (t-\tau)^{\frac{1}{a_j} - \frac{1}{b_j}} \|\phi_j\|_{L^{b_j}(0,t)} .$$

Therefore, if we let $t - \tau$ be sufficiently small so that

$$C \max_{1 \le j \le k} (t-\tau)^{\frac{1}{a_j} - \frac{1}{b_j}} < 1 ,$$

we deduce that $\|\phi_1\|_{L^{b_1}(0,t)} + \cdots + \|\phi_k\|_{L^{b_k}(0,t)} = 0$. We now let

$$\theta = \sup \Big\{ 0 < t < T; \sum_{j=1}^{k} \|\phi_j\|_{L^{b_j}(0,t)} = 0 \Big\} . \tag{4.2.5}$$

We deduce from what precedes that $\theta > 0$ (starting with $\tau = 0$). If $\theta < T$, then we let $\tau = \theta$ and we deduce that $\phi_1 = \cdots = \phi_k = 0$ a.e. on $(0, \theta + \varepsilon)$ for some $\varepsilon > 0$, which contradicts (4.2.5). Thus $\theta = T$ which shows the desired conclusion. The case $I = [-T, 0]$ is treated similarly (by changing t to $-t$). In the general case, we apply the above results to all $T > 0$ such that $[0, T] \subset I$, then to all $T > 0$ such that $[-T, 0] \subset I$, and we deduce that $\phi_1 = \cdots = \phi_k = 0$ a.e. on I. □

Proposition 4.2.1 can be extended to more general nonlinearities. In particular, we have the following result.

PROPOSITION 4.2.3. *Consider $g_1, \dots, g_k \in C(H^1(\mathbb{R}^N)), H^{-1}(\mathbb{R}^N))$ and let*

$$g = g_1 + \cdots + g_k .$$

Assume that each of the g_j's satisfies the assumption (3.3.7) for some exponents $r_j, \rho_j \in [2, 2N/(N-2))$ ($r_j, \rho_j \in [2, \infty]$ if $N = 1$); i.e., there exists C_j such that

$$\|g_j(u) - g_j(v)\|_{L^{\rho_j'}} \le C_j(M) \|u - v\|_{L^{r_j}} \tag{4.2.6}$$

for all $u, v \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1}, \|v\|_{H^1} \le M$. If $\varphi \in H^1(\mathbb{R}^N)$ and u_1, u_2 are two weak H^1-solutions of (4.1.1) on some interval $I \ni 0$, then $u_1 = u_2$.

Proposition 4.2.3 is a consequence of the following simple lemma.

LEMMA 4.2.4. *Let $I \ni 0$ be an interval and $k \geq 1$ be an integer. For every $1 \leq j \leq k$, let (q_j, r_j) and (γ_j, ρ_j) be admissible pairs and $f_j \in L^{\gamma_j'}(I, L^{\rho_j'}(\mathbb{R}^N))$. Finally, set*

$$w(t) = i \sum_{j=1}^{k} \int_0^t \mathcal{T}(t-s) f_j(s)ds \tag{4.2.7}$$

for all $t \in I$ (so that $w \in L^{q_j}(I, L^{r_j}(\mathbb{R}^N))$ for all $1 \leq j \leq k$ by Strichartz's estimates). If for every $1 \leq j \leq k$ there exist $1 \leq a_j < q_j$ and a constant C_j such that

$$\|f_j\|_{L^{\gamma_j'}(J, L^{\rho_j'})} \leq C_j \|w\|_{L^{a_j}(J, L^{r_j})} \tag{4.2.8}$$

for all bounded intervals J such that $0 \in J \subset I$, then $w = 0$.

PROOF. Letting

$$w_j(t) = i \int_0^t \mathcal{T}(t-s) f_j(s)ds$$

and applying k times Strichartz's estimate, we see that there exists a constant K_j such that

$$\sum_{\ell=1}^{k} \|w_j\|_{L^{q_\ell}(J, L^{r_\ell})} \leq K_j \|f_j\|_{L^{\gamma_j'}(J, L^{\rho_j'})}$$

for all bounded intervals J such that $0 \in J \subset I$. It follows from (4.2.7) that there exists a constant C such that

$$\sum_{j=1}^{k} \|w\|_{L^{q_j}(J, L^{r_j})} \leq C \sum_{j=1}^{k} \|f_j\|_{L^{\gamma_j'}(J, L^{\rho_j'})}$$

for all bounded intervals J such that $0 \in J \subset I$. Applying (4.2.8) we deduce that

$$\sum_{j=1}^{k} \|w\|_{L^{q_j}(J, L^{r_j})} \leq C \sum_{j=1}^{k} C_j \|w\|_{L^{a_j}(J, L^{r_j})},$$

and the result follows from Lemma 4.2.2. □

PROOF OF PROPOSITION 4.2.3. Let $u_1, u_2 \in L^\infty(I, H^1) \cap W^{1,\infty}(I, H^{-1})$ be two solutions of (4.1.1). By (4.1.2),

$$u_1(t) - u_2(t) = i \sum_{j=1}^{k} \int_0^t \mathcal{T}(t-s)[g_j(u_1(s)) - g_j(u_2(s))]ds \quad \text{for a.a. } t \in I.$$

The result follows from Lemma 4.2.4 applied with $w = u_1 - u_2$, $f_j = g_j(u_1) - g_j(u_2)$, and $(\gamma_j)_{1 \leq j \leq k}$ and $(q_j)_{1 \leq j \leq k}$ defined by

$$\frac{2}{\gamma_j} = N\Big(\frac{1}{2} - \frac{1}{\rho_j}\Big) \quad \text{and} \quad \frac{2}{q_j} = N\Big(\frac{1}{2} - \frac{1}{r_j}\Big),$$

so that (q_j, r_j) and (γ_j, ρ_j) are admissible pairs. (4.2.8) is indeed satisfied with $a_j = \gamma_j'$ since by (4.2.6)

$$\|f_j\|_{L^{\gamma_j'}(J, L^{\rho_j'})} \leq C \|w\|_{L^{\gamma_j'}(J, L^{r_j})},$$

and $\gamma_j' < 2 < q_j$. □

We note that the technique of proof of Proposition 4.2.1 does not work in the limiting case $\alpha = 4/(N-2)$, $N \geq 3$. Indeed, we would be lead to apply Lemma 4.2.2 with $a_1 = b_1$, in which case the conclusion of the lemma is clearly false. In fact, we do not know if the conclusion of Proposition 4.2.1 holds in the limiting case $\alpha = 4/(N-2)$. A slight modification of the method of proof, however, shows uniqueness of strong H^1-solutions. More precisely, we have the following result.

PROPOSITION 4.2.5. *Assume $N \geq 3$. Let $\lambda \in \mathbb{C}$ and $\alpha = 4/(N-2)$. If $\varphi \in H^1(\mathbb{R}^N)$ and u_1, u_2 are two strong H^1-solutions of* (4.1.3) *on some interval $I \ni 0$, then $u_1 = u_2$.*

PROOF. We may assume without loss of generality that $I = [0,T]$ for some $0 < T < \infty$. Given $M > 0$, set

$$\begin{aligned} f^M &= 1_{\{|u_1|+|u_2|>M\}}\left(|u_1|^{\frac{4}{N-2}}u_1 - |u_2|^{\frac{4}{N-2}}u_2\right), \\ f_M &= 1_{\{|u_1|+|u_2|\leq M\}}\left(|u_1|^{\frac{4}{N-2}}u_1 - |u_2|^{\frac{4}{N-2}}u_2\right), \end{aligned}$$

so that

$$|u_1|^{\frac{4}{N-2}}u_1 - |u_2|^{\frac{4}{N-2}}u_2 = f_M + f^M.$$

One easily verifies that there exists C independent of M such that

$$\begin{cases} |f_M| \leq CM^{\frac{4}{N-2}}|u_1 - u_2|, \\ |f^M| \leq C1_{\{|u_1|+|u_2|>M\}}(|u_1| + |u_2|)^{\frac{4}{N-2}}|u_1 - u_2|. \end{cases} \tag{4.2.9}$$

In order to show that $u_1 = u_2$, we use the endpoint Strichartz's estimate. We have

$$(u_1 - u_2)(t) = i\lambda \int_0^t \mathcal{T}(t-s)(f_M + f^M)ds,$$

so that for every $0 \leq \tau \leq T$,

$$\begin{aligned} &\|u_1 - u_2\|_{L^2((0,\tau),L^{\frac{2N}{N-2}})} + \|u_1 - u_2\|_{L^\infty((0,\tau),L^2)} \leq \\ &\qquad C\left(\|f_M\|_{L^1((0,\tau),L^2)} + \|f^M\|_{L^2((0,\tau),L^{\frac{2N}{N+2}})}\right). \end{aligned} \tag{4.2.10}$$

Using (4.2.9), we see that

$$\|f_M\|_{L^1((0,\tau),L^2)} \leq CM^{\frac{4}{N-2}}\|u_1 - u_2\|_{L^1((0,\tau),L^2)}, \tag{4.2.11}$$

and that

$$\begin{aligned} &\|f^M\|_{L^2((0,\tau),L^{\frac{2N}{N+2}})} \leq \\ &\qquad C\|1_{\{|u_1|+|u_2|>M\}}(|u_1| + |u_2|)\|_{L^\infty((0,\tau),L^{\frac{2N}{N-2}})}\|u_1 - u_2\|_{L^2((0,\tau),L^{\frac{2N}{N+2}})}. \end{aligned} \tag{4.2.12}$$

Finally, we observe that $|u_1| + |u_2| \in C([0,T], H^1(\mathbb{R}^N))$, so that we have $|u_1| + |u_2| \in C([0,T], L^{\frac{2N}{N-2}}(\mathbb{R}^N))$. It follows easily by dominated convergence that

$$\|1_{\{|u_1|+|u_2|>M\}}(|u_1| + |u_2|)\|_{L^\infty((0,T),L^{\frac{2N}{N-2}})} \underset{M\to\infty}{\longrightarrow} 0. \tag{4.2.13}$$

Applying (4.2.13) we see that, by choosing M large enough, we can absorb the right-hand side of (4.2.12) by the left-hand side of (4.2.10). Therefore, we deduce from (4.2.10)–(4.2.12) that there exists C such that

$$\|u_1 - u_2\|_{L^\infty((0,\tau),L^2)} \le C\|u_1 - u_2\|_{L^1(0,\tau),L^2)}$$

for every $0 \le \tau \le T$, and the result follows from Lemma 4.2.2. □

REMARK 4.2.6. Note that it is precisely for showing (4.2.13) that we use the assumption that u_1 and u_2 are strong H^1-solutions. For weak H^1-solutions, we only know that $|u_1| + |u_2| \in L^\infty((0,T), H^1(\mathbb{R}^N))$, which does not imply (4.2.13).

Proposition 4.2.5 can be extended to more general nonlinearities. We will not study the general case. Note, however, that an immediate adaptation of the proof of Proposition 4.2.5 yields the following result concerning local nonlinearities.

PROPOSITION 4.2.7. *Assume* $N \ge 3$. *Let* $g \in C(\mathbb{C},\mathbb{C})$ *with* $g(0) = 0$ *satisfy*

$$|g(z_1) - g(z_2)| \le C\big(1 + |z_1|^{\frac{4}{N-2}} + |z_2|^{\frac{4}{N-2}}\big)|z_1 - z_2|$$

for all $z_1, z_2 \in \mathbb{C}$. *If* $\varphi \in H^1(\mathbb{R}^N)$ *and* u_1, u_2 *are two strong* H^1*-solutions of* (4.1.1) *on some interval* $I \ni 0$, *then* $u_1 = u_2$.

We will construct in the following sections solutions of (4.1.1) that are not H^1-solutions, and we now study uniqueness of such solutions. We begin with a lemma about the equivalence of (4.1.1) and (4.1.2) for such solutions.

LEMMA 4.2.8. *Let* $I \ni 0$ *be an interval, let* $s, \sigma \in \mathbb{R}$, *and let* $g : H^s(\mathbb{R}^N) \to H^\sigma(\mathbb{R}^N)$ *be continuous and bounded on bounded sets. If* $u \in L^\infty(I, H^s(\mathbb{R}^N))$, *then both equations* (4.1.1) *and* (4.1.2) *make sense in* $H^\mu(\mathbb{R}^N)$ *for* $\mu = \min\{s-2, \sigma\}$. *Moreover,* u *satisfies equation* (4.1.1) *for a.a.* $t \in I$ *if and only if* u *satisfies the integral equation* (4.1.2) *for a.a.* $t \in I$.

PROOF. Let $u \in L^\infty(I, H^s(\mathbb{R}^N))$. Since $\Delta \in \mathcal{L}(H^s(\mathbb{R}^N), H^{s-2}(\mathbb{R}^N))$, we see that $\Delta u \in L^\infty(I, H^{s-2}(\mathbb{R}^N))$. Moreover, $g(u)$ is measurable $I \to H^\sigma(\mathbb{R}^N)$ because $g \in C(H^s(\mathbb{R}^N), H^\sigma(\mathbb{R}^N))$, and bounded because g is bounded on bounded sets, and so $g(u) \in L^\infty(I, H^\sigma(\mathbb{R}^N))$. Thus we see that both equations make sense in $H^\mu(\mathbb{R}^N)$. Since $(\mathcal{T}(t))_{t\in\mathbb{R}}$ is a group of isometries on $H^\mu(\mathbb{R}^N)$, the equivalence between the two then follows from the results of Section 1.6. □

According to the above lemma, under appropriate assumptions on g, we can address the question of uniqueness of solutions of (4.1.1) in $L^\infty(I, H^s(\mathbb{R}^N))$. We have the following result, which is an easy application of Lemma 4.2.4.

PROPOSITION 4.2.9. *Consider* $s \ge 0$. *Let*

(4.2.14) $g_1, \dots, g_k \in C(H^s(\mathbb{R}^N)), H^\sigma(\mathbb{R}^N))$ *be bounded on bounded sets,*

for some $\sigma \in \mathbb{R}$, *and let* $g = g_1 + \dots + g_k$. *Assume that there exist exponents* $r_j, \rho_j \in [2, 2N/(N-2))$ ($r_j, \rho_j \in [2,\infty]$ *if* $N = 1$) *and functions* $C_j \in C([0,\infty))$ *such that*

(4.2.15) $$\|g_j(u) - g_j(v)\|_{L^{\rho'_j}} \le C_j(M)\|u - v\|_{L^{r_j}},$$

for all $u, v \in H^s(\mathbb{R}^N)$ *such that* $\|u\|_{H^s}, \|v\|_{H^s} \le M$. *Let* $\varphi \in H^s(\mathbb{R}^N)$ *and* $u_1, u_2 \in L^\infty(I, H^s(\mathbb{R}^N))$ *be two solutions of* (4.1.1) *on some interval* $I \ni 0$. *If*

$$u_1 - u_2 \in \bigcap_{j=1}^{k} L^{q_j}(I, L^{r_j}(\mathbb{R}^N)), \tag{4.2.16}$$

where q_j *is such that* (q_j, r_j) *is an admissible pair, then* $u_1 = u_2$.

REMARK 4.2.10. The assumptions (4.2.14), (4.2.15), and (4.2.16) deserve some comments. (4.2.14) ensures that equation (4.1.1) makes sense for a function $u \in L^\infty(I, H^s(\mathbb{R}^N))$. (See Lemma 4.1.8.) Assumption (4.2.15) is a Lipschitz condition for the g_j's on bounded sets of $H^s(\mathbb{R}^N)$. It is rather natural, since a Lipschitz condition of some sort is necessary for a uniqueness property. Finally, (4.2.16) is a regularity assumption on the difference of the solutions u_1 and u_2. In practice, it is verified by requiring that $H^s(\mathbb{R}^N) \hookrightarrow L^{r_j}(\mathbb{R}^N)$ for all j's, so that both u_1 and u_2 belong to the prescribed space and in particular the difference $u_1 - u_2$. Note, however, that (4.2.16) is in principle weaker than assuming that both u_1 and u_2 belong to the prescribed space. For example, for the Navier-Stokes equation, the difference of two solutions has a better regularity in certain spaces than each of the solutions (see, e.g., [225]). However, it seems that no one could use such a property for the Schrödinger equation to take advantage of the fact that (4.2.16) concerns the difference of two solutions (see Furioli and Terraneo [120] for interesting comments on this problem).

PROOF OF PROPOSITION 4.2.9. The proof is identical to the proof of Proposition 4.2.3. □

REMARK 4.2.11. Given $s \ge 0$, we apply Proposition 4.2.9 to the model case $g(u) = \lambda |u|^\alpha u$ where $\alpha > 0$ and $\lambda \in \mathbb{C}$. There are three conditions to be checked, namely (4.2.14), (4.2.15), and (4.2.16). We note that, since g is a single power, we do not need to decompose $g = g_1 + \cdots + g_k$.

We investigate the condition (4.2.14). Suppose first $s \ge N/2$, so that $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for every $2 \le p < \infty$. It follows easily that (4.2.14) is satisfied with $\sigma = 0$. Suppose now $s < N/2$, so that $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for every $2 \le p \le 2N/(N-2s)$. We deduce easily that if $(N-2s)(\alpha+1) \le 2N$, then

$$g \in C\big(H^s(\mathbb{R}^N), L^{\frac{2N}{(N-2s)(\alpha+1)}}(\mathbb{R}^N)\big).$$

Condition (4.2.14) is then satisfied, for example, with $\sigma < -N/2$. If, on the other hand, $(N-2s)(\alpha+1) < 2N$, then $g(u)$ (which is a measurable function) need not be locally integrable, so that g does not map $H^s(\mathbb{R}^N)$ into any space of the type $H^\sigma(\mathbb{R}^N)$. Therefore, we see that (4.2.14) is satisfied if and only if

$$s \ge \frac{N(\alpha-1)}{2(\alpha+1)}. \tag{4.2.17}$$

(In particular, there is no condition if $\alpha \le 1$.)

We next investigate the condition (4.2.16). As observed in Remark 4.2.10, we require that $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, where r is as in (4.2.15). This is obviously satisfied if $s \geq N/2$. If $s < N/2$, then we need

$$2 \leq r \leq \frac{2N}{N-2s}. \tag{4.2.18}$$

We now turn to the condition (4.2.15) and we first study the case $s \geq N/2$ by using the inequality

$$\|g(u) - g(v)\|_{L^{r'}} \leq C\big(\|u\|^{\alpha}_{L^{\frac{\alpha r}{r-2}}} + \|v\|^{\alpha}_{L^{\frac{\alpha r}{r-2}}}\big)\|u - v\|_{L^r}.$$

We note that $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for all $2 \leq p < \infty$, so we see that (4.2.15) is satisfied with $\rho = r$ for all $r > 2$ sufficiently close to 2.

We then study the case $s < N/2$, so that $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for every $2 \leq p \leq 2N/(N-2s)$, and we use the inequality

$$\|g(u) - g(v)\|_{L^{\rho'}} \leq C\big(\|u\|^{\alpha}_{L^p} + \|v\|^{\alpha}_{L^p}\big)\|u - v\|_{L^r},$$

where $1 \leq p \leq \infty$ satisfies

$$\frac{\alpha}{p} = 1 - \frac{1}{\rho} - \frac{1}{r}.$$

We begin with the case $N = 1$. Since the admissible values of ρ are $2 \leq \rho \leq \infty$ and the admissible values of r are (by (4.2.18)) $2 \leq r \leq 2/(1-2s)$, we see that the admissible values of p are $2\alpha/(1+2s) \leq p \leq \infty$. Of course, we want $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, and this is compatible with the above restriction provided $2/(1-2s) \geq 2\alpha/(1+2s)$. We note that this is exactly (since $s < 1/2$) the condition (4.2.17). We now assume $N \geq 2$ and we begin by assuming $s \geq 1$. In this case (4.2.18) is not a further restriction on r, so that the admissible values of r and ρ are $2 \leq r, \rho < 2N/(N-2)$. Thus the admissible values of p are $N\alpha/2 < p \leq \infty$. We want $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, and this is compatible with the above restriction provided $2N/(N-2s) > N\alpha/2$, i.e., $\alpha < 4/(N-2s)$. If $s < 1$, then we have the further restriction (4.2.18) on r, so that the admissible values of p are $N\alpha/(1+s) < p \leq \infty$. We want $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, and this is compatible with the above restriction provided $2N/(N-2s) > N\alpha/(1+s)$, i.e., $\alpha < (2+2s)/(N-2s)$.

In conclusion, we see that if $s \geq N/2$ there is always uniqueness in $L^\infty(I, H^s(\mathbb{R}^N))$. If $s < N/2$ and $N = 1$, then there is uniqueness as soon as the equation makes sense, i.e., as soon as (4.2.17) holds, that is

$$\alpha \leq \frac{N+2s}{N-2s}. \tag{4.2.19}$$

If $s < N/2$ and $N \geq 2$, then there is uniqueness provided the equation makes sense, i.e., provided (4.2.19) holds, but under the additional assumption

$$\alpha < \frac{\min\{4, 2+2s\}}{N-2s}. \tag{4.2.20}$$

REMARK 4.2.12. Suppose $g \in C(\mathbb{C}, \mathbb{C})$ satisfies $g(0) = 0$ and

$$|g(z_1) - g(z_2)| \leq C\big(1 + |z_1|^\alpha + |z_2|^\alpha\big)|z_1 - z_2|$$

for some $\alpha > 0$. It follows that the conclusions of Remark 4.2.11 hold. More precisely, there is uniqueness in $L^\infty(I, H^s(\mathbb{R}^N))$ provided $s \geq N/2$, or provided

$s < N/2$, (4.2.19), and, if $N \geq 2$, (4.2.20). To see this, we decompose $g = g_1 + g_2$, where $g_1(0) = g_2(0) = 0$, g_1 is globally Lipschitz, and g_2 satisfies

$$|g_2(z_1) - g_2(z_2)| \leq C(|z_1|^\alpha + |z_2|^\alpha)|z_1 - z_2|.$$

(See Section 3.2.) We may apply Proposition 4.2.9, since g_2 is handled with exactly the argument of Remark 4.2.11 and g_1 clearly satisfies (4.2.15) with $r_1 = \rho_1 = 2$.

We now state an analogue of Proposition 4.2.7 in the case of H^s solutions.

PROPOSITION 4.2.13. *Assume $N \geq 3$ and $1 \leq s < N/2$. Let $g \in C(\mathbb{C}, \mathbb{C})$ with $g(0) = 0$ satisfy*

$$|g(z_1) - g(z_2)| \leq C\big(1 + |z_1|^{\frac{4}{N-2s}} + |z_2|^{\frac{4}{N-2s}}\big)|z_1 - z_2|$$

for all $z_1, z_2 \in \mathbb{C}$. If $\varphi \in H^s(\mathbb{R}^N)$ and u_1, u_2 are two solutions of (4.1.1) *in the class $C(I, H^s(\mathbb{R}^N))$ for some interval $I \ni 0$, then $u_1 = u_2$.*

PROOF. Since $N \geq 3$ and $s \geq 1$, (4.2.17) is satisfied. This implies that equation (4.1.1) makes sense for a function $u \in C(I, H^s(\mathbb{R}^N))$ (see Remarks 4.2.11 and 4.2.12). The proof is similar to the proof of Proposition 4.2.7. Note that we use the same admissible pairs $(\infty, 2)$ and $(2, 2N/(N-2))$, and that we use the property $u \in C(I, L^{\frac{2N}{N-2s}}(\mathbb{R}^N))$, so that

$$\big\|1_{\{|u_1|+|u_2|>M\}}(|u_1| + |u_2|)\big\|_{L^\infty((0,T), L^{\frac{2N}{N-2s}})} \to 0 \quad \text{as } M \to \infty.$$

□

REMARK 4.2.14. The observations of Remarks 4.2.11 and 4.2.12 and Proposition 4.2.13 are part of the work of Kato [206]. In the single power case, we observe that when $N = 1$ or when $s \geq N/2$ there is always uniqueness in $L^\infty(I, H^s(\mathbb{R}^N))$ as soon as the equation makes sense. When $N \geq 2$ and $0 \leq s < N/2$, there are some cases where the equation makes sense; but uniqueness is not a consequence of Remark 4.2.11, namely when

$$\frac{\min\{4, 2 + 2s\}}{N - 2s} \leq \alpha \leq \frac{N + 2s}{N - 2s}.$$

In fact, we will see in Section 4.9 that when $\alpha \leq 4/(N-2s)$, one can construct H^s solutions, while for $\alpha > 4/(N-2s)$ the existence problem is open. Even if one is willing to consider the restriction $\alpha \leq 4/(N-2s)$ as essential, there still are cases when uniqueness in $L^\infty(I, H^s(\mathbb{R}^N))$ does not follow from Remark 4.2.11, namely when $N \geq 2$, $0 \leq s < 1$ and

$$\frac{2 + 2s}{N - 2s} \leq \alpha \leq \frac{4}{N - 2s}.$$

This has been an open problem since the work of Kato [206] but there was a recent breakthrough by Furioli and Terraneo [120] who were able to fill part of the gap by using in particular negative order Besov spaces.

4.3. Local Existence in $H^1(\mathbb{R}^N)$

Consider $g_1, \dots, g_k \in C(H^1(\mathbb{R}^N)), H^{-1}(\mathbb{R}^N))$ and let

$$g = g_1 + \cdots + g_k \,.$$

Assume that each of the g_j's satisfies the assumptions (3.3.5)–(3.3.8) for some exponents r_j, ρ_j. Let

$$G = G_1 + \cdots + G_k \,,$$

and set

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx - G(u)$$

for $u \in H^1(\mathbb{R}^N)$. We will apply the results of Section 3.3 to establish the following result.

THEOREM 4.3.1. *If g is as above, then the initial-value problem* (4.1.1) *is locally well posed in $H^1(\mathbb{R}^N)$. Furthermore, there is conservation of charge and energy; i.e.,*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2} \,, \quad E(u(t)) = E(\varphi) \,,$$

for all $t \in (-T_{\min}, T_{\max})$, where u is the solution of (4.1.1) *with the initial value $\varphi \in H^1(\mathbb{R}^N)$.*

PROOF. By Theorem 3.3.9 we need only show uniqueness, which follows from Proposition 4.2.3. □

REMARK 4.3.2. Note that the only ingredient that we used for proving uniqueness (in Proposition 4.2.3) is Strichartz's estimate. In particular, it follows from Remark 2.7.7 that Theorem 4.3.1 still holds if one replaces $\mathbb{R}^N$ by $\mathbb{R}^N_+$, or by certain cones of $\mathbb{R}^N$.

We now give some applications of Theorem 4.3.1 to the nonlinearities introduced in Section 3.2.

COROLLARY 4.3.3. *Let $g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u$ be as in Example* 3.2.11. *Set*

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - G(u) \,,$$

where

$$G(u) = \int_{\mathbb{R}^N} \Big\{ \frac{1}{2} V(x)|u(x)|^2 + F(x, u(x)) + \frac{1}{4} (W \star |u|^2)(x)|u(x)|^2 \Big\} dx \,.$$

It follows that the initial-value problem (4.1.1) *is locally well posed in $H^1(\mathbb{R}^N)$. Moreover, there is conservation of charge and energy; i.e.,*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}, \,, \quad E(u(t)) = E(\varphi) \,,$$

for all $t \in (-T_{\min}, T_{\max})$, where u is the solution of (4.1.1) *with the initial value $\varphi \in H^1(\mathbb{R}^N)$.*

PROOF. Apply Theorem 4.3.1 (see Example 3.2.11). □

COROLLARY 4.3.4. *Let* $g(u) = \lambda|u|^\alpha u$ *with* $\lambda \in \mathbb{R}$ *and* $0 \le \alpha < \frac{4}{N-2}$ $(0 \le \alpha < \infty$ *if* $N = 1)$. *Set*

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - G(u)\,,$$

where

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{\lambda}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2}\, dx\,.$$

It follows that the initial-value problem (4.1.3) *is locally well posed in* $H^1(\mathbb{R}^N)$. *Moreover, there is conservation of charge and energy; i.e.,*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}, \quad E(u(t)) = E(\varphi)$$

for all $t \in (-T_{\min}, T_{\max})$, *where* u *is the solution of* (4.1.3) *with the initial value* $\varphi \in H^1(\mathbb{R}^N)$.

4.4. Kato's Method

If $g(u) = \lambda|u|^\alpha u$ with $\operatorname{Im}\lambda \neq 0$, then we may not apply Theorem 3.3.9 because g satisfies neither (3.3.5) nor (3.3.8). T. Kato [203] introduced a method, based on a fixed point argument and Strichartz's estimates, by which one can solve the problem (4.1.1) for g as above. Besides, that method provides a simple, direct proof of the local well-posedness result for a certain class of nonlinearities. We begin with a typical result based on the fixed point method (see [203, 204] and Theorem 4.4.6 below for more general results).

Let $f \in C(\mathbb{C}, \mathbb{C})$ satisfy

$$f(0) = 0 \tag{4.4.1}$$

and

$$|f(u) - f(v)| \le L(K)|u - v| \tag{4.4.2}$$

for all $u, v \in \mathbb{C}$ such that $|u|, |v| \le K$, with

$$\begin{cases} L(t) \in C([0,\infty)) & \text{if } N = 1 \\ L(t) \le C(1 + t^\alpha) \text{ with } 0 \le \alpha < \frac{4}{N-2} & \text{if } N \ge 2. \end{cases} \tag{4.4.3}$$

Set

$$g(u)(x) = f(u(x)) \tag{4.4.4}$$

for all measurable $u : \mathbb{R}^N \to \mathbb{C}$ and a.a. $x \in \mathbb{R}^N$.

THEOREM 4.4.1. *Let* $f \in C(\mathbb{C}, \mathbb{C})$ *satisfy* (4.4.1)–(4.4.3) *and let* g *be defined by* (4.4.4). *If* f *(considered as a function* $\mathbb{R}^2 \to \mathbb{R}^2$*) is of class* C^1, *then the initial-value problem* (4.1.1) *is locally well posed in* $H^1(\mathbb{R}^N)$.

REMARK 4.4.2. Since we assume neither (3.3.5) nor (3.3.8), we cannot expect conservation of charge and energy. If, in addition to the hypotheses of Theorem 4.4.1, we assume (3.3.5) (respectively, (3.3.8)), then there is conservation of energy (respectively, conservation of charge). See [203, 204] and Theorem 4.4.6 below.

PROOF OF THEOREM 4.4.1. We consider the case $N \geq 2$, the proof in the case $N = 1$ being easily adapted. Let $\theta \in C_c^\infty(\mathbb{C}, \mathbb{R})$ be such that $\theta(z) = 1$ for $|z| \leq 1$. Setting

$$f_1(u) = \theta(u) f(u),$$
$$f_2(u) = (1 - \theta(u)) f(u),$$

one easily verifies that

$$(4.4.5) \qquad \begin{aligned} |f_1(u) - f_1(v)| &\leq C|u - v|, \\ |f_2(u) - f_2(v)| &\leq C(|u|^\alpha + |v|^\alpha)|u - v|, \end{aligned}$$

where α is given by (4.4.3). Set $g_\ell(u)(x) = f_\ell(u(x))$ for $\ell = 1, 2$ and let

$$r = \alpha + 2.$$

Using (4.4.5), we deduce from Hölder's inequality that

$$(4.4.6) \qquad \begin{aligned} \|g_1(u) - g_1(v)\|_{L^2} &\leq C\|u - v\|_{L^2}, \\ \|g_2(u) - g_2(v)\|_{L^{r'}} &\leq C\big(\|u\|_{L^r}^\alpha + \|v\|_{L^r}^\alpha\big)\|u - v\|_{L^r}, \end{aligned}$$

and from Remark 1.3.1(vii) that

$$(4.4.7) \qquad \begin{aligned} \|\nabla g_1(u)\|_{L^2} &\leq C\|\nabla u\|_{L^2}, \\ \|\nabla g_2(u)\|_{L^{r'}} &\leq C\|u\|_{L^r}^\alpha \|\nabla u\|_{L^r}. \end{aligned}$$

We now proceed in three steps.

STEP 1. Local existence. Fix $M, T > 0$, to be chosen later, and let q be such that (q, r) is an admissible pair. Consider the set

$$(4.4.8) \qquad \begin{aligned} E = \big\{u \in L^\infty((-T, T), H^1(\mathbb{R}^N)) \cap L^q((-T, T), W^{1,r}(\mathbb{R}^N)); \\ \|u\|_{L^\infty((-T,T),H^1)} \leq M,\ \|u\|_{L^q((-T,T),W^{1,r})} \leq M\big\} \end{aligned}$$

equipped with the distance

$$(4.4.9) \qquad \mathrm{d}(u, v) = \|u - v\|_{L^q((-T,T),L^r)} + \|u - v\|_{L^\infty((-T,T),L^2)}.$$

We claim that (E, d) is a complete metric space. Indeed, we need only show that E is closed in $L^q((-T, T), L^r(\mathbb{R}^N))$. Consider $(u_n)_{n \geq 0} \subset E$ such that $u_n \to u$ in $L^q((-T, T), L^r(\mathbb{R}^N))$. In particular, there exists a subsequence, which we still denote by $(u_n)_{n \geq 0}$, such that $u_n(t) \to u(t)$ in $L^r(\mathbb{R}^N)$ for a.a. $t \in (-T, T)$. Applying Theorem 1.2.5 twice, we deduce that

$$u \in L^\infty((-T, T), H^1(\mathbb{R}^N)) \cap L^q((-T, T), W^{1,r}(\mathbb{R}^N))$$

and that

$$\|u\|_{L^\infty(-T,T),H^1)} \leq \liminf_{n \to \infty} \|u_n\|_{L^\infty(-T,T),H^1)} \leq M,$$
$$\|u\|_{L^q(-T,T),W^{1,r})} \leq \liminf_{n \to \infty} \|u_n\|_{L^q(-T,T),W^{1,r})} \leq M;$$

and so $u \in E$.

Consider now $u \in E$. Since g_1 is continuous $L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, it follows that $g_1(u) : (-T, T) \to L^2(\mathbb{R}^N)$ is measurable, and we deduce easily that $g_1(u) \in L^\infty((-T, T), L^2(\mathbb{R}^N))$. Similarly, since g_2 is continuous $L^r(\mathbb{R}^N) \to L^{r'}(\mathbb{R}^N)$, we see that $g_2(u) \in L^q((-T, T), L^{r'}(\mathbb{R}^N))$. Using inequalities (4.4.6) and (4.4.7) and Remark 1.2.2(iii), we deduce the following: $g_1(u) \in L^\infty((-T,T),H^1(\mathbb{R}^N))$, $g_2(u) \in L^q((-T,T),W^{1,r'}(\mathbb{R}^N))$,

$$\|g_1(u)\|_{L^\infty((-T,T),H^1)} \le C\|u\|_{L^\infty((-T,T),H^1)},$$
$$\|g_2(u)\|_{L^q((-T,T),W^{1,r'})} \le C\|u\|^\alpha_{L^\infty((-T,T),L^r)}\|u\|_{L^q((-T,T),W^{1,r'})},$$

and

$$\|g_1(u)-g_1(v)\|_{L^\infty((-T,T),L^2)} \le C\|u-v\|_{L^\infty((-T,T),L^2)},$$
$$\|g_2(u)-g_2(v)\|_{L^q((-T,T),L^{r'})} \le C\big(\|u\|^\alpha_{L^\infty((-T,T),L^r)} + \|v\|^\alpha_{L^\infty((-T,T),L^r)}\big)\|u-v\|_{L^q((-T,T),L^r)}.$$

Using the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ and Hölder's inequality in time, we deduce from the above estimates that

$$\|g_1(u)\|_{L^1((-T,T),H^1)}+\|g_2(u)\|_{L^{q'}((-T,T),W^{1,r'})} \le C\big(T+T^{\frac{q-q'}{qq'}}\big)(1+M^\alpha)M \tag{4.4.10}$$

and

$$\|g_1(u)-g_1(v)\|_{L^1((-T,T),L^2)} + \|g_2(u)-g_2(v)\|_{L^{q'}((-T,T),L^{r'})} \le C\big(T+T^{\frac{q-q'}{qq'}}\big)(1+M^\alpha)\mathrm{d}(u,v). \tag{4.4.11}$$

Given $\varphi \in H^1(\mathbb{R}^N)$ and $u \in E$, let $\mathcal{H}(u)$ be defined by

$$\mathcal{H}(u)(t) = \mathcal{T}(t)\varphi + i\int_0^t \mathcal{T}(t-s)g(u(s))ds. \tag{4.4.12}$$

It follows from (4.4.10) and Strichartz's estimates that

$$\mathcal{H}(u) \in C([-T,T],H^1(\mathbb{R}^N)) \cap L^q((-T,T),W^{1,r}(\mathbb{R}^N)), \tag{4.4.13}$$

and

$$\|\mathcal{H}(u)\|_{L^\infty((-T,T),H^1)} + \|\mathcal{H}(u)\|_{L^q((-T,T),W^{1,r})} \le C\|\varphi\|_{H^1} + C\big(T+T^{\frac{q-q'}{qq'}}\big)(1+M^\alpha)M. \tag{4.4.14}$$

Also, we deduce from (4.4.11) that

$$\|\mathcal{H}(u)-\mathcal{H}(v)\|_{L^\infty((-T,T),L^2)} + \|\mathcal{H}(u)-\mathcal{H}(v)\|_{L^q((-T,T),L^r)} \le C\big(T+T^{\frac{q-q'}{qq'}}\big)(1+M^\alpha)\mathrm{d}(u,v). \tag{4.4.15}$$

Finally, note that

$$\frac{q-q'}{qq'} = 1-\frac{2}{q} = N\frac{4-(N-2)\alpha}{2N(\alpha+2)} > 0.$$

We now proceed as follows. Given $\varphi \in H^1(\mathbb{R}^N)$, we set

$$M = \frac{1}{2C}\|\varphi\|_{H^1},$$

and we choose T small enough so that

$$C\big(T+T^{\frac{q-q'}{qq'}}\big)(1+M^\alpha) \le \frac{1}{2}.$$

Note that T depends on φ only through $\|\varphi\|_{H^1}$. It then follows from (4.4.14) and (4.4.15) that

$$\|\mathcal{H}(u)\|_{L^\infty((-T,T),H^1)} + \|\mathcal{H}(u)\|_{L^q((-T,T),W^{1,r})} \leq M\,;$$

i.e., $\mathcal{H}(u) \in E$ and

$$\mathrm{d}(\mathcal{H}(u), \mathcal{H}(v)) \leq \frac{1}{2}\mathrm{d}(u,v)\,.$$

In particular, $\mathcal{H}$ is a strict contraction on E. By Banach's fixed-point theorem, $\mathcal{H}$ has a unique fixed point $u \in E$; i.e., u satisfies (4.1.2). By (4.4.13), $\mathcal{H}(u) \in C([-T,T], H^1(\mathbb{R}^N))$, and so $u \in C([-T,T], H^1(\mathbb{R}^N))$. Applying Proposition 3.1.3, we deduce that u is a strong H^1-solution of (4.1.1) on $[-T,T]$.

STEP 2. Uniqueness and the blowup alternative. Uniqueness follows from Proposition 4.2.3. We then proceed as in the proof of Theorem 3.3.9: using uniqueness, we define the maximal solution; and since the solution u of Step 1 is constructed on an interval depending on $\|\varphi\|_{H^1}$, we deduce the blowup alternative.

STEP 3. Continuous dependence. Let $\varphi \in H^1(\mathbb{R}^N)$; consider $(\varphi_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$ such that $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$; and let u_n be the maximal solution of (4.1.1) corresponding to the initial value φ_n. We claim that there exists $T > 0$ depending on $\|\varphi\|_{H^1}$ such that u_n is defined on $[-T,T]$ for n large enough and $u_n \to u$ in $C([-T,T], H^1(\mathbb{R}^N))$ as $n \to \infty$. The result follows by iterating this property in order to cover any compact subset of $(-T_{\min}, T_{\max})$.

We now prove the claim. Since $\|\varphi_n\|_{H^1} \leq 2\|\varphi\|_{H^1}$ for n sufficiently large, we deduce from Step 1 that there exists $T = T(\|\varphi\|_{H^1})$ such that u and u_n are defined on $[-T,T]$ for $n \geq n_0$ and

$$\|u\|_{L^\infty((-T,T),H^1)} + \sup_{n\geq n_0} \|u_n\|_{L^\infty((-T,T),H^1)} \leq C\|\varphi\|_{H^1}\,. \tag{4.4.16}$$

Note that $u_n(t) - u(t) = \mathcal{T}(t)(\varphi_n - \varphi) + \mathcal{H}(u_n)(t) - \mathcal{H}(u)(t)$. Therefore, applying (4.4.15) we obtain

$$\begin{aligned} &\|u_n - u\|_{L^\infty((-T,T),L^2)} + \|u_n - u\|_{L^q((-T,T),L^r)} \\ &\quad \leq C\|\varphi_n - \varphi\|_{H^1} \\ &\qquad + C\big(T + T^{\frac{q-q'}{qq'}}\big)\big(\|u_n - u\|_{L^\infty((-T,T),L^2)} + \|u_n - u\|_{L^q((-T,T),L^r)}\big)\,, \end{aligned}$$

where C depends on $\|\varphi\|_{H^1}$. By choosing T possibly smaller, but still depending on $\|\varphi\|_{H^1}$, we may assume that $C(T + T^{\frac{q-q'}{qq'}}) \leq 1/2$ and we conclude that

$$\|u_n - u\|_{L^\infty((-T,T),L^2)} + \|u_n - u\|_{L^q((-T,T),L^r)} \leq 2C\|\varphi_n - \varphi\|_{H^1}\,. \tag{4.4.17}$$

Note that ∇ commutes with $\mathcal{T}(t)$, and so

$$\nabla u(t) = \mathcal{T}(t)\nabla\varphi + i\int_0^t \mathcal{T}(t-s)\nabla g(u)ds\,.$$

A similar identity holds for u_n. We now use the assumption $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, which implies that $\nabla g(u) = f'(u)\nabla u$. Therefore, we may write

$$\nabla(u_n - u)(t) = \mathcal{T}(t)\nabla(u_n - u) + i\int_0^t \mathcal{T}(t-s)f'(u_n)\nabla(u_n - u)ds$$
$$+ i\int_0^t \mathcal{T}(t-s)(f'(u_n) - f'(u))\nabla u\, ds.$$

Note that f_1 and f_2 are also C^1, so that $f' = f_1' + f_2'$. Therefore, arguing as in Step 1 and using (4.4.16), we obtain the estimate

$$\begin{aligned}
&\|\nabla(u_n - u)\|_{L^\infty((-T,T),L^2)} + \|\nabla(u_n - u)\|_{L^q((-T,T),L^r)} \\
&\quad \le C\big[\|\varphi_n - \varphi\|_{H^1} \\
&\qquad + \big(T + T^{\frac{q-q'}{qq'}}\big)\big(\|\nabla(u_n - u)\|_{L^\infty((-T,T),L^2)} + \|\nabla(u_n - u)\|_{L^q((-T,T),L^r)}\big) \\
&\qquad + \big\|(f_1'(u_n) - f_1'(u))\nabla u\big\|_{L^1((-T,T),L^2)} \\
&\qquad + \big\|(f_2'(u_n) - f_2'(u))\nabla u\big\|_{L^{q'}((-T,T),L^{r'})}\big],
\end{aligned}$$

where C depends on $\|\varphi\|_{H^1}$. By choosing T possibly smaller, but still depending on $\|\varphi\|_{H^1}$, we deduce that

$$\begin{aligned}
&\|\nabla(u_n - u)\|_{L^\infty((-T,T),L^2)} + \|\nabla(u_n - u)\|_{L^q((-T,T),L^r)} \\
&\quad \le C\big[\|\varphi_n - \varphi\|_{H^1} + \|(f_1'(u_n) - f_1'(u))\nabla u\|_{L^1((-T,T),L^2)} \\
&\qquad + \|(f_2'(u_n) - f_2'(u))\nabla u\|_{L^{q'}((-T,T),L^{r'})}\big].
\end{aligned}$$

Therefore, if we show that

$$\begin{aligned}
&\|(f_1'(u_n) - f_1'(u))\nabla u\|_{L^1((-T,T),L^2)} \\
&\qquad + \|(f_2'(u_n) - f_2'(u))\nabla u\|_{L^{q'}((-T,T),L^{r'})} \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned} \tag{4.4.18}$$

we obtain that

$$\|\nabla(u_n - u)\|_{L^\infty((-T,T),L^2)} + \|\nabla(u_n - u)\|_{L^q((-T,T),L^r)} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.4.19}$$

which, combined with (4.4.17), yields the desired convergence. We prove (4.4.18) by contradiction, and we assume that there exist $\varepsilon > 0$, and a subsequence, which we still denote by $(u_n)_{n\ge 0}$ such that

$$\begin{aligned}
&\|(f_1'(u_n) - f_1'(u))\nabla u\|_{L^1((-T,T),L^2)} \\
&\qquad + \|(f_2'(u_n) - f_2'(u))\nabla u\|_{L^{q'}((-T,T),L^{r'})} \ge \varepsilon.
\end{aligned} \tag{4.4.20}$$

By using (4.4.17) and by possibly extracting a subsequence, we may assume that $u_n \to u$ a.e. on $(-T,T)\times\mathbb{R}^N$ and that there exists $w \in L^q((-T,T), L^r(\mathbb{R}^N))$ such that $|u_n| \le w$ a.e. on $(-T,T)\times\mathbb{R}^N$. In particular, $(f_1'(u_n) - f_1'(u))\nabla u$ and $(f_2'(u_n) - f_2'(u))\nabla u$ both converge to 0 a.e. on $(-T,T)\times\mathbb{R}^N$. Since

$$|(f_1'(u_n) - f_1'(u))\nabla u| \le C|\nabla u| \in L^1((-T,T), L^2(\mathbb{R}^N)),$$

and

$$\begin{aligned}
|(f_2'(u_n) - f_2'(u))\nabla u| &\le C\big(|u_n|^\alpha + |u|^\alpha\big)|\nabla u| \\
&\le C(|w|^\alpha + |u|^\alpha)|\nabla u| \in L^{q'}((-T,T), L^{r'}(\mathbb{R}^N)),
\end{aligned}$$

we obtain from the dominated convergence a contradiction with (4.4.20). □

REMARK 4.4.3. It follows follows easily from (4.4.10) and Strichartz's estimates that

$$u \in L^{\gamma}_{\mathrm{loc}}((-T_{\min}, T_{\max}), W^{1,\rho}(\mathbb{R}^N))$$

for every admissible pair (γ, ρ).

REMARK 4.4.4. (4.4.19) and (4.4.17) imply that the solution depends continuously on the initial value not only in $C(I, H^1(\mathbb{R}^N))$ but also in $L^q(I, W^{1,r}(\mathbb{R}^N))$ (and more generally in $L^{\gamma}(I, W^{1,\rho}(\mathbb{R}^N))$, where (γ, ρ) is any admissible pair).

REMARK 4.4.5. Let $f \in C(\mathbb{C}, \mathbb{C})$ satisfy (4.4.1)–(4.4.3) (i.e., f is as in Theorem 4.4.1 except that we do not assume that f is C^1). Since we used the C^1 assumption only at the end of Step 3 of the proof of Theorem 4.4.1, there is local existence and the blowup alternative. The full statement of continuous dependence might fail. However, there is a weaker form of continuous dependence. First, with the notation of Step 3, u_n is defined on an interval $[-T, T]$ for n large, with T depending on $\|\varphi\|_{H^1}$. Also, we still have estimates (4.4.16) and (4.4.17). Using Gagliardo-Nirenberg's inequality and a covering argument, we deduce that $u_n \to u$ in $C([-T, T], L^p(\mathbb{R}^N))$ for all $2 \le p < 2N/(N-2)$.

The method of proof of Theorem 4.4.1 can be applied to more general nonlinearities. More precisely, let $g \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$ and suppose that there exists $2 \le r, \rho < 2N/(N-2)$ ($2 \le r, \rho < \infty$ if $N = 1$) such that

$$\|g(u) - g(v)\|_{L^{\rho'}} \le C(M)\|u - v\|_{L^r} \tag{4.4.21}$$

for all $u, v \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1}, \|v\|_{H^1} \le M$. Suppose further that

$$\|g(u)\|_{W^{1,\rho'}} \le C(M)(1 + \|u\|_{W^{1,r}}) \tag{4.4.22}$$

for all $u \in H^1(\mathbb{R}^N) \cap W^{1,r}(\mathbb{R}^N)$ such that $\|u\|_{H^1} \le M$.

THEOREM 4.4.6. *Let $g = g_1 + \cdots + g_k$, where each of the g_j's satisfies* (4.4.21)–(4.4.22) *for some exponents r_j, ρ_j. For every $\varphi \in H^1(\mathbb{R}^N)$, there exists a unique, strong H^1-solution u of* (4.1.1), *defined on a maximal time interval $(-T_{\min}, T_{\max})$. Moreover,*

$$u \in L^{a}_{\mathrm{loc}}((-T_{\min}, T_{\max}), W^{1,b}(\mathbb{R}^N))$$

for every admissible pair (a, b). In addition, the following properties hold:

(i) *There is the blowup alternative; i.e., $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_{\max}$ if $T_{\max} < \infty$ and as $t \downarrow -T_{\min}$ if $T_{\min} < \infty$.*

(ii) *u depends continuously on φ in the following sense: There exists $T > 0$ depending on $\|\varphi\|_{H^1}$ such that if $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ and if u_n is the corresponding solution of* (4.1.1), *then u_n is defined on $[-T, T]$ for n large enough and $u_n \to u$ in $C([-T, T], L^p(\mathbb{R}^N))$ for all $2 \le p < 2N/(N-2)$.*

(iii) *If $(g(w), iw)_{H^{-1}, H^1} = 0$ for all $w \in H^1(\mathbb{R}^N)$, then there is conservation of charge; i.e., $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in (-T_{\min}, T_{\max})$.*

(iv) *If each of the g_j's satisfies* (3.3.5), *then there is conservation of energy; i.e., $E(u(t)) = E(\varphi)$ for all $t \in (-T_{\min}, T_{\max})$, where E is defined by* (3.3.9) *with $G = G_1 + \cdots + G_k$.*

PROOF. We set

$$r = \max\{r_1, \dots, r_k, \rho_1, \dots, \rho_k\},$$

and we consider the corresponding admissible pair (q, r). Given $M, T > 0$ to be chosen later, we consider the complete metric space (E, d) defined by (4.4.8)–(4.4.9). We now proceed in three steps.

STEP 1. Existence, uniqueness, regularity, the blowup alternative, and continuous dependence. We first claim that if $\varphi \in H^1(\mathbb{R}^N)$, then the mapping $\mathcal{H}$ defined by (4.4.12) is a strict contraction on E for appropriate choices of M and T.

Given $1 \le j \le k$, we consider q_j, γ_j such that (q_j, r_j) and (γ_j, ρ_j) are admissible pairs. It follows from Hölder's inequality that

$$\|w\|_{W^{1,r_j}} \le \|w\|_{H^1}^{\frac{2(r-r_j)}{r_j(r-2)}} \|w\|_{W^{1,r}}^{\frac{r(r_j-2)}{r_j(r-2)}},$$

so that

$$\|w\|_{L^{q_j}((-T,T),W^{1,r_j})} \le \|w\|_{L^\infty((-T,T),H^1)}^{\frac{2(r-r_j)}{r_j(r-2)}} \|w\|_{L^q((-T,T),W^{1,r})}^{\frac{r(r_j-2)}{r_j(r-2)}}.$$

In particular, if $u \in E$, then $u \in L^{q_j}((-T,T), W^{1,r_j}(\mathbb{R}^N))$ for all $1 \le j \le k$ and

$$\|u\|_{L^{q_j}((-T,T),W^{1,r_j})} \le M^{\frac{2(r-r_j)}{r_j(r-2)}} M^{\frac{r(r_j-2)}{r_j(r-2)}} = M. \tag{4.4.23}$$

Next, it follows from (4.4.21)–(4.4.22) that g_j is continuous $H^1(\mathbb{R}^N) \to L^{\rho_j'}(\mathbb{R}^N)$. We deduce that if $u \in E$, then $g_j(u) : (-T,T) \to L^{\rho_j'}(\mathbb{R}^N)$ is measurable, and it follows easily that $g_j(u) \in L^\infty((-T,T), L^{\rho_j'}(\mathbb{R}^N))$. Applying Remark 1.2.2(iii) and (4.4.22)–(4.4.23), we conclude that $g_j(u) \in L^{q_j}((-T,T), W^{1,\rho_j'}(\mathbb{R}^N))$ and

$$\|g_j(u)\|_{L^{q_j}((-T,T),W^{1,\rho_j'})} \le C_M\big(T^{\frac{1}{q_j}} + \|u\|_{L^{q_j}((-T,T),W^{1,r_j})}\big) \le C_M\big(T^{\frac{1}{q_j}} + M\big),$$

where C_M depends on M. It follows that

$$\|g_j(u)\|_{L^{\gamma_j'}((-T,T),W^{1,\rho_j'})} \le C_M\big(T^{\frac{1}{q_j}} + M\big) T^{\frac{q_j-\gamma_j'}{q_j\gamma_j'}}. \tag{4.4.24}$$

Applying now Strichartz's inequalities, we deduce from (4.4.24) that if $T \le 1$, then

$$\mathcal{H}(u) \in L^q((-T,T), W^{1,r}(\mathbb{R}^N)) \cap C([-T,T], H^1(\mathbb{R}^N))$$

and

$$\|\mathcal{H}(u)\|_{L^q((-T,T),W^{1,r})} + \|\mathcal{H}(u)\|_{L^\infty((-T,T),H^1)} \le K\|\varphi\|_{H^1} + KC_M(1+M)T^\sigma,$$

where

$$\sigma = \min_{1\le j\le k} \frac{q_j - \gamma_j'}{q_j\gamma_j'} > 0.$$

We now choose M, T so that $M \ge 2K\|\varphi\|_{H^1}$ and $KC_M(1+M)T^\sigma \le M$ and we see that $\mathcal{H}(u) \in E$ for all $u \in E$. (Note that T depends on φ through $\|\varphi\|_{H^1}$.) Applying now (4.4.21), it is not difficult to show by similar estimates that, by possibly choosing T smaller (but still depending on $\|\varphi\|_{H^1}$),

$$\mathrm{d}(\mathcal{H}(u), \mathcal{H}(v)) \le \frac{1}{2}\mathrm{d}(u, v) \tag{4.4.25}$$

for all $u, v \in E$. So $\mathcal{H}$ has a fixed point $u \in E$. This proves the existence part. Uniqueness follows from Proposition 4.2.3. The $L^a_{\text{loc}}((-T_{\min}, T_{\max}), W^{1,b}(\mathbb{R}^N))$ regularity follows from (4.4.24) and Strichartz's estimates. The blowup alternative is proved as in Theorem 4.4.1 and continuous dependence as in Remark 4.4.5.

STEP 2. Property (iii). Since equation (4.1.1) makes sense in $H^{-1}(\mathbb{R}^N)$ for a.a. $t \in (-T_{\min}, T_{\max})$, we may multiply it (in the H^{-1}-H^1 duality) by iu, and we obtain

$$(u_t, u)_{H^{-1},H^1} = (-\Delta u, iu)_{H^{-1},H^1} + (g(u), iu)_{H^{-1},H^1} = 0\,.$$

Since $u \in L^\infty_{\text{loc}}((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N))$ and $u_t \in L^\infty_{\text{loc}}((-T_{\min}, T_{\max}), H^{-1}(\mathbb{R}^N))$, we deduce that

$$\frac{d}{dt}\|u(t)\|^2_{L^2} = 2(u_t, u)_{H^{-1},H^1} = 0\,,$$

and the result follows.

STEP 3. Property (iv). We first assume that $\varphi \in H^2(\mathbb{R}^N)$. Given $\varepsilon > 0$, we set $I_\varepsilon = (I - \varepsilon\Delta)^{-2}$. The reader is referred to Propositions 1.5.2 and 1.5.3 for all the relevant properties of I_ε. We define

$$g_{j,\varepsilon}(w) = I_\varepsilon g_j(I_\varepsilon w)$$

for $1 \le j \le k$ and $w \in H^1(\mathbb{R}^N)$, and we set

$$g_\varepsilon = \sum_{j=1}^{k} g_{j,\varepsilon} \quad \text{and} \quad G_\varepsilon(w) = G(I_\varepsilon w)\,.$$

We observe that the $g_{j,\varepsilon}$'s satisfy the same estimates as the g_j's, uniformly in $\varepsilon > 0$ and that

$$g_\varepsilon = G'_\varepsilon\,. \tag{4.4.26}$$

We denote by u_ε the solutions of (4.1.1) with g replaced by g_ε. It follows from the estimates of Step 1 that there exists $T = T(\|\varphi\|_{H^1})$ such that u_ε is defined on $[-T, T]$ and

$$\sup_{\substack{-T\le t\le T\\ \varepsilon>0}} \|u_\varepsilon(t)\|_{H^1} \le M = M(\|\varphi\|_{H^1})\,. \tag{4.4.27}$$

Since u_ε is continuous $[-T, T] \to H^1(\mathbb{R}^N)$ so is $I_\varepsilon u_\varepsilon$. Therefore, $g(I_\varepsilon u_\varepsilon)$ is continuous $[-T, T] \to H^{-1}(\mathbb{R}^N)$, thus $g_\varepsilon(u_\varepsilon)$ is continuous $[-T, T] \to H^2(\mathbb{R}^N)$. Since $\varphi \in H^2(\mathbb{R}^N)$, we deduce that

$$u_\varepsilon \in C([-T, T], H^2(\mathbb{R}^N)) \cap C^1([-T, T], L^2(\mathbb{R}^N))\,.$$

Therefore, we may take the L^2 scalar product of the equation with

$$\partial_t u_\varepsilon \in C([-T, T], L^2(\mathbb{R}^N))$$

and obtain

$$\langle i\partial_t u_\varepsilon, i\partial_t u_\varepsilon\rangle_{L^2} + \langle \Delta u_\varepsilon, \partial_t u_\varepsilon\rangle_{L^2} + \langle g_\varepsilon(u_\varepsilon), \partial_t u_\varepsilon\rangle_{L^2} = 0\,.$$

Using (4.4.26), we deduce that

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 - G_\varepsilon(u_\varepsilon)\right\} = 0\,,$$

so that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 - G_\varepsilon(u_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \varphi|^2 - G_\varepsilon(\varphi) \tag{4.4.28}$$

for all $t \in [-T, T]$. We claim that, after possibly choosing T smaller (but still depending on $\|\varphi\|_{H^1}$),

$$u_\varepsilon \underset{\varepsilon\downarrow 0}{\longrightarrow} u \tag{4.4.29}$$

in $C([-T,T], L^p(\mathbb{R}^N))$ for all $2 \le p < 2N/(N-2)$. Indeed for every j we write

$$g_{j,\varepsilon}(u_\varepsilon) - g_j(u) = g_{j,\varepsilon}(u_\varepsilon) - g_{j,\varepsilon}(u) + I_\varepsilon(g_j(I_\varepsilon u) - g_j(u)) + (I_\varepsilon - I)g_j(u),$$

and we deduce that, with the notation of Step 1,

$$\begin{aligned}\|g_{j,\varepsilon}(u_\varepsilon) - g_j(u)\|_{L^{\gamma_j'}((-T,T),L^{\rho_j'})} &\le \|g_{j,\varepsilon}(u_\varepsilon) - g_{j,\varepsilon}(u)\|_{L^{\gamma_j'}((-T,T),L^{\rho_j'})} \\ &\quad + \|g_j(I_\varepsilon u) - g_j(u)\|_{L^{\gamma_j'}((-T,T),L^{\rho_j'})} + \|(I_\varepsilon - I)g_j(u)\|_{L^{\gamma_j'}((-T,T),L^{\rho_j'})}.\end{aligned}$$

Since $g_j(u) \in L^{\gamma_j'}((-T,T), L^{\rho_j'}(\mathbb{R}^N))$, we have

$$\|(I_\varepsilon - I)g_j(u)\|_{L^{\gamma_j'}((-T,T),L^{\rho_j'})} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0.$$

Next, since $I_\varepsilon u \to u$ in $C([-T,T], H^1(\mathbb{R}^N))$, we deduce from (4.4.21) that

$$\|g_j(I_\varepsilon u) - g_j(u)\|_{L^{\gamma_j'}((-T,T),L^{\rho_j'})} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0.$$

We also deduce from (4.4.21) applied to $g_{j,\varepsilon}$ and (4.4.27) that (see the estimates of Step 1)

$$\|g_{j,\varepsilon}(u_\varepsilon) - g_{j,\varepsilon}(u)\|_{L^{\gamma_j'}((-T,T),L^{\rho_j'})} \le CT^\sigma \|u_\varepsilon - u\|_{L^q((-T,T),L^r)}.$$

Using the above estimates and Strichartz's inequalities, we conclude that

$$\|u_\varepsilon - u\|_{L^\infty((-T,T),L^2)} + \|u_\varepsilon - u\|_{L^q((-T,T),L^r)} \le a_\varepsilon + CT^\sigma \|u_\varepsilon - u\|_{L^q((-T,T),L^r)}$$

with $a_\varepsilon \to 0$ as $\varepsilon \downarrow 0$. By choosing T sufficiently small, we deduce that

$$\|u_\varepsilon - u\|_{L^\infty((-T,T),L^2)} \to 0 \quad \text{as } \varepsilon \downarrow 0,$$

and (4.4.29) follows by applying (4.4.27) and Gagliardo-Nirenberg's inequality.

Next, we deduce easily from (4.4.21)–(4.4.22) that (see the proof of (3.3.14))

$$|G(u) - G(v)| \le C(M)(\|u - v\|_{L^2} + \|u - v\|_{L^r}) \tag{4.4.30}$$

for all $u, v \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1}, \|v\|_{H^1} \le M$. In particular,

$$|G_\varepsilon(\varphi) - G(\varphi)| \le C(\|I_\varepsilon\varphi - \varphi\|_{L^2} + \|I_\varepsilon\varphi - \varphi\|_{L^r}) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0. \tag{4.4.31}$$

Similarly, one shows using (4.4.29) and (4.4.30) that

$$|G_\varepsilon(u_\varepsilon) - G(u)| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0 \tag{4.4.32}$$

for all $-T \leq t \leq T$. We now let $\varepsilon \downarrow 0$ in (4.4.28). Since

$$\|\nabla u(t)\|_{L^2} \leq \liminf_{\varepsilon \downarrow 0} \|\nabla u_n(t)\|_{L^2},$$

we deduce by using (4.4.31) and (4.4.32) that

$$E(u(t)) \leq E(\varphi) \tag{4.4.33}$$

for all $-T \leq t \leq T$.

We now consider $\varphi \in H^1(\mathbb{R}^N)$. We approximate φ in $H^1(\mathbb{R}^N)$ by a sequence $(\varphi_n)_{n\geq 1} \subset H^2(\mathbb{R}^N)$, and we denote by u_n the corresponding solutions of (4.1.1). We note that u_n satisfies (4.4.33). Letting $n \to \infty$, using continuous dependence (property (ii)) and the argument just above, we deduce that u satisfies (4.4.33). This means that $E(u(t))$ has a local maximum at $t = 0$. The same property applied after replacing φ by $u(t_0)$, where $t_0 \in (-T_{\min}, T_{\max})$ is arbitrary, implies that $E(u(t))$ has a local maximum at $t = t_0$. Since $E(u(t))$ is a continuous function of t, it must be constant. This completes the proof. □

COROLLARY 4.4.7. *Let g be as in Theorem* 4.4.6. *If each of the g_j's satisfies* (3.3.5), *then the initial-value problem* (4.1.1) *is locally well posed in* $H^1(\mathbb{R}^N)$.

PROOF. By Theorem 4.4.6, we need only prove the continuous dependence; i.e., that if $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ and if u_n and u are the corresponding solutions of (4.1.1), then for every interval $[-S,T] \subset (-T_{\min}(\varphi), T_{\max}(\varphi))$, $u_n \to u$ in $C([-S,T], H^1(\mathbb{R}^N))$. We claim that there exists $T > 0$ depending on $\|\varphi\|_{H^1}$ such that u_n is defined on $[-T,T]$ for n large enough and $u_n \to u$ in $C([-T,T], H^1(\mathbb{R}^N))$ as $n \to \infty$. The result follows by iterating this property in order to cover any compact subset of $(-T_{\min}, T_{\max})$. We now prove the claim. By Theorem 4.4.6(ii), we know that there exists $T > 0$ depending on $\|\varphi\|_{H^1}$ such that u_n is defined on $[-T,T]$ for n large enough and $u_n \to u$ in $C([-T,T], L^p(\mathbb{R}^N))$ for all $2 \leq p < 2N/(N-2)$. It follows that (see (4.4.27)) $G(u_n) \to G(u)$ in $C([-T,T])$. By conservation of energy, $\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}$ in $C([-T,T])$, so that (see Proposition 1.3.14) $\nabla u_n \to \nabla u$ in $C([-T,T], L^2(\mathbb{R}^N))$. This completes the proof. □

REMARK 4.4.8. We may apply Theorem 4.4.6 to the case

$$g(u) = Vu + f(u(\cdot)) + (W \star |u|^2)u,$$

where $V, \nabla V \in L^\delta(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\delta \geq 1, \delta > N/2$, f is as in Theorem 4.4.1 (for example, $f(z) = \lambda |z|^\alpha z$ with $\lambda \in \mathbb{C}$ and $(N-2)\alpha < 4$), and $W \in L^\sigma(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\sigma \geq 1$, $\sigma > N/4$. This follows easily from the estimates of Section 3.2. Note that in this case, even though the assumptions of Corollary 4.4.7 are possibly not satisfied, the initial-value problem (4.1.1) is, however, locally well-posed in $H^1(\mathbb{R}^N)$. We need only prove the continuous dependence, and this follows from the argument used in Step 3 of the proof of Theorem 4.4.1. The term $\nabla[V(u_n - u) + (W \star |u_n|^2)u_n - (W \star |u|^2)u]$ is easily estimated by using the formula

$$\nabla[Vu + (W \star |u|^2)u] = \\ V\nabla u + \nabla V u + (W \star |u|^2)\nabla u + (W \star u\nabla\overline{u})u + (W \star \nabla u\overline{u})u$$

together with Hölder and Young's inequalities. Note, in addition, that there is conservation of charge provided V and W are real valued and $\operatorname{Im}(f(z)\overline{z}) = 0$ for all $z \in \mathbb{C}$. Moreover, there is conservation of energy provided V and W are real valued, W is even, and $f(z) = z\theta(|z|)/|z|$ for all $z \neq 0$ with $\theta : (0,\infty) \to \mathbb{R}$.

4.5. A Critical Case in $H^1(\mathbb{R}^N)$

In this section we assume $N \geq 3$. If we consider the model case $g(u) = \lambda|u|^\alpha u$ with $\lambda \in \mathbb{R}$ and $\alpha > 0$, then it follows from Corollary 4.3.4 that the initial-value problem (4.1.3) is locally well posed in $H^1(\mathbb{R}^N)$ if $\alpha < 4/(N-2)$. If $\alpha > 4/(N-2)$, then g does not map $H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$, so we may consider the problem out of the reach of our method. (See Section 9.4 for some partial results in that case.) In the limiting case $\alpha = 4/(N-2)$, $g \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$, the energy is well defined on $H^1(\mathbb{R}^N)$, and the various notions of H^1-solutions make sense. On the other hand, the methods we presented do not apply at several steps. However, since this is a borderline case, we may think that an appropriate refinement of the method will yield some local well-posedness result. This is indeed the case, and below is such a result. (See Cazenave and Weissler [69].)

THEOREM 4.5.1. *Assume $N \geq 3$. Let $g(u) = \lambda|u|^{\frac{4}{N-2}}u$ with $\lambda \in \mathbb{R}$. For every $\varphi \in H^1(\mathbb{R}^N)$, there exists a unique strong H^1-solution u of* (4.1.3) *defined on the maximal interval $(-T_{\min}, T_{\max})$ with $0 < T_{\max}, T_{\min} \leq \infty$. Moreover, the following properties hold:*

(i) *There is conservation of charge and energy.*

(ii) *$u \in L^q_{\text{loc}}(-T_{\min}, T_{\max}), W^{1,p}(\mathbb{R}^N))$ for every admissible pair (q, r).*

(iii) *If $T_{\max} < \infty$ (respectively, $T_{\min} < \infty$), then $\|\nabla u\|_{L^q((0,T_{\max}),L^r)} = +\infty$ (respectively, $\|\nabla u\|_{L^q((-T_{\min},0),L^r)} = +\infty$) for every admissible pair (q,r) with $2 < r < N$.*

(iv) *u depends continuously on φ as follows. The functions $T_{\max}, T_{\min}$ are lower semicontinuous $H^1(\mathbb{R}^N) \to (0,\infty]$. Moreover, if $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ and if u_n is the maximal solution of* (4.1.3) *with the initial value φ_n, then $u_n \to u$ in $L^p((-S,T), H^1(\mathbb{R}^N))$ for every $p < \infty$ and every interval $[-S,T] \subset (-T_{\min}, T_{\max})$.*

REMARK 4.5.2. Here are some comments on Theorem 4.5.1.

(i) We do not know whether there is uniqueness in the sense of Definition 3.1.4, i.e., uniqueness of weak H^1-solutions. In our proof of uniqueness, it is essential that we consider strong H^1-solutions.

(ii) We do not know whether the usual blowup alternative holds (i.e., the blowup of $\|u(t)\|_{H^1}$). In particular, we cannot deduce global existence results from the a priori estimates of $\|u(t)\|_{H^1}$ that follow from the conservation laws when $\lambda < 0$.

(iii) The statement of continuous dependence is weaker than usual, since $u_n \to u$ in $L^p((-S,T), H^1(\mathbb{R}^N))$ for every $p < \infty$, but possibly not for $p = \infty$. In the case $\lambda < 0$, then there is also convergence for $p = \infty$; see Remark 4.5.4(iii).

There are at least two methods for proving the existence part in Theorem 4.5.1. One can use a variation of Kato's method. This provides a simple proof, but it is then delicate to establish the conservation of energy. Instead, one can truncate the nonlinearity g and obtain solutions of the truncated problem for which there is conservation of energy. Next, one uses the Strichartz estimates to pass to the limit. This is the method we follow here.

We begin by introducing the truncated problem. Given $n \in \mathbb{N}$, let

$$g_n(u) = \begin{cases} g(u) & \text{if } |u| \le n \\ \lambda n^{\frac{4}{N-2}} u & \text{if } |u| \ge n. \end{cases} \tag{4.5.1}$$

In particular, $g_n : \mathbb{C} \to \mathbb{C}$ is globally Lipschitz. Set

$$G_n(s) = \int_0^s g_n(\sigma) d\sigma\,, \tag{4.5.2}$$

so that $|G_n(s)| \le C_n s^2$ for all $s \ge 0$, and let $E_n \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ be defined by

$$E_n(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G_n(u) \tag{4.5.3}$$

for all $u \in H^1(\mathbb{R}^N)$. Given $T > 0$ and $u : [0,T] \to H^1(\mathbb{R}^N)$, define $\mathcal{H}_n(u)$ by

$$\mathcal{H}_n(u)(t) = \mathcal{T}(t)\varphi + i \int_0^t \mathcal{T}(t-s) g_n(u(s)) ds \tag{4.5.4}$$

for $0 \le t \le T$. $\mathcal{H}$ is defined similarly, by replacing g_n by g in (4.5.4). Finally, let

$$\rho = \frac{2N^2}{N^2 - 2N + 4}, \quad \gamma = \frac{2N}{N-2}, \tag{4.5.5}$$

so that (γ, ρ) is an admissible pair. We will use the following lemma.

LEMMA 4.5.3. *If (q,r) is any admissible pair, then*

$$\begin{aligned} &\|\mathcal{H}_n(u) - \mathcal{H}_n(v)\|_{L^q((0,T),L^r)} \le \\ &\quad C\big(\|\nabla u\|_{L^\gamma((0,T),L^\rho)} + \|\nabla v\|_{L^\gamma((0,T),L^\rho)}\big)^{\frac{4}{N-2}} \|u - v\|_{L^\gamma((0,T),L^\rho)}\,, \end{aligned} \tag{4.5.6)}$$

$$\|\nabla \mathcal{H}_n(u)\|_{L^q((0,T),L^r)} \le C\|\mathcal{T}(\cdot)\nabla\varphi\|_{L^q((0,T),L^r)} + C\|\nabla u\|_{L^\gamma((0,T),L^\rho)}^{\frac{N+2}{N-2}}\,, \tag{4.5.7}$$

$$\begin{aligned} &\|\mathcal{H}_n(u) - \mathcal{H}(u)\|_{L^q((0,T),L^r)} \le \\ &\quad C T^{\frac{N-2}{2N}} n^{-\frac{4}{N(N-2)}} \|\nabla u\|_{L^\gamma((0,T),L^\rho)}^{\frac{4}{N-2}} \|u\|_{L^\infty((0,T),H^1)}^{\frac{N^2-2N+4}{N(N-2)}}\,, \end{aligned} \tag{4.5.8}$$

for some constant C independent of n, T, and φ.

PROOF. It is clear that $|g_n(u) - g_n(v)| \le C(|u|^{\frac{4}{N-2}} + |v|^{\frac{4}{N-2}})|u - v|$ for some constant C independent of n. Therefore,

$$\begin{aligned} &\|g_n(u) - g_n(v)\|_{L^{\gamma'}((0,T),L^{\rho'})} \\ &\qquad \le C\Big(\|u\|_{L^\gamma((0,T),L^{\frac{N\gamma}{N-2}})} + \|v\|_{L^\gamma((0,T),L^{\frac{N\gamma}{N-2}})}\Big)^{\frac{4}{N-2}} \|u - v\|_{L^\gamma((0,T),L^\rho)}\,, \end{aligned}$$

by Hölder's inequality in space, then in time. Since $\|w\|_{L^{\frac{N\gamma}{N-2}}} \le C\|\nabla w\|_{L^\rho}$ by Sobolev's inequality, (4.5.6) follows by applying Strichartz's estimate. (4.5.7) is proved similarly, by using the inequality $|\nabla g_n(u)| \le C|u|^{\frac{4}{N-2}}|\nabla u|$. Next,

$$|g_n(u) - g(u)| \le C|u|^{\frac{4}{N-2}} |1_{\{|u|>n\}} u|\,.$$

We deduce that

$$(4.5.9)\qquad \|g_n(u)-g(u)\|_{L^{\gamma'}((0,T),L^{\rho'})} \le C\|\nabla u\|^{\frac{4}{N-2}}_{L^\gamma((0,T),L^\rho)}\|1_{\{|u|>n\}}u\|_{L^\gamma((0,T),L^\rho)}\,.$$

Finally,

$$(4.5.10)\qquad \|1_{\{|u|>n\}}u\|_{L^\rho} \le n^{-\frac{4}{N(N-2)}}\|u\|^{\frac{N^2-2N+4}{N(N-2)}}_{L^{\frac{2N}{N-2}}} \le Cn^{-\frac{4}{N(N-2)}}\|u\|^{\frac{N^2-2N+4}{N(N-2)}}_{H^1}\,.$$

(4.5.8) follows from (4.5.9), (4.5.10), and Hölder's inequality in time. □

PROOF OF THEOREM 4.5.1. We consider only positive times, the problem for $t<0$ being treated by the same method. We proceed in six steps.

STEP 1. Uniqueness. This follows from Proposition 4.2.5.

STEP 2. Approximate solutions. Since g_n defined by (4.5.1) is globally Lipschitz $\mathbb{C}\to\mathbb{C}$, there exists a unique, global solution $u_n\in C([0,\infty),H^1(\mathbb{R}^N))$ of the problem

$$(4.5.11)\qquad u_n(t)=\mathcal{H}_n(u)(t)$$

for all $t\ge 0$. Moreover, there is conservation of charge and energy,

$$(4.5.12)\qquad \|u_n(t)\|_{L^2}=\|\varphi\|_{L^2},\quad E_n(u_n(t))=E_n(\varphi)$$

for all $t\ge 0$. See, for example, Corollary 4.3.3 and Corollary 6.1.2 below. Furthermore, it follows from Remark 4.4.3 or Theorem 4.4.6 that

$$(4.5.13)\qquad u_n\in L^q((0,T),W^{1,r}(\mathbb{R}^N))$$

for every admissible pair (q,r) and every $T>0$. Consider now any admissible pair (q,r) and any $T>0$. We deduce from (4.5.13), (4.5.11), and (4.5.7) that

$$(4.5.14)\qquad \|\nabla u_n\|_{L^q((0,T),L^r)} \le \|\mathcal{T}(\cdot)\nabla\varphi\|_{L^q((0,T),L^r)} + C\|\nabla u_n\|^{\frac{N+2}{N-2}}_{L^\gamma((0,T),L^\rho)}\,.$$

Similarly, we deduce from (4.5.6) that

$$(4.5.15)\qquad \|u_n\|_{L^q((0,T),L^r)} \le C\|\varphi\|_{L^2} + C\|\nabla u_n\|^{\frac{4}{N-2}}_{L^\gamma((0,T),L^\rho)}\|u_n\|_{L^\gamma((0,T),L^\rho)}\,.$$

Finally, given $\ell\ge n$, we may write

$$u_n-u_\ell=[\mathcal{H}_n(u_n)-\mathcal{H}_n(u_\ell)]+[\mathcal{H}_n(u_\ell)-\mathcal{H}(u_\ell)]+[\mathcal{H}(u_\ell)-\mathcal{H}_\ell(u_\ell)]\,,$$

and we deduce from (4.5.6) and (4.5.8) that

$$(4.5.16)\qquad \begin{aligned}&\|u_n-u_\ell\|_{L^q((0,T),L^r)}\\ &\le C\big(\|\nabla u_n\|_{L^\gamma((0,T),L^\rho)}+\|\nabla u_\ell\|_{L^\gamma((0,T),L^\rho)}\big)^{\frac{4}{N-2}}\\ &\quad\Big(\|u_n-u_\ell\|_{L^\gamma((0,T),L^\rho)}+T^{\frac{N-2}{2N}}n^{-\frac{4}{N(N-2)}}\|u_\ell\|^{\frac{N^2-2N+4}{N(N-2)}}_{L^\infty((0,T),H^1)}\Big)\,.\end{aligned}$$

Note that the constant C in (4.5.14), (4.5.15), and (4.5.16) may depend on the admissible pair (q,r), but is independent of n, ℓ, and T.

STEP 3. Passage to the limit. We will solve the equation (4.1.4) (which is equivalent to (4.1.3)), by letting $n\to\infty$ in (4.5.11). Consider K larger than the constant C appearing in (4.5.14), (4.5.15), and (4.5.16) for the particular choice of the admissible pair $(q,r)=(\gamma,\rho)$. Fix $\delta>0$ small enough so that

$$K(4\delta)^{\frac{4}{N-2}} < \frac{1}{2}. \tag{4.5.17}$$

We claim that if $0<T\le\infty$ is such that

$$\|\mathcal{T}(\cdot)\nabla\varphi\|_{L^\gamma((0,T),L^\rho)} \le \delta, \tag{4.5.18}$$

then

$$\sup_{n\ge 0}\|\nabla u_n\|_{L^\gamma((0,T),L^\rho)} \le 2\delta \tag{4.5.19}$$

and

$$\sup_{n\ge 0}\|u_n\|_{L^q((0,T),W^{1,r})} < \infty \tag{4.5.20}$$

for every admissible pair (q,r). (Note that, given $\varphi\in H^1(\mathbb{R}^N)$, (4.5.18) is satisfied if $T>0$ is sufficiently small. Indeed, $\mathcal{T}(\cdot)\nabla\varphi\in L^\gamma((0,\infty),L^\rho(\mathbb{R}^N))$ by Strichartz's estimate, so that by dominated convergence $\|\mathcal{T}(\cdot)\nabla\varphi\|_{L^\gamma((0,T),L^\rho)}\to 0$ as $T\downarrow 0$.) Set $\theta_n(t)=\|\nabla u_n\|_{L^\gamma((0,t),L^\rho)}$. It follows from (4.5.14) that for every $0\le t\le T$,

$$\theta_n(t)\le \delta + C\theta_n(t)^{\frac{N+2}{N-2}}.$$

If $\theta_n(t)=2\delta$ for some $t\in[0,T]$, then

$$2\delta \le \delta + C(2\delta)^{\frac{N+2}{N-2}} < 2\delta,$$

by (4.5.17), which is absurd. Since θ_n is a continuous function with $\theta_n(0)=0$, we conclude that $\theta_n(t)<2\delta$ for all $t\in[0,T)$, which proves (4.5.19). Applying now (4.5.14) for any admissible pair (q,r), we find that

$$\sup_{n\ge 0}\|\nabla u_n\|_{L^q((0,T),L^r)} < \infty. \tag{4.5.21}$$

Applying (4.5.15) with $(q,r)=(\gamma,\rho)$ and using (4.5.17) and (4.5.19), we obtain

$$\|u_n\|_{L^\gamma((0,T),L^\rho)} \le C\|\varphi\|_{L^2} + \frac{1}{2}\|u_n\|_{L^\gamma((0,T),L^\rho)},$$

and so $\|u_n\|_{L^\gamma((0,T),L^\rho)}\le 2C\|\varphi\|_{L^2}$. We then apply (4.5.15) for any admissible pair (q,r) and we deduce that

$$\sup_{n\ge 0}\|u_n\|_{L^q((0,T),L^r)} < \infty. \tag{4.5.22}$$

(4.5.20) now follows from (4.5.21) and (4.5.22). We now deduce from (4.5.19), from (4.5.20) applied with $(q,r)=(\infty,2)$, and from (4.5.16) applied with $(q,r)=(\gamma,\rho)$, that

$$\|u_n-u_\ell\|_{L^\gamma((0,\tau),L^\rho)} \le \frac{1}{2}\Big(\|u_n-u_\ell\|_{L^\gamma((0,\tau),L^\rho)} + C\tau^{\frac{N-2}{2N}}n^{-\frac{4}{N(N-2)}}\Big)$$

for all $\ell\ge n$ and for all $0\le\tau\le T$, $\tau<\infty$. (Note that we used again (4.5.17).) It follows that $(u_n)_{n\ge 0}$ is a Cauchy sequence in $L^\gamma((0,\tau),L^\rho(\mathbb{R}^N))$. Applying again (4.5.16), but with an arbitrary admissible pair (q,r), we conclude that $(u_n)_{n\ge 0}$

is a Cauchy sequence in $L^q((0,\tau),L^r(\mathbb{R}^N))$. If we denote by u its limit, then for every admissible pair (q,r), $u\in L^q((0,T),W^{1,r}(\mathbb{R}^N))$ by (4.5.20) and

$$u_n \underset{n\to\infty}{\longrightarrow} u \tag{4.5.23}$$

in $L^q((0,\tau),L^r(\mathbb{R}^N))$ for all $0\le\tau\le T$, $\tau<\infty$. By using Lemma 4.5.3 we may let $n\to\infty$ in (4.5.11), and we obtain that u satisfies (4.1.4) for all $0\le t\le T$, $t<\infty$. Since $g(u)\in L^{\gamma'}((0,T),W^{1,\rho'}(\mathbb{R}^N))$, we deduce from Strichartz's estimate that $u\in C([0,\tau],H^1(\mathbb{R}^N))$ for every $0<\tau\le T$, $\tau<\infty$. In particular, u is a strong H^1-solution of (4.1.3) by Proposition 3.1.3.

STEP 4. The conservation laws. We deduce from the conservation of mass for u_n (see (4.5.12)) and from (4.5.23) applied with $(q,r)=(\infty,2)$ that $\|u(t)\|_{L^2}\equiv\|\varphi\|_{L^2}$. We now show the conservation of energy. Applying (4.5.23) with $(q,r)=(\infty,2)$ and using (4.5.20), we deduce easily that $u_n\to u$ in $L^q((0,\tau),L^{\frac{2N}{N-2}}(\mathbb{R}^N))$ for every $\tau<\infty$, $\tau\le T$, and every $q<\infty$. In particular, there exists a subsequence, which we still denote by $(u_n)_{n\ge0}$, such that $u_n(t)\to u(t)$ in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ for a.a. $t\in(0,T)$. It follows that $G_n(u_n)\to G(u)$ in $L^1(\mathbb{R}^N)$ for a.a. $t\in(0,T)$. Using the conservation of energy for u_n and the lower semicontinuity of the gradient term, we deduce that $E(u(t))\le E(\varphi)$ for a.a. $t\in(0,T)$, hence for all $t\in(0,T)$ by continuity of $u(t)$ in $H^1(\mathbb{R}^N)$. Considering the reverse equation, one shows the converse inequality.

STEP 5. The blowup alternative. By uniqueness, we may consider the maximal solution, defined on the interval $[0,T_{\max})$. We show the blowup alternative by contradiction, so we assume that $T_{\max}<\infty$ and $u\in L^q((0,T_{\max}),W^{1,r}(\mathbb{R}^N))$ for some admissible pair (q,r) with $2<r<N$. Let $b\in(2,\frac{2N}{N-2})$ be defined by

$$\frac{2}{b}=1-\frac{4}{N-2}\left(\frac{1}{r}-\frac{1}{N}\right),$$

and let a be such that (a,b) is an admissible pair. Since $|\nabla g(u)|\le C|u|^{\frac{4}{N-2}}|\nabla u|$, one easily verifies by using the Sobolev inequality $\|u\|_{L^{\frac{Nr}{N-r}}}\le C\|\nabla u\|_{L^r}$ that

$$\|g(u)\|_{L^{a'}((s,t),W^{1,b'})}\le C\|\nabla u\|^{\frac{4}{N-2}}_{L^q((s,t),L^r)}\|u\|_{L^a((s,t),W^{1,b})}, \tag{4.5.24}$$

with C independent of $0\le s\le t<T_{\max}$. Since

$$u(s+\tau)=\mathcal{T}(\tau)u(s)+i\int_0^\tau \mathcal{T}(\tau-\sigma)g(u(s+\sigma))d\sigma, \tag{4.5.25}$$

we deduce from Strichartz's estimate that

$$\|u\|_{L^a((s,t),W^{1,b})}\le C\|u(s)\|_{H^1}+C\|\nabla u\|^{\frac{4}{N-2}}_{L^q((s,t),L^r)}\|u\|_{L^a((s,t),W^{1,b})},$$

with C independent of $0\le s\le t<T_{\max}$. Fix s close enough to $T_{\max}$ so that $C\|\nabla u\|^{\frac{4}{N-2}}_{L^q((s,T_{\max}),L^r)}\le 1/2$. It follows that

$$\|u\|_{L^a((s,t),W^{1,b})}\le 2C\|u(s)\|_{H^1}$$

for all $s<t<T_{\max}$, and so $u\in L^a((s,T_{\max}),W^{1,b}(\mathbb{R}^N))$. Therefore, $u\in L^a((0,T_{\max}),W^{1,b}(\mathbb{R}^N))$ and, applying again (4.5.24), we conclude that $g(u)\in$

$L^{a'}((0,T_{\max}),W^{1,b'}(\mathbb{R}^N))$, so that $u \in L^\gamma((0,T_{\max}),W^{1,\rho}(\mathbb{R}^N))$ by Strichartz's estimate. We finally deduce from (4.5.25) and Lemma 4.5.3 that

$$\|\mathcal{T}(\cdot)u(t)\|_{L^\gamma((0,T_{\max}-t),W^{1,\rho})} \leq \|u\|_{L^\gamma((t,T_{\max}),W^{1,\rho})} + C\|u\|^{\frac{N+2}{N-2}}_{L^\gamma((t,T_{\max}),W^{1,\rho})},$$

where C is independent of $t \in [0,T_{\max}]$. Therefore, we may choose t close enough to $T_{\max}$ so that $\|\mathcal{T}(\cdot)u(t)\|_{L^\gamma((0,T_{\max}-t),W^{1,\rho})} < \delta$ with δ given by (4.5.17). Therefore, there exists $\varepsilon > 0$ such that $\|\mathcal{T}(\cdot)u(t)\|_{L^\gamma((0,T_{\max}+\varepsilon-t),W^{1,\rho})} < \delta$. We deduce from Step 3 that the solution u can be extended to the interval $[0,T_{\max}+\varepsilon]$, which contradicts the maximality.

STEP 6. Continuous dependence. Fix $0 < T < T_{\max}$ and let $\delta > 0$ satisfy (4.5.17). Since $u \in C([0,T],H^1(\mathbb{R}^N))$, $\bigcup_{0\leq t\leq T}\{u(t)\}$ is a compact set of $H^1(\mathbb{R}^N)$. Therefore, it follows from Strichartz's estimate that there exists $\tau > 0$ such that

$$\sup_{0\leq t\leq T}\|\mathcal{T}(\cdot)u(t)\|_{L^\gamma((0,\tau),W^{1,\rho})} \leq \frac{\delta}{2}. \tag{4.5.26}$$

Suppose now $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$. It follows in particular from (4.5.26) and Strichartz's estimate that $\|\mathcal{T}(\cdot)\varphi_n\|_{L^\gamma((0,\tau),W^{1,\rho})} < \delta$ for n large enough. Therefore, by Step 3, the solution u_n of (4.1.3) with the initial value φ_n exists on $[0,\tau]$ and $\|\nabla u_n\|_{L^\gamma((0,\tau),L^\rho)} \leq 2\delta$. Arguing as in Step 4, we deduce that $u_n \to u$ in $L^q((0,\tau),L^r(\mathbb{R}^N))$ and that u_n is in a bounded subset of $L^q((0,\tau),W^{1,r}(\mathbb{R}^N))$ for every admissible pair (q,r). Therefore (see (4.5.6)),

$$u_n \to u \tag{4.5.27}$$

in $C([0,\tau],L^2(\mathbb{R}^N))$. Choosing $r > 2$ arbitrarily close to 2, so that $q < \infty$ is arbitrarily large, and applying Gagliardo-Nirenberg's inequality, we obtain that $u_n \to u$ in $L^q((0,\tau),L^{\frac{2N}{N-2}}(\mathbb{R}^N))$ for every $q < \infty$. Since $E(u_n(t)) = E(\varphi_n) \to E(\varphi) = E(u(t))$, we see that $\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}$ in $L^q(0,\tau)$ for every $q < \infty$. On the other hand, since u_n is bounded in $C([0,\tau],H^1(\mathbb{R}^N))$, we deduce from (4.5.27) that $u_n(t) \rightharpoonup u(t)$ in $H^1(\mathbb{R}^N)$ for every $t \in [0,\tau]$. Using the L^q convergence of the norm, one concludes easily that $u_n \to u$ in $L^q((0,\tau),H^1(\mathbb{R}^N))$ for every $q < \infty$. In particular, there exists $(t_n)_{n\geq 0} \subset [\tau/2,\tau]$ such that $\|u_n(t_n) - u(t_n)\|_{H^1} \to 0$. Repeating the above argument (using (4.5.26) and (4.5.27)), we deduce that u_n exists on $[0,3\tau/2]$ for n large enough and that $u_n \to u$ in $L^q((0,3\tau/2),H^1(\mathbb{R}^N))$ for every $q < \infty$. We may now iterate the same process to cover the interval $[0,T]$. □

REMARK 4.5.4. Here are some further comments on Theorem 4.5.1.

(i) If $\|\nabla\varphi\|_{L^2}$ is small enough, then we may take $T = \infty$ in Step 3 of the proof of Theorem 4.5.1. Indeed, $\|\mathcal{T}(\cdot)\nabla\varphi\|_{L^\gamma(\mathbb{R},L^\rho)} \leq C\|\nabla\varphi\|_{L^2}$. Therefore, the solution is global in that case, i.e., $T_{\max} = T_{\min} = \infty$.

(ii) It is clear from Step 5 of the proof of Theorem 4.5.1 that the blowup alternative can be improved, in the sense that if $T_{\max} < \infty$, then for every admissible pair (q,r) with $2 < r < N$, $\|u\|_{L^q((0,T_{\max}),L^{\frac{Nr}{N-r}})} = \infty$. In addition, using the same type of estimates as in the proof of uniqueness (see Step 1), one can show that if $T_{\max} < \infty$, then

$$\lim_{t\uparrow T_{\max}}\lim_{M\uparrow\infty}\|u^M\|_{L^\infty\left((t,T_{\max}),L^{\frac{2N}{N-2}}\right)} > 0,$$

where $u^M = u1_{\{|u|>M\}}$. In fact, the limit is not only positive, but bounded from below by a positive number independent of the solution. This indicates a concentration phenomenon in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$.

(iii) In the case $\lambda < 0$, the continuous dependence statement (iv) can be improved. More precisely, $u_n \to u$ in $C([-S,T], H^1(\mathbb{R}^N))$ for every interval $[-S,T] \subset (-T_{\min}, T_{\max})$. In other words, there is the usual continuous dependence property. The proof is in fact simpler. We have $u_n(t) \to u(t)$ strongly in $L^2(\mathbb{R}^N)$, weakly in $H^1(\mathbb{R}^N)$, and $E(u_n(t)) \to E(u(t))$ for all $t \in [0,\tau]$. Since $\lambda < 0$, both terms in the energy are lower semicontinuous so that indeed $\|u_n(t)\|_{L^{\frac{2N}{N-2}}} \to \|u(t)\|_{L^{\frac{2N}{N-2}}}$ and $\|\nabla u_n(t)\|_{L^2} \to \|\nabla u(t)\|_{L^2}$. Thus $u_n(t) \to u(t)$ strongly in $H^1(\mathbb{R}^N)$ and one concludes as above.

(iv) In the case $\lambda < 0$, one would expect that the solution is global for every initial value. This is only known if we assume further that $\|\nabla\varphi\|_{L^2}$ is small (see (i) above) or if φ is spherically symmetric (see Bourgain [39]).

4.6. L^2 Solutions

In this section we construct solutions of some nonlinear Schrödinger equations for initial data in $L^2(\mathbb{R}^N)$. Such results were first obtained by Y. Tsutsumi [343] (see also Cazenave and Weissler [69, 70] and Kato [204]). We assume that

$$g : L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \to L^{r'}(\mathbb{R}^N) \tag{4.6.1}$$

for some

$$r \in \left[2, \frac{2N}{N-2}\right) \quad (r \in [2,\infty] \text{ if } N = 1). \tag{4.6.2}$$

Furthermore, we assume that there exists $\alpha > 0$ such that, for every $M > 0$, there exists $K(M) < \infty$ such that

$$\|g(v) - g(u)\|_{L^{r'}} \le K(M)(\|u\|_{L^r}^{\alpha} + \|v\|_{L^r}^{\alpha})\|v-u\|_{L^r} \tag{4.6.3}$$

for all $u, v \in L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ such that $\|u\|_{L^2}, \|v\|_{L^2} \le M$. We have the following result.

THEOREM 4.6.1. *Assume* (4.6.1)–(4.6.3) *and set*

$$\frac{2}{q} = N\left(\frac{1}{2} - \frac{1}{r}\right)$$

so that (q,r) is an admissible pair. If $\alpha + 2 < q$, then for every $\varphi \in L^2(\mathbb{R}^N)$, there exist $T_{\max}, T_{\min} \in (0,\infty]$ and a unique, maximal solution $u \in C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N)) \cap L^q_{\mathrm{loc}}((-T_{\min}, T_{\max}), L^r(\mathbb{R}^N))$ of problem (4.1.1). *Moreover, the following properties hold:*

(i) (Blowup alternative) *If $T_{\max} < \infty$ (respectively, if $T_{\min} < \infty$), then $\|u(t)\|_{L^2} \to \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow -T_{\min}$).*

(ii) *$u \in L^{\gamma}_{\mathrm{loc}}((-T_{\min}, T_{\max}), L^{\rho}(\mathbb{R}^N))$ for every admissible pair (γ, ρ).*

(iii) *u depends continuously on φ in the following sense: The mappings $\varphi \mapsto T_{\min}, T_{\max}$ are lower semicontinuous $L^2(\mathbb{R}^N) \to (0,\infty]$. If $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N)$ and if u_n denotes the solution of* (4.1.1) *with the initial value φ_n,* *then* $u_n \to u$ *in* $L^\gamma((-S,T),L^\rho(\mathbb{R}^N))$ *for every admissible pair* (γ,ρ) *and every* $-T_{\min} < -S < 0 < T < T_{\max}$.

(iv) *If* $(g(w), iw)_{L^{r'},L^r} = 0$ *for all* $w \in L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, *then* $T_{\min} = T_{\max} = +\infty$ *and* $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ *for all* $t \in \mathbb{R}$.

REMARK 4.6.2. Consider u as in Theorem 4.6.1; i.e., $u \in C((-T_{\min},T_{\max}), L^2(\mathbb{R}^N)) \cap L^q_{\mathrm{loc}}((-T_{\min},T_{\max}),L^r(\mathbb{R}^N))$. It follows that $\Delta u \in C((-T_{\min},T_{\max}), H^{-2}(\mathbb{R}^N))$ and $g(u) \in L^{\frac{q}{\alpha+1}}_{\mathrm{loc}}((-T_{\min},T_{\max}),L^{r'}(\mathbb{R}^N))$ by (4.6.3). Since $L^{r'}(\mathbb{R}^N) \hookrightarrow H^{-2}(\mathbb{R}^N)$, we see that $\Delta u + g(u) \in L^{\frac{q}{\alpha+1}}_{\mathrm{loc}}((-T_{\min},T_{\max}),H^{-2}(\mathbb{R}^N))$. It follows that equation (4.1.1) makes sense in $\mathcal{D}'((-T_{\min},T_{\max}),H^{-2}(\mathbb{R}^N))$. In particular, $u_t \in L^{\frac{q}{\alpha+1}}_{\mathrm{loc}}((-T_{\min},T_{\max}),H^{-2}(\mathbb{R}^N))$ and (4.1.1) makes sense in $H^{-2}(\mathbb{R}^N)$ for a.a. $t \in (-T_{\min},T_{\max})$.

For the proof of Theorem 4.6.1, we will use the following lemma.

LEMMA 4.6.3. *Let g satisfy* (4.6.1)–(4.6.3) *and let I be an open interval of $\mathbb{R}$. If u is measurable both as a function $I \to L^2(\mathbb{R}^N)$ and as a function $I \to L^r(\mathbb{R}^N)$, then $g(u)$ is measurable $I \to L^{r'}(\mathbb{R}^N)$.*

PROOF. Note that for a.a. $t \in I$, $u(t) \in L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, so that $g(u(t)) \in L^{r'}(\mathbb{R}^N)$ is well defined. Consider a function $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $\varphi(x) = 1$ for $|x| \le 1$ and set $\varphi_n(x) = \varphi(x/n)$ for $n \ge 1$ and $x \in \mathbb{R}^N$. Using the dominated convergence theorem, we see that $\varphi_n u(t) \to u(t)$ in $L^2(\mathbb{R}^N)$ and in $L^r(\mathbb{R}^N)$ as $n \to \infty$ for a.a. $t \in I$. In particular, $g(\varphi_n u) \to g(u)$ in $L^{r'}(\mathbb{R}^N)$ as $n \to \infty$ for a.a. $t \in I$. Therefore, we need only show that for any given $n \ge 1$, $g(\varphi_n u)$ is measurable $I \to L^{r'}(\mathbb{R}^N)$. To see this, we observe that, since u is measurable $I \to L^r(\mathbb{R}^N)$, there exists a sequence $(u_k)_{k\ge 0} \subset C(I,L^r(\mathbb{R}^N))$ such that $u_k(t) \to u(t)$ in $L^r(\mathbb{R}^N)$ as $k \to \infty$ for a.a. $t \in I$. Since $\varphi_n u_k(t)$ is supported in a fixed compact subset of $K_n \subset \mathbb{R}^N$ and $L^r(K_n) \hookrightarrow L^2(K_n)$, it follows that $\varphi_n u_k$ is continuous $I \to L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, and so $g(\varphi_n u_k)$ is continuous $I \to L^{r'}(\mathbb{R}^N)$. Since $\varphi_n u_k \to \varphi_n u$ as $k \to \infty$ in $L^r(\mathbb{R}^N)$, hence in $L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ for a.a. $t \in I$, we have $g(\varphi_n u_k) \to g(\varphi_n u)$ in $L^{r'}(\mathbb{R}^N)$ for a.a. $t \in I$. So $g(\varphi_n u)$ is measurable $I \to L^{r'}(\mathbb{R}^N)$, which completes the proof. □

PROOF OF THEOREM 4.6.1. For the existence part, we use a fixed point argument as in Section 4.4. For the conservation of charge, we need a regularization process. We proceed in five steps.

STEP 1. Existence. Fix $T, M > 0$ and set

$$E = \big\{u \in L^\infty((-T,T),L^2(\mathbb{R}^N)) \cap L^q((-T,T),L^r(\mathbb{R}^N)); \\ \|u\|_{L^\infty((-T,T),L^2)} + \|u\|_{L^q((-T,T),L^r)} \le M\big\}. \tag{4.6.4}$$

It follows that E is a complete metric space when equipped with the distance

$$\mathrm{d}(u,v) = \|u-v\|_{L^\infty((-T,T),L^2)} + \|u-v\|_{L^q((-T,T),L^r)}. \tag{4.6.5}$$

Consider $u \in E$. It follows from Lemma 4.6.1 that $g(u) : I \to L^{r'}(\mathbb{R}^N)$ is measurable. Moreover, we deduce from (4.6.1) and (4.6.3) that for a.a. $t \in (-T,T)$,

$$\|g(u(t))\|_{L^{r'}} \le \|g(0)\|_{L^{r'}} + K(M)\|u(t)\|_{L^r}^{\alpha+1}.$$

Therefore, by Hölder's inequality in time,

$$\|g(u)\|_{L^{q'}(-T,T),L^{r'})} \le CT^{\frac{1}{q'}}\|g(0)\|_{L^{r'}} + CK(M)\|u\|^{\alpha+1}_{L^{(\alpha+1)q'}((-T,T),L^r)}$$
$$\le CT^{\frac{1}{q'}}\|g(0)\|_{L^{r'}} + CT^{\frac{q-(\alpha+2)}{q}}K(M)\|u\|^{\alpha+1}_{L^q((-T,T),L^r)};$$

and so

$$\|g(u)\|_{L^{q'}((-T,T),L^{r'})} \le CT^{\frac{1}{q'}}\|g(0)\|_{L^{r'}} + CT^{\frac{q-(\alpha+2)}{q}}K(M)M^{\alpha+1}. \tag{4.6.6}$$

Similarly, one shows that for $u, v \in E$,

$$\|g(u) - g(v)\|_{L^{q'}((-T,T),L^{r'})} \le CT^{\frac{q-(\alpha+2)}{q}}K(M)M^{\alpha}\,\mathrm{d}(u,v). \tag{4.6.7}$$

Applying (4.6.6), (4.6.7), and Strichartz's estimate, we see that

$$\mathcal{G}(u)(t) = i\int_0^t \mathcal{T}(t-s)g(u(s))ds$$

is well defined, that $\mathcal{G}(u) \in C([-T,T], L^2(\mathbb{R}^N)) \cap L^\gamma((-T,T), L^\rho(\mathbb{R}^N))$ for every admissible pair (γ,ρ), and that

$$\|\mathcal{G}(u)\|_{L^\gamma((-T,T),L^\rho)} \le CT^{\frac{1}{q'}}\|g(0)\|_{L^{r'}} + CT^{\frac{q-(\alpha+2)}{q}}K(M)M^{\alpha+1}, \tag{4.6.8}$$

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{L^\gamma((-T,T),L^\rho)} \le CT^{\frac{q-(\alpha+2)}{q}}K(M)M^{\alpha}\,\mathrm{d}(u,v). \tag{4.6.9}$$

Given $\varphi \in L^2(\mathbb{R}^N)$, set now $\mathcal{H}(u)(t) = \mathcal{T}(t)\varphi + \mathcal{G}(u)(t)$. We deduce from (4.6.8) and Strichartz's estimate that for every $u \in E$,

$$\|\mathcal{H}(u)\|_{L^\infty((-T,T),L^2)} + \|\mathcal{H}(u)\|_{L^q((-T,T),L^r)} \le$$
$$C\|\varphi\|_{L^2} + CT^{\frac{1}{q'}}\|g(0)\|_{L^{r'}} + CT^{\frac{q-(\alpha+2)}{q}}K(M)M^{\alpha+1}.$$

Choosing $M = 2C\|\varphi\|_{L^2}$, we see that if T is sufficiently small (depending on $\|\varphi\|_{L^2}$), $\mathcal{H}(u) \in E$ for all $u \in E$. Moreover, we deduce from (4.6.9) and Strichartz's estimate that, by possibly choosing T smaller (but still depending on $\|\varphi\|_{L^2}$),

$$\mathrm{d}(\mathcal{H}(u), \mathcal{H}(v)) \le \frac{1}{2}\mathrm{d}(u,v)$$

for all $u, v \in E$. Thus $\mathcal{H}$ has a unique fixed point $u \in E$. Note that $g(u) \in L^{q'}((-T,T), L^{r'}(\mathbb{R}^N)) \hookrightarrow L^{q'}((-T,T), H^{-1}(\mathbb{R}^N))$. It follows (see Section 1.6) that $u \in C([-T,T], H^{-1}(\mathbb{R}^N)) \cap W^{1,1}((-T,T), H^{-3}(\mathbb{R}^N))$ and u satisfies (4.1.1) in $H^{-3}(\mathbb{R}^N)$ for a.a. $t \in (-T,T)$. This proves local existence.

Step 2. Uniqueness. We first note that uniqueness is a local property, so that we need only establish it on possibly small intervals. To see this, we argue for positive times, the argument for negative times being the same. Suppose we know that if $u, v \in C([0,T], L^2(\mathbb{R}^N)) \cap L^q((0,T), L^r(\mathbb{R}^N))$ are any two solutions of (4.1.1), then $u = v$ on $(0,\tau)$ for $0 < \tau \le T$ sufficiently small. We may then define $0 < \theta \le T$ by

$$\theta = \sup\{0 < \tau < T; u = v \text{ on } (0,\tau)\}.$$

It follows that $u = v$ on $[0,\theta]$. If $\theta = T$, uniqueness follows, so we assume by contradiction that $\theta < T$. We see that $u_1(\cdot) = u(\theta + \cdot)$ and $v_1(\cdot) = v(\theta + \cdot)$ are two solutions of (4.1.1) with φ replaced by $u(\theta) = v(\theta)$ on the interval $(0, T-\theta)$. By uniqueness for small time, we deduce that $u_1 = v_1$ on some interval $[0,\varepsilon]$ with

$0 < \varepsilon \le T - \theta$. This means that $u = v$ on $[0, \theta + \varepsilon]$, contradicting the definition of θ.

We now show uniqueness for small time. The proof of (4.6.9) indeed shows that

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{L^q(I,L^r)} \le C|I|^{\frac{q-(\alpha+2)}{q}} K(\|u\|_{L^\infty(I,L^2)} + \|v\|_{L^\infty(I,L^2)}) \times (\|u\|_{L^q(I,L^r)} + \|v\|_{L^q(I,L^r)})^\alpha \|u - v\|_{L^q(I,L^r)} .$$

Since $\mathcal{G}(u) - \mathcal{G}(v) = u - v$, we deduce that if $|I|$ is sufficiently small; i.e., if T is sufficiently small, then

$$\|u - v\|_{L^q(I,L^r)} \le \frac{1}{2}\|u - v\|_{L^q(I,L^r)} ,$$

i.e., $u = v$ on I.

STEP 3. The blowup alternative and continuous dependence. Arguing as in the proof of Theorem 3.3.9, we define the maximal solution by using the uniqueness property; and since T depends on $\|\varphi\|_{L^2}$, we deduce the blowup alternative. Next, using again (4.6.9), we deduce easily that, if $\varphi, \psi \in L^2(\mathbb{R}^N)$ and if u, v are the corresponding solutions of (4.1.1), then for some T depending on $\|\varphi\|_{L^2}, \|\psi\|_{L^2}$, $d(u,v) \le C\|\varphi - \psi\|_{L^2}$. Continuous dependence (property (iii)) follows easily (see the proof of Theorem 3.3.9).

STEP 4. Proof of property (ii). Let $I \ni 0$ be a bounded interval and let $u \in L^\infty(I, L^2(\mathbb{R}^N)) \cap L^q(I, L^r(\mathbb{R}^N))$ be a solution of (4.1.1). We need to show that $u \in L^\gamma(I, L^\rho(\mathbb{R}^N))$ for every admissible pair (q, r). We note that the argument of proof of (4.6.6) shows that

$$\|g(u)\|_{L^{q'}((I,L^{r'})} \le C|I|^{\frac{1}{q'}} \|g(0)\|_{L^{r'}} + C|I|^{\frac{q-(\alpha+2)}{q}} K(\|u\|_{L^\infty(I,L^2)}) \|u\|^{\alpha+1}_{L^q(I,L^r)} .$$

In particular, $g(u) \in L^{q'}((I, L^{r'}(\mathbb{R}^N))$ and the result follows from Strichartz's estimates.

STEP 5. Proof of property (iv). Fix $\varepsilon > 0$ and let $J_\varepsilon = (I - \varepsilon\Delta)^{-1}$. (The reader is referred to Proposition 1.5.2 and 1.5.3 for all the relevant properties of J_ε.) We define the nonlinearity g_ε by

$$g_\varepsilon(w) = J_\varepsilon g(J_\varepsilon w))$$

for all $w \in L^2(\mathbb{R}^N)$. We observe that $J_\varepsilon w \in H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ so that $g(J_\varepsilon w) \in L^{r'}(\mathbb{R}^N)$. Since $L^{r'}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$, we have $g_\varepsilon(u) \in H^1(\mathbb{R}^N)$. Moreover, since J_ε is a contraction in $L^p(\mathbb{R}^N)$ for all $1 \le p \le \infty$, we see that g_ε satisfies the assumption (4.6.3) uniformly in $\varepsilon > 0$. Moreover,

$$(g_\varepsilon(w), iw)_{L^2} = (g_\varepsilon(w), iw)_{L^{r'},L^r} = (g(J_\varepsilon w), iJ_\varepsilon w)_{L^{r'},L^r} = 0 .$$

We now proceed as follows. Consider $\varphi \in H^1(\mathbb{R}^N)$ and let u be the corresponding solution of (4.1.1). Let u_ε be the solutions of (4.1.1) with g replaced by g_ε. Since g_ε satisfies the assumption (4.6.3) uniformly in $\varepsilon > 0$, we deduce from the estimates of Step 1 that u_ε is defined on some interval $[-T, T]$ with T independent of $\varepsilon > 0$. Since $g(J_\varepsilon u_\varepsilon) \in L^{q'}(-T,T), L^{r'}(\mathbb{R}^N))$ (see the estimates of Step 1), it follows that $g_\varepsilon(u_\varepsilon) \in L^1((-T,T), H^1(\mathbb{R}^N))$. This implies that $u_\varepsilon \in C([-T,T], H^1(\mathbb{R}^N))$.

Therefore, we may take the $H^{-1} - H^1$ duality product of equation (4.1.1) (with g replaced by g_ε) by iu_ε, and we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(t)\|_{L^2}^2 = -(\Delta u_\varepsilon, iu_\varepsilon)_{H^{-1},H^1} - (g_\varepsilon(u_\varepsilon), iu_\varepsilon)_{H^{-1},H^1} = 0\,,$$

so that

$$\|u_\varepsilon(t)\|_{L^2} = \|\varphi\|_{L^2} \tag{4.6.10}$$

for $|t| \le T$. We claim that, after possibly choosing T smaller,

$$u_\varepsilon \underset{\varepsilon\downarrow 0}{\longrightarrow} u \tag{4.6.11}$$

in $L^\infty((-T,T), L^2(\mathbb{R}^N))$. Indeed, we write

$$g_\varepsilon(u_\varepsilon) - g(u) = g_\varepsilon(u_\varepsilon) - g_\varepsilon(u) + J_\varepsilon(g(J_\varepsilon u) - g(u)) + (J_\varepsilon - I)g(u)\,,$$

and we deduce that

$$\begin{aligned}
&\|g_\varepsilon(u_\varepsilon) - g(u)\|_{L^{q'}((-T,T),L^{r'})} \\
&\quad \le \|g_\varepsilon(u_\varepsilon) - g_\varepsilon(u)\|_{L^{q'}((-T,T),L^{r'})} \\
&\qquad + \|g(J_\varepsilon u) - g(u)\|_{L^{q'}((-T,T),L^{r'})} + \|(J_\varepsilon - I)g(u)\|_{L^{q'}((-T,T),L^{r'})}\,.
\end{aligned}$$

Since $g(u) \in L^{q'}((-T,T), L^{r'}(\mathbb{R}^N))$, we have

$$\|(J_\varepsilon - I)g(u)\|_{L^{q'}((-T,T),L^{r'})} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0\,.$$

Next, since $J_\varepsilon u \to u$ in $L^q((-T,T), L^r(\mathbb{R}^N))$, we deduce from (4.6.7) that

$$\|g(J_\varepsilon u) - g(u)\|_{L^{q'}((-T,T),L^{r'})} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0\,.$$

We also deduce from (4.6.7) (applied to g_ε) that

$$\|g_\varepsilon(u_\varepsilon) - g_\varepsilon(u)\|_{L^{q'}((-T,T),L^{r'})} \le CT^{\frac{q-(\alpha+2)}{q}}\|u_\varepsilon - u\|_{L^q((-T,T),L^r)}\,.$$

Using the above estimates and Strichartz's inequalities, we conclude that

$$\begin{aligned}
&\|u_\varepsilon - u\|_{L^\infty((-T,T),L^2)} + \|u_\varepsilon - u\|_{L^q((-T,T),L^r)} \\
&\qquad\qquad \le a_\varepsilon + CT^{\frac{q-(\alpha+2)}{q}}\|u_\varepsilon - u\|_{L^q((-T,T),L^r)}\,,
\end{aligned}$$

with $a_\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Choosing T sufficiently small, we obtain the claim (4.6.11). We then deduce from (4.6.10) and (4.6.11) that $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for $|t| \le T$. Replacing φ by $u(t_0)$ for any $t_0 \in (-T_{\max}, T_{\min})$, we obtain that $\|u(t)\|_{L^2}$ is locally constant, hence constant. Finally, in the general case $\varphi \in L^2(\mathbb{R}^N)$, we approximate φ in $L^2(\mathbb{R}^N)$ by $(\varphi_n)_{n\ge 0} \subset H^1(\mathbb{R}^N)$ and we use the continuous dependence to obtain the conservation of charge. Global existence follows from the blowup alternative. □

THEOREM 4.6.4. *Let $g = g_1 + \cdots + g_k$, where each of the g_j's satisfies* (4.6.1)–(4.6.3) *for some exponents r_j, α_j. Set*

$$\frac{2}{q_j} = N\Big(\frac{1}{2} - \frac{1}{r_j}\Big)\,,$$

and let $r = \max\{r_1, \dots, r_k\}$ and $q = \min\{q_1, \dots, q_k\}$. If $2+\alpha_j < q_j$ for $j = 1, \dots, k$, then all the conclusions of Theorem 4.6.1 *hold.*

PROOF. Fix $M > 0$ and consider (E, d) defined by (4.6.4)–(4.6.5). We see (cf. the proof of (4.4.23)) that

$$\|w\|_{L^{q_j}((-T,T),L^{r_j})} \le \|w\|_{L^\infty((-T,T),L^2)}^{\frac{2(r-r_j)}{r_j(r-2)}} \|w\|_{L^q((-T,T),L^r)}^{\frac{r(r_j-2)}{r_j(r-2)}} .$$

In particular, $\|u\|_{L^{q_j}((-T,T,L^{r_j})} \le M$ for all $u \in E$ and $\|u - v\|_{L^{q_j}((-T,T,L^{r_j})} \le \mathrm{d}(u,v)$ for all $u, v \in E$. We deduce (see the proof of (4.6.6)–(4.6.7)) that

$$\|g_j(u)\|_{L^{q_j'}(-T,T),L^{r_j'})} \le CT^{\frac{1}{q_j'}} \|g_j(0)\|_{L^{r'}} + CT^{\frac{q_j-(\alpha_j+2)}{q_j}} K_j(M) M^{\alpha_j+1} ,$$

$$\|g_j(u) - g_j(v)\|_{L^{q_j'}(-T,T),L^{r_j'})} \le CT^{\frac{q_j-(\alpha_j+2)}{q_j}} K_j(M) M^{\alpha_j} \mathrm{d}(u,v) .$$

It follows that

$$\mathcal{G}_j(u)(t) = i \int_0^t \mathcal{T}(t-s) g_j(u(s)) ds$$

is well defined, that $\mathcal{G}_j(u) \in C([-T,T], L^2(\mathbb{R}^N)) \cap L^\gamma(-T,T), L^\rho(\mathbb{R}^N))$ for every admissible pair (γ, ρ), and that

$$\|\mathcal{G}_j(u)\|_{L^\gamma(-T,T),L^\rho)} \le CT^{\frac{1}{q_j'}} \|g_j(0)\|_{L^{r'}} + CT^{\frac{q_j-(\alpha_j+2)}{q_j}} K_j(M) M^{\alpha_j+1} ,$$

$$\|\mathcal{G}_j(u) - \mathcal{G}_j(v)\|_{L^\gamma(-T,T),L^\rho)} \le CT^{\frac{q_j-(\alpha_j+2)}{q_j}} K_j(M) M^{\alpha_j} \mathrm{d}(u,v) .$$

Given $\varphi \in L^2(\mathbb{R}^N)$, set now $\mathcal{H}(u)(t) = \mathcal{T}(t)\varphi + \mathcal{G}_1(u)(t) + \cdots + \mathcal{G}_k(u)(t)$. We deduce that for every $u \in E$,

$$\|\mathcal{H}(u)\|_{L^\infty(-T,T),L^2)} + \|\mathcal{H}(u)\|_{L^q(-T,T),L^r)} \le$$
$$C\|\varphi\|_{L^2} + C \sum_{j=1}^{k} \left(T^{\frac{1}{q_j'}} \|g_j(0)\|_{L^{r'}} + T^{\frac{q_j-(\alpha_j+2)}{q_j}} K_j(M) M^{\alpha_j+1} \right) .$$

Choosing $M = 2C\|\varphi\|_{L^2}$, we see that if T is sufficiently small (depending on $\|\varphi\|_{L^2}$), $\mathcal{H}(u) \in E$ for all $u \in E$. Similarly, one shows that, by possibly choosing T smaller (but still depending on $\|\varphi\|_{L^2}$), $\mathrm{d}(\mathcal{H}(u), \mathcal{H}(v)) \le \mathrm{d}(u,v)/2$ for all $u, v \in E$. Thus $\mathcal{H}$ has a unique fixed point $u \in E$. The rest of the proof of Theorem 4.6.1 is easily adapted. □

Let us now give an example, of application of Theorem 4.6.4. Consider $V \in L^\delta(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\delta \ge 1$, $\delta > N/2$ and $W \in L^\sigma(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\sigma \ge 1$, $\sigma > N/2$. Let $f : \mathbb{R}^N \times \mathbb{C} \to \mathbb{C}$ be measurable in $x \in \mathbb{R}^N$ and continuous in $z \in \mathbb{C}$. Suppose that $f(x, 0) = 0$ for all $x \in \mathbb{R}^N$ and that

$$|f(x, z_1) - f(x, z_2)| \le C(1 + |z_1| + |z_2|)^\beta |z_1 - z_2|$$

for some $0 \le \beta < 4/N$. Set

$$g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u .$$

We have the following result.

COROLLARY 4.6.5. *If g is as above, then the conclusions of Theorem 4.6.4 hold. Moreover, if V and W are real valued and if $f(x,z)\overline{z} \in \mathbb{R}$ for all $z \in \mathbb{C}$ and $x \in \mathbb{R}^N$, then there is conservation of charge and all solutions are global.*

PROOF. We let $V = V_1 + V_2$ with $V_1 \in L^\delta(\mathbb{R}^N)$ and $V_2 \in L^\infty(\mathbb{R}^N)$, $W = W_1 + W_2$ with $W_1 \in L^\sigma(\mathbb{R}^N)$ and $W_2 \in L^\infty(\mathbb{R}^N)$. We need only show that each of the terms $V_1 u$, $V_2 u$, $f(\cdot, u(\cdot))$, $(W_1 \star |u|^2)u$ and $(W_2 \star |u|^2)u$ satisfies the assumptions of Theorem 4.6.4. It is immediate that $V_1 u$ (respectively, $V_2 u$) satisfies the assumptions with $K(M) \equiv C$, $\alpha = 0$, and $r = 2\delta/(\delta - 1)$ (respectively, $r = 2$). Applying Hölder's and Young's inequalities, one easily verifies that $(W_1 \star |u|^2)u$ (respectively, $(W_2 \star |u|^2)u$) satisfy the assumptions with $K(M) = M^2$, $\alpha = 0$, and $r = 2\sigma/(\sigma - 1)$ (respectively, $r = 2$). Finally, one may write $f(x, z) = f_1(x, z) + f_2(x, z)$, where f_1 is Lipschitz continuous in z, uniformly in x, and

$$|f_2(x, z_1) - f_2(x, z_2)| \le C(|z_1|^\beta + |z_2|^\beta)|z_1 - z_2|$$

for a.a. $x \in \mathbb{R}^N$ and all $z_1, z_2 \in \mathbb{C}$. One easily verifies that $f_1(\cdot, u(\cdot))$ (respectively, $f_2(\cdot, u(\cdot))$) satisfies the assumptions with $K(M) \equiv C$ and with $\alpha = 0$ and $r = 2$ (respectively, $\alpha = \beta$ and $r = \beta + 2$). □

4.7. A Critical Case in $L^2(\mathbb{R}^N)$

If we consider the model case $g(u) = \lambda |u|^\alpha u$ with $\lambda \in \mathbb{C}$ and $\alpha > 0$, it follows from Corollary 4.6.5 that the initial-value problem (4.1.3) is locally well posed (in an appropriate sense) in $L^2(\mathbb{R}^N)$ if $\alpha < 4/N$. In the limiting case $\alpha = 4/N$, the method we presented does not apply at several steps. However, an appropriate refinement of this method yields local well-posedness, and below is a typical result. (See Cazenave and Weissler [69].)

THEOREM 4.7.1. *Let $g(u) = \lambda |u|^\alpha u$ with $\lambda \in \mathbb{C}$ and $\alpha = 4/N$. For every $\varphi \in L^2(\mathbb{R}^N)$, there exist $T_{\max}, T_{\min} \in (0, \infty]$ and a unique, maximal solution $u \in C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N)) \cap L^{\alpha+2}_{\mathrm{loc}}((-T_{\min}, T_{\max}), L^{\alpha+2}(\mathbb{R}^N))$ of (4.1.3). Moreover, the following properties hold:*

(i) (Blowup alternative.) *If $T_{\max} < \infty$ (respectively, $T_{\min} < \infty$), then $\|u\|_{L^q((0,T_{\max}),L^r)} = \infty$ (respectively, $\|u\|_{L^q((-T_{\min},0),L^r)} = \infty$) for every admissible pair (q, r) with $r \ge \alpha + 2$.*

(ii) *$u \in L^q_{\mathrm{loc}}((-T_{\min}, T_{\max}), L^r(\mathbb{R}^N))$ for every admissible pair (q, r).*

(iii) *If, in addition, $\varphi \in H^1(\mathbb{R}^N)$, then $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N))$.*

(iv) (Conservation of charge.) *If $\lambda \in \mathbb{R}$, then $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in (-T_{\min}, T_{\max})$.*

(v) (Continuous dependence.) *The mappings $\varphi \mapsto T_{\min}, T_{\max}$ are lower-semicontinuous $L^2(\mathbb{R}^N) \to (0, \infty]$. If $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N)$ and if u_n denotes the corresponding solutions of (4.1.3), then $u_n \to u$ in $L^q(I, L^r(\mathbb{R}^N))$ for every interval $I \Subset (-T_{\min}, T_{\max})$ and every admissible pair (q, r).*

REMARK 4.7.2. Arguing as in Remark 4.6.2, we see that if u is as in Theorem 4.7.1, then equation (4.1.3) makes sense in $H^{-2}(\mathbb{R}^N)$ for a.a. $t \in (-T_{\min}, T_{\max})$.

PROOF OF THEOREM 4.7.1. Consider an interval $I \subset \mathbb{R}$ with $0 \in I$ and let $u, v \in L^{\alpha+2}(I, L^{\alpha+2}(\mathbb{R}^N))$. It follows from the estimate

$$||u|^\alpha u - |v|^\alpha v| \le (\alpha + 1)(|u|^\alpha + |v|^\alpha)|u - v|$$

and Hölder's inequality that

$$\begin{aligned}(4.7.1)\qquad &\||u|^\alpha u - |v|^\alpha v\|_{L^{\frac{\alpha+2}{\alpha+1}}(I,L^{\frac{\alpha+2}{\alpha+1}})} \le \\ &(\alpha+1)\big(\|u\|^\alpha_{L^{\alpha+2}(I,L^{\alpha+2})} + \|v\|^\alpha_{L^{\alpha+2}(I,L^{\alpha+2})}\big)\|u-v\|_{L^{\alpha+2}(I,L^{\alpha+2})}\,.\end{aligned}$$

Setting

$$\mathcal{G}(u)(t) = \int_0^t \mathcal{T}(t-s)|u(s)|^\alpha u(s)ds\,,$$

it follows from (4.7.1) and Strichartz's estimates that

$$\mathcal{G}(u) \in C(I, L^2(\mathbb{R}^N)) \cap L^q(I, L^r(\mathbb{R}^N))$$

for every admissible pair (q,r), and that

$$(4.7.2)\qquad \|\mathcal{G}(u)\|_{L^q(I,L^r)} \le C\|u\|^{\alpha+1}_{L^{\alpha+2}(I,L^{\alpha+2})}$$

and

$$\begin{aligned}(4.7.3)\qquad &\|\mathcal{G}(u) - \mathcal{G}(v)\|_{L^q(I,L^r)} \le \\ &C\big(\|u\|^\alpha_{L^{\alpha+2}(I,L^{\alpha+2})} + \|v\|^\alpha_{L^{\alpha+2}(I,L^{\alpha+2})}\big)\|u-v\|_{L^{\alpha+2}(I,L^{\alpha+2})}\end{aligned}$$

for some constant C independent of I. We now proceed in four steps.

STEP 1. There exists $\delta > 0$ such that if $\varphi \in L^2(\mathbb{R}^N)$ satisfies

$$(4.7.4)\qquad \|\mathcal{T}(\cdot)\varphi\|_{L^{\alpha+2}(I,L^{\alpha+2})} < \delta$$

for some interval $I \subset \mathbb{R}$ containing 0, then there exists a unique solution $u \in C(I, L^2(\mathbb{R}^N)) \cap L^{\alpha+2}(I, L^{\alpha+2}(\mathbb{R}^N))$ of (4.1.3). In addition, $u \in L^q(I, L^r(\mathbb{R}^N))$ for any admissible pair (q,r). Moreover, if $\varphi, \psi \in L^2(\mathbb{R}^N)$ both satisfy (4.7.4) and if u, v denote the corresponding solutions of (4.1.3), then

$$(4.7.5)\qquad \|u-v\|_{L^\infty(I,L^2)} + \|u-v\|_{L^{\alpha+2}(I,L^{\alpha+2})} \le K\|\varphi-\psi\|_{L^2}$$

for some constant K independent of T, u, and v.

Indeed, fix $\delta > 0$, to be chosen later, and let $\varphi \in L^2(\mathbb{R}^N)$ satisfy (4.7.4). Consider the set

$$E = \big\{u \in L^{\alpha+2}(I, L^{\alpha+2}(\mathbb{R}^N));\, \|u\|_{L^{\alpha+2}(I,L^{\alpha+2})} \le 2\delta\big\}\,,$$

so that (E, d) is a complete metric space with $\mathrm{d}(u,v) = \|u-v\|_{L^{\alpha+2}(I,L^{\alpha+2})}$. For $u \in E$, set

$$\mathcal{H}(u)(t) = \mathcal{T}(t)\varphi + i\lambda\int_0^t \mathcal{T}(t-s)|u(s)|^\alpha u(s)ds$$

for $t \in I$. It follows easily from (4.7.2), (4.7.3), and (4.7.4) that if δ is small enough (independently of φ and I), then $\mathcal{H}$ is a strict contraction on E. Thus $\mathcal{H}$ has a fixed point u, which is the unique solution of (4.1.3) in E.

Applying (4.7.2) and Strichartz's estimates, we see that $u \in C(I, L^2(\mathbb{R}^N)) \cap L^q(I, L^r(\mathbb{R}^N))$ for every admissible pair (q,r). (4.7.5) follows easily from (4.7.3) and Strichartz's estimates.

We now show uniqueness (without any smallness assumption). Let $I \ni 0$ and consider two solutions $u, v \in L^{\alpha+2}(I, L^{\alpha+2}(\mathbb{R}^N))$ of (4.1.3). Uniqueness being a local property, we need only show that if $0 \in J \subset I$ with $|J|$ sufficiently small, then $u = v$ on J (see Step 2 of the proof of Theorem 4.6.1). We note that

$$\|u\|^{\alpha}_{L^{\alpha+2}(J,L^{\alpha+2})} + \|v\|^{\alpha}_{L^{\alpha+2}(J,L^{\alpha+2})} \to 0 \quad \text{as } |J| \downarrow 0. \tag{4.7.6}$$

It follows from (4.7.3) that

$$\|u - v\|_{L^{\alpha+2}(J,L^{\alpha+2})} \leq C\big(\|u\|^{\alpha}_{L^{\alpha+2}(J,L^{\alpha+2})} + \|v\|^{\alpha}_{L^{\alpha+2}(J,L^{\alpha+2})}\big)\|u - v\|_{L^{\alpha+2}(J,L^{\alpha+2})} .$$

We deduce from (4.7.6) that if $|J|$ is sufficiently small, then

$$C(\|u\|^{\alpha}_{L^{\alpha+2}(J,L^{\alpha+2})} + \|v\|^{\alpha}_{L^{\alpha+2}(J,L^{\alpha+2})}) < 1 ,$$

and we conclude that $\|u - v\|_{L^{\alpha+2}(J,L^{\alpha+2})} = 0$, i.e., $u = v$ on J.

STEP 2. For u as in Step 1, we show that if $\varphi \in H^1(\mathbb{R}^N)$, then $u \in C(I, H^1(\mathbb{R}^N))$. It follows from Corollary 4.3.4 that (4.1.3) has a solution $v \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N))$. This v is in particular an L^2 solution, so that by uniqueness u and v coincide as long as they are both defined. Therefore, we need only show that $I \subset (-T_{\min}, T_{\max})$. Assuming $I = (a, b)$, suppose $b > T_{\max}$. Since the equation (4.1.3) is invariant by space translations and since the gradient is the limit of the finite differences quotient, we deduce easily from (4.7.5) that

$$\|\nabla v\|_{L^{\infty}((0,T_{\max}),L^2)} \leq C\|\nabla\varphi\|_{L^2} ,$$

which contradicts the blowup alternative for the H^1 solutions. Thus $T_{\max} > b$ and one shows by the same argument that $a > -T_{\min}$.

STEP 3. For u as in Step 1, we show that there is conservation of charge. Indeed, let $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N)$, with $\varphi_n \in H^1(\mathbb{R}^N)$. It follows from Strichartz's estimates that for n large enough, φ_n satisfies (4.7.4), so that by (4.7.5), $u_n \to u$ in $C(I, L^2(\mathbb{R}^N))$, where u_n is the solution associated to φ_n. On the other hand, we deduce from Step 2 that u_n is an H^1 solution, so that $\|u_n(t)\|_{L^2} = \|\varphi_n\|_{L^2}$ for all $t \in I$. Passing to the limit as $n \to \infty$, we obtain $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in I$.

STEP 4. Let $\varphi \in L^2(\mathbb{R}^N)$. Since $\mathcal{T}(\cdot)\varphi \in L^{\alpha+2}(\mathbb{R}, L^{\alpha+2}(\mathbb{R}^N))$ by Strichartz's estimates, we have $\|\mathcal{T}(\cdot)\varphi\|_{L^{\alpha+2}((-T,T),L^{\alpha+2})} \to 0$ as $T \downarrow 0$. Therefore, (4.7.4) is satisfied for T small enough, and we can apply Step 1 to construct a unique local solution. Using uniqueness, we define the maximal solution $u \in C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N))$ (as in the proof of Theorem 3.3.9). It remains to establish the blowup alternative and the continuous dependence. We argue for positive times, the argument for negative times being the same. We show the blowup alternative by contradiction. Suppose that $T_{\max} < \infty$ and that $\|u\|_{L^{\alpha+2}((0,T_{\max}),L^{\alpha+2})} < \infty$. Let $0 \leq t \leq t + s < T_{\max}$. It follows that

$$\mathcal{T}(s)u(t) = u(t + s) - i\lambda \int_0^s \mathcal{T}(s - \sigma)|u(t + \sigma)|^{\alpha}u(t + \sigma)d\sigma .$$

By (4.7.2), there exists C such that

$$\|\mathcal{T}(\cdot)u(t)\|_{L^{\alpha+2}((0,T_{\max}-t),L^{\alpha+2})} \leq \|u\|_{L^{\alpha+2}((t,T_{\max}),L^{\alpha+2})} + C\|u\|^{\alpha+1}_{L^{\alpha+2}((t,T_{\max}),L^{\alpha+2})} .$$

Therefore for t close enough to $T_{\max}$,

$$\|\mathcal{T}(\cdot)u(t)\|_{L^{\alpha+2}((0,T_{\max}-t),L^{\alpha+2})} < \frac{\delta}{2}.$$

By Step 1, u can be extended after $T_{\max}$, which is a contradiction. This shows that

$$\|u\|_{L^{\alpha+2}((0,T_{\max}),L^{\alpha+2})} = \infty.$$

Let now (q,r) be an admissible pair such that $r > \alpha+2$. It follows from Hölder's inequality that for any $T < T_{\max}$,

$$\|u\|_{L^{\alpha+2}((0,T),L^{\alpha+2})} \le \|u\|^{\mu}_{L^{\infty}((0,T),L^2)}\|u\|^{1-\mu}_{L^q((0,T),L^r)} \le \|\varphi\|^{\mu}_{L^2}\|u\|^{1-\mu}_{L^q((0,T),L^r)},$$

with $\mu = \frac{2(r-\alpha-2)}{(\alpha+2)(r-2)}$. Letting $T \uparrow T_{\max}$, we obtain $\|u\|_{L^q((0,T_{\max}),L^r)} = \infty$.

Finally, we show the continuous dependence. Consider $T < T_{\max}$. Since $u \in C([0,T],L^2(\mathbb{R}^N))$, it follows from Strichartz's estimates and an obvious compactness argument that there exists $\tau > 0$ such that

$$\|\mathcal{T}(\cdot)u(t)\|_{L^{\alpha+2}((0,\tau),L^{\alpha+2})} \le \frac{\delta}{2}$$

for all $t \in [0,T]$, where δ is as in (4.7.4). Fix an integer n such that $T \le n\tau$, let $K \ge 1$ be the constant in (4.7.5), and let M be such that $\|\mathcal{T}(\cdot)v\|_{L^{\alpha+2}(\mathbb{R},L^{\alpha+2})} \le M\|v\|_{L^2}$. Let $\varepsilon > 0$ be small enough so that $MK^{n-1}\varepsilon < \delta/2$. We claim that if $\|\varphi-\psi\|_{L^2} \le \varepsilon$, then $T_{\max}(\psi) > T$ and $\|u-v\|_{C([0,T],L^2)} + \|u-v\|_{L^{\alpha+2}((0,T),L^{\alpha+2})} \le nK^n\|\varphi-\psi\|_{L^2}$, where v is the solution corresponding to the initial value ψ. Indeed, if $\|\varphi-\psi\|_{L^2} \le \varepsilon$, then

$$\begin{aligned}\|\mathcal{T}(\cdot)\psi\|_{L^{\alpha+2}((0,T/n),L^{\alpha+2})} &\le \|\mathcal{T}(\cdot)\varphi\|_{L^{\alpha+2}((0,T/n),L^{\alpha+2})} \\ &\quad + \|\mathcal{T}(\cdot)(\varphi-\psi)\|_{L^{\alpha+2}((0,T/n),L^{\alpha+2})} \\ &\le \frac{\delta}{2} + M\varepsilon < \delta.\end{aligned}$$

Therefore, it follows from Step 1 that $T_{\max}(\psi) > T/n$ and that

$$\|u-v\|_{C([0,T/n],L^2)} + \|u-v\|_{L^{\alpha+2}((0,T/n),L^{\alpha+2})} \le K\|\varphi-\psi\|_{L^2}.$$

In particular, $\|u(T/n) - v(T/n)\|_{L^2} \le K\varepsilon$. The claim follows by iterating this argument n times. This completes the proof. □

REMARK 4.7.3. The blowup alternative in Theorem 4.7.1 is not very handy, since it does not concern the L^2 norm of u. In fact, despite of the conservation of charge when $\lambda \in \mathbb{R}$, $T_{\min}$ and $T_{\max}$ can be finite in some cases. For example, assume $\lambda > 0$ and let $\varphi \in H^1(\mathbb{R}^N)$ be such that $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$ and $E(\varphi) < 0$. It follows from Theorem 6.5.4 below that u blows up in H^1 for both $t > 0$ and $t < 0$. Therefore, $T_{\max} < \infty$ and $T_{\min} < \infty$, by Theorem 4.7.1(iii).

REMARK 4.7.4. We conjecture that if $\lambda < 0$, then $T_{\min} = T_{\max} = \infty$ for all $\varphi \in L^2(\mathbb{R}^N)$. However, we only have the following partial result. Assume $\lambda < 0$, and suppose that $\varphi \in L^2(\mathbb{R}^N)$ is such that $x\varphi(x) \in L^2(\mathbb{R}^N)$. It follows that $T_{\max} = T_{\min} = \infty$ and, in addition, $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ for every admissible pair (q,r). Indeed, consider a sequence $(\varphi_n)_{n\ge 0} \subset H^1(\mathbb{R}^N)$, with $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $L^2(\mathbb{R}^N)$ and $x\varphi_n(x)$ bounded in $L^2(\mathbb{R}^N)$. The corresponding solutions satisfy $u_n \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ and $|\cdot|u_n \in C(\mathbb{R}, L^2(\mathbb{R}^N))$ (see Lemma 6.5.2 below), and from the pseudoconformal conservation law (see Theorem 7.2.1 below), we see that $\|u_n(t)\|_{L^{\alpha+2}}^{\alpha+2} \le Ct^{-2}$ for all $t \in \mathbb{R}$. By continuous dependence, this implies that $\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \le Ct^{-2}$ for a.a. $t \in (-T_{\min}, T_{\max})$. In particular, we see that if $T_{\max} < \infty$, then $u \in L^{\alpha+2}((0,T_{\max}), L^{\alpha+2}(\mathbb{R}^N))$, which contradicts the blowup alternative (note that $u \in L^{\alpha+2}((0,T), L^{\alpha+2}(\mathbb{R}^N))$ for all $0 < T < T_{\max}$). We see as well that $T_{\min} = \infty$. In addition, it is clear that the above estimate implies that $u \in L^{\alpha+2}(\mathbb{R}, L^{\alpha+2}(\mathbb{R}^N))$. The estimate for an arbitrary admissible pair follows easily from Strichartz's estimates.

REMARK 4.7.5. There exists $\eta > 0$ such that if

$$\|\mathcal{T}(\cdot)\varphi\|_{L^{\alpha+2}(\mathbb{R},L^{\alpha+2})} < \eta\,, \tag{4.7.7}$$

then $T_{\min} = T_{\max} = \infty$. Moreover, $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ for every admissible pair (q,r). This follows easily from Step 1 of the proof of Theorem 4.7.1 (see in particular (4.7.4)). However, this conclusion does not hold in general for large data. Indeed, if $\lambda > 0$, there exist nontrivial solutions (standing waves) of the form $u(t,x) = e^{i\omega t}\phi(x)$, with $\phi \in H^1(\mathbb{R}^N)$, $\phi \ne 0$ (see Section 7.2). These solutions obviously do not belong to $L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ if $q < \infty$. On the other hand, by Strichartz's estimates, (4.7.7) is satisfied if $\|\varphi\|_{L^2} < \mu$ for μ small enough.

4.8. H^2 Solutions

In this section we construct H^2 solutions by a fixed-point argument, and we follow the proof of Kato [203, 204]. See also Y. Tsutsumi [340]. We note that obtaining H^2 estimates by differentiating twice the equation in space would require that the nonlinearity is sufficiently smooth (see Section 4.9 below). Instead, we differentiate the equation once in time, and then deduce H^2 estimates by the equation.

Let $g : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. Assume there exist $0 \le s < 2$ and $2 \le r, \rho < 2N/(N-2)$ ($2 \le r,\rho < \infty$ if $N = 1$) such that

$$g \in C(H^s(\mathbb{R}^N), L^2(\mathbb{R}^N)) \text{ is bounded on bounded sets} \tag{4.8.1}$$

and

$$\|g(u) - g(v)\|_{L^{\rho'}} \le L(M)\|u - v\|_{L^r} \tag{4.8.2}$$

for all $u, v \in H^2(\mathbb{R}^N)$ such that $\|u\|_{H^s}, \|v\|_{H^s} \le M$.

THEOREM 4.8.1. *Let $g = g_1 + \cdots + g_k$, where each of the g_j's satisfies the conditions (4.8.1)–(4.8.2) for some exponent s_j, r_j, ρ_j and some function $L_j(M)$. For every $\varphi \in H^2(\mathbb{R}^N)$, there exist $T_{\max}, T_{\min} > 0$ and a unique, maximal solution $u \in C((-T_{\min}, T_{\max}), H^2(\mathbb{R}^N)) \cap C^1((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N))$ of (4.1.1). Moreover, the following properties hold:*

(i) *$u \in W^{1,q}_{\mathrm{loc}}((-T_{\min}, T_{\max}), L^r(\mathbb{R}^N))$ for every admissible pair (q,r).*
(ii) (Blowup alternative) *If $T_{\max} < \infty$ (respectively, $T_{\min} < \infty$), then $\|u(t)\|_{H^2} \to \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow -T_{\min}$).*
(iii) *u depends continuously on φ in the following sense. There exists $T > 0$ depending on $\|\varphi\|_{H^2}$ such that if $\varphi_n \to \varphi$ in $H^2(\mathbb{R}^N)$ and if u_n is the corresponding solution of (4.1.1), then u_n is defined on $[-T,T]$ for*

n large enough and $\|u_n\|_{L^\infty((-T,T),H^2)}$ is bounded. Moreover $u_n \to u$ in $C([-T,T],H^s(\mathbb{R}^N))$ as $n\to\infty$ for all $0\le s<2$.

(iv) *If $(g(w),iw)_{L^2}=0$ for all $w\in H^2(\mathbb{R}^N)$, then there is conservation of charge; i.e., $\|u(t)\|_{L^2}=\|\varphi\|_{L^2}$ for all $t\in(-T_{\min},T_{\max})$.*

(v) *If for every j there exists $G_j\in C^1(H^2(\mathbb{R}^N),\mathbb{R})$ such that $g_j=G_j'$, then there is conservation of energy; i.e., $E(u(t))=E(\varphi)$ for all $t\in(-T_{\min},T_{\max})$, where E is defined by (3.3.9) with $G=G_1+\cdots+G_k$.*

For the proof of Theorem 4.8.1 we will use the following lemmas.

LEMMA 4.8.2. *Let $\varphi\in H^2(\mathbb{R}^N)$ and set $v(t)=\mathcal{T}(t)\varphi$. It follows that $v\in C(\mathbb{R},H^2(\mathbb{R}^N))$ and $\|v\|_{L^\infty(\mathbb{R},H^2)}\le\|\varphi\|_{H^2}$. Moreover, if (q,r) is any admissible pair, then $v\in C^1(\mathbb{R},L^2(\mathbb{R}^N))\cap W^{1,q}(\mathbb{R},L^r(\mathbb{R}^N))$ and there exists C independent of φ such that $\|v\|_{W^{1,q}(\mathbb{R},L^r)}\le C\|\varphi\|_{H^2}$.*

PROOF. The result follows from Strichartz's estimates and the identity $v_t=i\Delta v=i\mathcal{T}(t)\Delta\varphi$. ☐

LEMMA 4.8.3. *Let g satisfy (4.8.1). It follows that $g(u)\in L^\infty(J,L^2(\mathbb{R}^N))$ for every interval $J\subset\mathbb{R}$ and every $u\in L^\infty(J,H^s(\mathbb{R}^N))$. Moreover, there exists a continuous function $K:(0,\infty)\to(0,\infty)$ such that*

$$\|g(u)\|_{L^\infty(J,L^2)}\le K(M) \tag{4.8.3}$$

for every $u\in L^\infty(J,H^s(\mathbb{R}^N))$ with $\|u\|_{L^\infty(J,H^s)}\le M$.

PROOF. This is an immediate consequence of (4.8.1). ☐

LEMMA 4.8.4. *Let g satisfy (4.8.1)–(4.8.2). Let q and γ be such that (q,r) and (γ,ρ) are admissible pairs. It follows that $\frac{d}{dt}g(u)\in L^{\gamma'}(J,L^{\rho'}(\mathbb{R}^N))$. Moreover,*

$$\left\|\frac{d}{dt}g(u)\right\|_{L^{\gamma'}(J,L^{\rho'})}\le L(M)\|u_t\|_{L^{\gamma'}(J,L^r)}, \tag{4.8.4}$$

and in particular

$$\left\|\frac{d}{dt}g(u)\right\|_{L^{\gamma'}(J,L^{\rho'})}\le L(M)|J|^{1-\frac{1}{\gamma}-\frac{1}{q}}\|u_t\|_{L^q(J,L^r)} \tag{4.8.5}$$

for every interval $J\subset\mathbb{R}$ and every $u\in L^\infty(J,H^s(\mathbb{R}^N))$ with $u_t\in L^q(JL^r(\mathbb{R}^N))$ and $\|u\|_{L^\infty(J,H^s)}\le M$.

PROOF. (4.8.4) is a consequence of (4.8.2) and Proposition 1.3.12 (applied to $f(t)=g(u(t))-g(u(t_0))$, where $t_0\in J$ is fixed). ☐

LEMMA 4.8.5. *Let $J\ni 0$ be a bounded interval, let (γ,ρ) be an admissible pair, and consider $f\in L^\infty(J,L^2(\mathbb{R}^N))$ such that $f_t\in L^{\gamma'}(J,L^{\rho'}(\mathbb{R}^N))$. If*

$$v(t)=i\int_0^t\mathcal{T}(t-s)f(s)ds\quad\textit{for all } t\in J,$$

then $v \in L^\infty(J, H^2(\mathbb{R}^N)) \cap C^1(J, L^2(\mathbb{R}^N)) \cap W^{1,a}(J, L^b(\mathbb{R}^N))$ *for every admissible pair* (a, b), *and*

$$\|v\|_{L^a(J,L^b)} \le C\|f\|_{L^1(J,L^2)}, \tag{4.8.6}$$

$$\|v_t\|_{L^a(J,L^b)} \le C\|f(0)\|_{L^2} + C\|f_t\|_{L^{\gamma'}(J,L^{\rho'})}, \tag{4.8.7}$$

$$\|\Delta v\|_{L^\infty(J,L^2)} \le \|f\|_{L^\infty(J,L^2)} + C\|f(0)\|_{L^2} + C\|f_t\|_{L^{\gamma'}(J,L^{\rho'})}, \tag{4.8.8}$$

where C *is independent of* J *and* f. *If, in addition,* $f \in C(J, L^2(\mathbb{R}^N))$, *then* $v \in C(J, H^2(\mathbb{R}^N))$.

PROOF. Since $L^{\rho'}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$, we see that $f \in W^{1,1}(J, H^{-1}(\mathbb{R}^N)) \hookrightarrow C(J, H^{-1}(\mathbb{R}^N))$. It follows that

$$\begin{aligned} v_t(t) = i\frac{d}{dt}\int_0^t \mathcal{T}(s)f(t-s)ds &= i\mathcal{T}(t)f(0) + i\int_0^t \mathcal{T}(s)f_t(t-s)ds \\ &= i\mathcal{T}(t)f(0) + i\int_0^t \mathcal{T}(t-s)f_t(s)ds. \end{aligned}$$

(The above formula is trivial if $f \in C^1(J, H^{-1}(\mathbb{R}^N))$ and follows by density for $f \in W^{1,1}(J, H^{-1}(\mathbb{R}^N))$.) The result is then a consequence of Strichartz's estimates. Note that we use the equation $iv_t + \Delta v + f = 0$ to obtain (4.8.8) and the H^2 regularity. □

PROOF OF THEOREM 4.8.1. We first note that by Lemma 4.2.8, the problems (4.1.1) and (4.1.2) are equivalent. We then proceed in four steps.

STEP 1. Local existence. We construct local solutions by a fixed-point argument. We set

$$s = \max\{s_1, \dots, s_k\} < 2,$$
$$r = \max\{r_1, \dots, r_k, \rho_1, \dots, \rho_k\},$$

and we consider the corresponding admissible pair (q, r). Given $M, T > 0$ to be chosen later, we set $I = (-T, T)$ and we consider

$$\begin{aligned} E = \{&u \in L^\infty(I, H^s(\mathbb{R}^N)) \cap W^{1,\infty}(I, L^2(\mathbb{R}^N)) \cap W^{1,q}(I, L^r(\mathbb{R}^N)); \\ &u(0) = \varphi \text{ and } \|u\|_{L^\infty(I,H^s)} + \|u\|_{W^{1,\infty}(I,L^2)} + \|u\|_{W^{1,q}(I,L^r)} \le M\}. \end{aligned} \tag{4.8.9}$$

It follows that (E, d) is a complete metric space, where the distance d is defined by

$$\mathrm{d}(u, v) = \|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^q(I,L^r)}. \tag{4.8.10}$$

(This is established by the argument of Step 1 of the proof of Theorem 4.4.1, using Remark 1.3.13(ii) in addition to Theorem 1.2.5.) Moreover, $E \neq \varnothing$ since $u(t) \equiv \varphi$ clearly belongs to E. We now consider $\mathcal{H}$ defined by

$$\mathcal{H}(u)(t) = \mathcal{T}(t)\varphi + \mathcal{G}(u)(t),$$

where

$$\mathcal{G}(u)(t) = i\int_0^t \mathcal{T}(t-s)g(u(s))ds$$

for all $u \in E$ and all $t \in I$. Since $g(u) \in L^\infty(I, L^2(\mathbb{R}^N))$ by Lemma 4.8.3, it follows that $\mathcal{G}(u) \in C(I, L^2(\mathbb{R}^N))$ is well defined.

We first note that by (4.8.3),

$$\|g_j(u)\|_{L^\infty(J,L^2)} \le K(M) \tag{4.8.11}$$

for every $u \in E$ (with $K = \max\{K_1, \dots K_k\}$). Next, given $1 \le j \le k$, we consider q_j, γ_j such that (q_j, r_j) and (γ_j, ρ_j) are admissible pairs. It follows from Hölder's inequality that

$$\|w\|_{L^{q_j}(I,L^{r_j})} \le \|w\|_{L^\infty(I,L^2)}^{\frac{2(r-r_j)}{r_j(r-2)}} \|w\|_{L^q(I,L^r)}^{\frac{r(r_j-2)}{r_j(r-2)}}.$$

In particular, if $u \in E$, then $u \in W^{1,q_j}(I, L^{r_j}(\mathbb{R}^N))$ for all $1 \le j \le k$ and

$$\|u\|_{W^{1,q_j}(I,L^{r_j})} \le M^{\frac{2(r-r_j)}{r_j(r-2)}} M^{\frac{r(r_j-2)}{r_j(r-2)}} = M. \tag{4.8.12}$$

Therefore, we deduce from (4.8.5) and (4.8.12) that

$$\left\| \frac{d}{dt} g_j(u) \right\|_{L^{\gamma'_j}(I,L^{\rho'_j})} \le F(M) T^{1-\frac{1}{\gamma_j}-\frac{1}{q_j}} \tag{4.8.13}$$

with $F(M) = M \max\{L_1(M), \dots, L_k(M)\}$. Applying now Lemma 4.8.5 to each of the g_j's, we conclude that $\mathcal{G}(u) \in L^\infty(I, H^2(\mathbb{R}^N)) \cap W^{1,a}(I, L^b(\mathbb{R}^N))$ for every admissible pair (a, b) and every $u \in E$. Moreover, it follows from (4.8.6) and (4.8.11) that

$$\|\mathcal{G}(u)\|_{L^\infty(I,L^2)} \le C_0 T K(M) \tag{4.8.14}$$

for some C_0 independent of M and T. Similarly, it follows from (4.8.6), (4.8.7), (4.8.11), and (4.8.13) that, by possibly choosing C_0 larger,

$$\begin{aligned} &\|\mathcal{G}(u)\|_{W^{1,\infty}(I,L^2)} + \|\mathcal{G}(u)\|_{W^{1,q}(I,L^r)} \le \\ &\qquad C_0\Big(TK(M) + \sum_{j=1}^k \|g_j(\varphi)\|_{L^2} + F(M) \sum_{j=1}^k T^{1-\frac{1}{\gamma_j}-\frac{1}{q_j}} \Big). \end{aligned} \tag{4.8.15}$$

Also, it follows from (4.8.8), (4.8.14), (4.8.11), and (4.8.13) that, by possibly choosing C_0 larger,

$$\begin{aligned} &\|\mathcal{G}(u)\|_{L^\infty(I,H^2)} \le \\ &\qquad C_0\Big((1+T)K(M) + \sum_{j=1}^k \|g_j(\varphi)\|_{L^2} + F(M) \sum_{j=1}^k T^{1-\frac{1}{\gamma_j}-\frac{1}{q_j}} \Big) \end{aligned} \tag{4.8.16}$$

with C independent of M and T. Applying now Lemma 4.8.2, we deduce from (4.8.14)–(4.8.16) that, by possibly choosing C_0 larger,

$$\|\mathcal{H}(u)\|_{L^\infty(I,L^2)} \le C_0(TK(M) + \|\varphi\|_{H^2}), \tag{4.8.17}$$

$$\begin{aligned} &\|\mathcal{H}(u)\|_{W^{1,\infty}(I,L^2)} + \|\mathcal{H}(u)\|_{W^{1,q}(I,L^r)} \le \\ &\quad C_0\Big(\|\varphi\|_{H^2} + \sum_{j=1}^k \|g_j(\varphi)\|_{L^2} + TK(M) + F(M) \sum_{j=1}^k T^{1-\frac{1}{\gamma_j}-\frac{1}{q_j}} \Big), \end{aligned} \tag{4.8.18}$$

and

$$(4.8.19)\qquad \|\mathcal{H}(u)\|_{L^\infty(I,H^2)} \le C_0\Big(\|\varphi\|_{H^2} + \sum_{j=1}^{k} \|g_j(\varphi)\|_{L^2} + (1+T)K(M) + F(M)\sum_{j=1}^{k} T^{1-\frac{1}{\gamma_j}-\frac{1}{q_j}}\Big).$$

We now let

$$M = 4C_0\Big(\|\varphi\|_{H^2} + \sum_{j=1}^{k} \|g_j(\varphi)\|_{L^2}\Big).$$

It follows from (4.8.18) that, by choosing T sufficiently small,

$$\|\mathcal{H}(u)\|_{W^{1,\infty}(I,L^2)} + \|\mathcal{H}(u)\|_{W^{1,q}(I,L^r)} \le \frac{M}{2}.$$

Next, using (4.8.17), (4.8.19), and the elementary interpolation estimate

$$(4.8.20)\qquad \|u\|_{H^s} \le \|u\|_{H^2}^{\frac{s}{2}} \|u\|_{L^2}^{\frac{2-s}{2}},$$

we see that, by choosing T possibly smaller,

$$\|\mathcal{H}(u)\|_{L^\infty(I,H^s)} \le \frac{M}{2}.$$

It follows that $\mathcal{H} : E \to E$. A similar, though simpler, argument shows that, after choosing T possibly smaller, $\mathcal{H}$ is a strict contraction on (E, d). Therefore, $\mathcal{H}$ has a fixed point $u \in E$, which is a solution of (4.1.1). It remains to show that $u \in C(I, H^2(\mathbb{R}^N)) \cap C^1(I, L^2(\mathbb{R}^N))$ and that $u \in W^{1,a}(I, L^b(\mathbb{R}^N))$ for every admissible pair (a, b). This follows from Lemmas 4.8.2 and 4.8.5. The only point which is not immediate is that $g(u) \in C(I, L^2(\mathbb{R}^N))$. To see this, we observe that $u \in C(I, L^2(\mathbb{R}^N))$. Moreover, by (4.8.19), $u \in L^\infty(I, H^2(\mathbb{R}^N))$. Applying the inequality (4.8.20), we deduce that $u \in C(I, H^s(\mathbb{R}^N))$, so that $g(u) \in C(I, L^2(\mathbb{R}^N))$ by (4.8.1).

STEP 2. Uniqueness, the blowup alternative, and continuous dependence. Uniqueness follows from Proposition 4.2.9. For the blowup alternative, we proceed as in the proof of Theorem 3.3.9: using uniqueness, we define the maximal solution; and since the solution u of Step 1 is constructed on an interval depending on $\|\varphi\|_{H^2}$ (as is easily verified), we deduce the blowup alternative. Arguing as in Remark 4.4.5, we see that there is boundedness in $L^\infty((-T,T), H^2(\mathbb{R}^N))$ and continuous dependence in $L^\infty((-T,T), L^2(\mathbb{R}^N))$ for some $T > 0$ depending on $\|\varphi\|_{H^2}$. Applying (4.8.20), we deduce the continuous dependence in $L^\infty((-T,T), H^s(\mathbb{R}^N))$ for every $s < 2$.

STEP 3. Property (iv). Since equation (4.1.1) makes sense in $L^2(\mathbb{R}^N)$ for all $t \in (-T_{\min}, T_{\max})$, we may multiply it (in the L^2 scalar product) by iu, and we obtain

$$(u_t, u)_{L^2} = (-\Delta u, iu)_{L^2} + (g(u), iu)_{L^2} = 0.$$

Therefore,

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2(u_t, u)_{H^{-1},H^1} = 0,$$

and the result follows.

STEP 4. Property (v). Since equation (4.1.1) makes sense in $L^2(\mathbb{R}^N)$ for all $t \in (-T_{\min}, T_{\max})$, we may multiply it (in the L^2 scalar product) by u_t, and we obtain

$$(iu_t, u_t)_{L^2} = (-\Delta u, u_t)_{L^2} + (g(u), u_t)_{L^2}.$$

Since $(iu_t, u_t)_{L^2} = 0$, we deduce that $\frac{d}{dt}E(u(t)) = 0$ and the result follows. Note that the identity

$$\frac{d}{dt}G(u) = (g(u), u_t)_{L^2}$$

holds in principle for $u \in C^1((-T_{\min}, T_{\max}), H^2(\mathbb{R}^N))$. However, it is equivalent to

$$G(u(t)) = G(u(0)) + \int_0^t (g(u(s)), u_t(s))_{L^2}\, ds \quad \text{for all } t \in (-T_{\min}, T_{\max}).$$

This last identity is easily established for u as in the statement by an obvious density argument. This completes the proof. □

We now give an application of Theorem 4.8.1 in a model case.

COROLLARY 4.8.6. *Let $V \in L^\delta(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\delta \geq 1$, $\delta > N/2$ and $W \in L^\sigma(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\sigma \geq 1$, $\sigma > N/6$. Let $f \in C(\mathbb{C}, \mathbb{C})$ satisfy $f(0) = 0$ and*

$$|f(u) - f(v)| \leq L(|u| + |v|)|u - v|$$

for all $u, v \in \mathbb{C}$ with $L \in C([0,\infty), \mathbb{R})$ if $N \leq 3$ and $L(t) \leq C(1 + t^\alpha)$ with $\alpha \geq 0$ and $(N-4)\alpha < 4$ if $N \geq 4$. Finally, set

$$g(u) = Vu + f(u(\cdot)) + (W \star |u|^2)u.$$

It follows that all the conclusions of Theorem 4.8.1 hold. If, in addition, $f \in C^1(\mathbb{C}, \mathbb{C})$ (in the real sense), there is continuous dependence in a stronger sense as follows. The mappings $\varphi \mapsto T_{\min}, T_{\max}$ are lower semicontinuous $H^2(\mathbb{R}^N) \to (0, \infty]$. If $\varphi_n \to \varphi$ in $H^2(\mathbb{R}^N)$ and if u_n denotes the solution of (4.1.1) with the initial value φ_n, then $u_n \to u$ in $C([-S, T], H^2(\mathbb{R}^N))$ and in $W^{1,q}((-S,T), L^r(\mathbb{R}^N))$ for every admissible pair (q, r) and every $-T_{\min} < -S < 0 < T < T_{\max}$.

PROOF. The fact that g satisfies the assumptions of Theorem 4.8.1 follows easily from the estimates of Section 3.2. The stronger continuous dependence property when f is C^1 is proved by the argument used in Step 3 of the proof of Theorem 4.4.1, except that we differentiate the equation in time instead of space. Doing so we first obtain continuous dependence in $W^{1,q}(I, L^r(\mathbb{R}^N))$ for every admissible pair (q, r), then in $C(I, H^2(\mathbb{R}^N))$ by the equation. Note that the term $\partial_t[V(u_n - u) + (W \star |u_n|^2)u_n - (W \star |u|^2)u]$ is easily estimated by using the formula $\partial_t[Vu + (W \star |u|^2)u] = V\partial_t u + (W \star |u|^2)\partial_t u + (W \star u\partial_t\overline{u})u + (W \star \partial_t u\overline{u})u$ together with Hölder and Young's inequalities. □

REMARK 4.8.7. Here are some comments on Corollary 4.8.6.

(i) If g is in Corollary 4.8.6, then there is conservation of charge provided V and W are real valued and $\operatorname{Im}(f(z)\overline{z}) = 0$ for all $z \in \mathbb{C}$.

(ii) If g is in Corollary 4.8.6, then there is conservation of energy provided V and W are real valued, W is even and $f(z) = z\theta(|z|)/|z|$ for all $z \neq 0$ with $\theta : (0, \infty) \to \mathbb{R}$.

(iii) Corollary 4.8.6 applies in particular to the case $g(u) = \lambda|u|^\alpha u$ with $\lambda \in \mathbb{C}$, $\alpha \geq 0$, and $(N-4)\alpha < 4$. A similar result holds in the H^2 "critical" case $N \geq 5$ and $\alpha = 4/(N-4)$; see Cazenave and Weissler [70], theorem 1.4.

4.9. H^s Solutions, $s < N/2$

In this section we study the existence of solutions of the nonlinear Schrödinger equation (4.1.1) in the Sobolev space $H^s(\mathbb{R}^N)$ for $s \geq 0$. (We note that the cases $s = 0$, $s = 1$, and $s = 2$ have been studied in the preceding sections.) In principle, a local existence result can be established by a fixed point argument by using Strichartz's estimates in the Sobolev spaces $H^{s,r}(\mathbb{R}^N)$ (see Remark 2.3.8) along with estimates of $\|g(u)\|_{H^{s,r}}$. This is the program carried out by Kato [206] and it makes use of a delicate estimate of $\|g(u)\|_{H^{s,r}}$ (Lemma A3 in [206]). Here, we rather use the Besov space $B^s_{r,2}(\mathbb{R}^N)$ as an auxiliary space because estimates of $\|g(u)\|_{B^s_{r,2}}$ are much simpler to obtain (see Cazenave and Weissler [70]). We also note that, regardless of the auxiliary space, the case $s > 1$ tends to be more complicated as it requires more regularity of the nonlinearity. Thus we restrict ourselves to $s \in (0,1)$ and we comment on the case $s > 1$ at the end of the section. Also, consider the case $s < N/2$ (we comment on the limiting case $s = N/2$ at the end of the section). When $s > N/2$, the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ allows a simpler treatment of the equation (provided the nonlinearity is sufficiently smooth, though); see Section 4.10 below. Note that if we consider a nonlinearity of the form $g(u) = \lambda|u|^\alpha u$, then the results of this section provide local existence in $H^s(\mathbb{R}^N)$ under the condition $\alpha \leq 4/(N-2s)$ (and, also, a regularity assumption). Thus, in principle, any power $\alpha > 0$ can be handled in the H^s framework with $s < N/2$. Furthermore, and for the sake of simplicity, we only consider local nonlinearities. The first result of this section is the following.

THEOREM 4.9.1. *Let $0 < s < \min\{1, N/2\}$. Let $g \in C(\mathbb{C},\mathbb{C})$, and assume that $g(0) = 0$ and that there exists*

$$0 \leq \alpha < \frac{4}{N-2s} \tag{4.9.1}$$

such that

$$|g(u) - g(v)| \leq C(1 + |u|^\alpha + |v|^\alpha)|u - v| \quad \textit{for all } u, v \in \mathbb{C}. \tag{4.9.2}$$

Let (γ, ρ) be the admissible pair defined by

$$\rho = \frac{N(\alpha+2)}{N+s\alpha}, \quad \gamma = \frac{4(\alpha+2)}{\alpha(N-2s)}. \tag{4.9.3}$$

Given $\varphi \in H^s(\mathbb{R}^N)$, there exist $T_{\max}, T_{\min} \in (0,\infty]$ and a unique, maximal solution $u \in C((-T_{\min}, T_{\max}), H^s(\mathbb{R}^N)) \cap L^\gamma_{\mathrm{loc}}((-T_{\min}, T_{\max}), B^s_{\rho,2}(\mathbb{R}^N))$ of the problem (4.1.1). Moreover, the following properties hold:

(i) *$u \in L^q_{\mathrm{loc}}((-T_{\min}, T_{\max}), B^s_{r,2}(\mathbb{R}^N))$ for every admissible pair (q, r).*

(ii) *(Blowup alternative) If $T_{\max} < \infty$ (respectively, if $T_{\min} < \infty$), then $\|u(t)\|_{H^s} \to \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow -T_{\min}$).*

(iii) *u depends continuously on φ in the following sense. There exists $0 < T < T_{\max}, T_{\min}$ such that if $\varphi_n \to \varphi$ in $H^s(\mathbb{R}^N)$ and if u_n denotes the solution* *of* (4.1.1) *with the initial value* φ_n, *then* $0 < T < T_{\max}(\varphi_n), T_{\min}(\varphi_n)$ *for all sufficiently large* n *and* u_n *is bounded in* $L^q((-T,T), B^s_{r,2}(\mathbb{R}^N))$ *for any admissible pair* (q,r). *Moreover,* $u_n \to u$ *in* $L^q((-T,T), L^r(\mathbb{R}^N))$ *as* $n \to \infty$. *In particular,* $u_n \to u$ *in* $C([-T,T], H^{s-\varepsilon}(\mathbb{R}^N))$ *for all* $\varepsilon > 0$.

REMARK 4.9.2. We decompose $g = g_1 + g_2$ where $g_1(0) = g_2(0) = 0$, g_1 is globally Lipschitz $\mathbb{C} \to \mathbb{C}$, and

$$|g_2(z_1) - g_2(z_2)| \le C(|z_1|^\alpha + |z_2|^\alpha)|z_1 - z_2| \tag{4.9.4}$$

for all $z_1, z_1 \in \mathbb{C}$ (see Section 3.2). Let now $I \ni 0$ be a bounded interval and let $u \in L^\infty(I, H^s(\mathbb{R}^N)) \cap L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N))$. In particular, $u \in L^\infty(I, L^2(\mathbb{R}^N))$ so that $g_1(u) \in L^\infty(I, L^2(\mathbb{R}^N))$. Next, we note that $\rho \ge 2$ so that $B^s_{\rho,2}(\mathbb{R}^N) \hookrightarrow H^{s,\rho}(\mathbb{R}^N)$. Since $2s < N$, we see $s\rho < N$, and it follows that

$$B^s_{\rho,2}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \tag{4.9.5}$$

for all $\rho \le p \le N\rho/(N - s\rho) = N(\alpha+2)/(N-2s)$. Using (4.9.4) we easily deduce that $g_2(u) \in L^\gamma(I, L^p(\mathbb{R}^N))$ for all

$$\max\left\{1, \frac{\rho}{\alpha+1}\right\} \le p \le \frac{N(\alpha+2)}{(N-2s)(\alpha+1)}.$$

In particular, we see that $g_2(u) \in L^\gamma(I, L^{\rho'}(\mathbb{R}^N))$. Since $L^{\rho'}(\mathbb{R}^N) \hookrightarrow H^{-\sigma}(\mathbb{R}^N)$ for $\sigma = \alpha(N-2s)/2(\alpha+2)$, we deduce that $g_2(u) \in L^\gamma(I, H^{-\sigma}(\mathbb{R}^N))$. Therefore, $g(u) \in L^\gamma(I, H^{-\sigma}(\mathbb{R}^N))$, so that equation (4.1.1) makes sense. Moreover, we see that equation (4.1.1) is equivalent to (4.1.2) and that (4.1.2) makes sense in $H^{-\sigma}(\mathbb{R}^N)$.

REMARK 4.9.3. In Theorem 4.9.1, uniqueness is stated in $C(I, H^s(\mathbb{R}^N)) \cap L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N))$. If, in addition to (4.9.1), we assume $\alpha \le (N+2s)/(N-2s)$, then equation (4.1.1) makes sense for any $u \in L^\infty(I, H^s(\mathbb{R}^N))$, without assuming that u belongs to the auxiliary space $L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N))$. Assuming, in addition, $\alpha \le (1+2s)/(1-2s)$ if $N = 1$ or $\alpha < (2+2s)/(N-2s)$ if $N \ge 2$, we know that there is uniqueness in $L^\infty(I, H^s(\mathbb{R}^N))$ (see Section 4.2, especially Remark 4.2.12 for these properties). In this case, we deduce in particular from Theorem 4.9.1 that if $u \in L^\infty(I, H^s(\mathbb{R}^N))$ is a solution of equation (4.1.1), then $L^q(I, B^s_{r,2}(\mathbb{R}^N))$ for every admissible pair (q,r).

For the proof of Theorem 4.9.1, we will use the following nonlinear estimates in Besov spaces.

PROPOSITION 4.9.4. *Let* $g \in C(\mathbb{C}, \mathbb{C})$. *Assume that* $g(0) = 0$ *and that there exists* $\alpha > 0$ *such that*

$$|g(u) - g(v)| \le C(|u|^\alpha + |v|^\alpha)|u - v| \quad \text{for all } u, v \in \mathbb{C}. \tag{4.9.6}$$

Let $0 < s < 1$, $1 \le q \le \infty$, $1 \le p \le r \le \infty$. *If* $\sigma = \alpha pr/(r-p)$, *then*

$$\|g(u)\|_{\dot{B}^s_{p,q}} \le C\|u\|^\alpha_{L^\sigma} \|u\|_{\dot{B}^s_{r,q}} \tag{4.9.7}$$

and

$$\|g(u)\|_{B^s_{p,q}} \le C\|u\|^\alpha_{L^\sigma} \|u\|_{B^s_{r,q}} \tag{4.9.8}$$

for all $u \in B^s_{r,q}(\mathbb{R}^N)$. *Here, we use the convention that* $\|u\|_{L^\sigma} = (\int_{\mathbb{R}^N} |u|^\sigma)^{\frac{1}{\sigma}}$ *for* $0 < \sigma < 1$.

PROOF. It follows from Hölder's inequality that

$$\| |u|^\alpha v \|_{L^p} \le \|u\|^\alpha_{L^\sigma} \|v\|_{L^r}. \tag{4.9.9}$$

Since $|g(u)| \le C|u|^{\alpha+1}$ by (4.9.6), we deduce that

$$\|g(u)\|_{L^p} \le C\|u\|^\alpha_{L^\sigma}\|u\|_{L^r}. \tag{4.9.10}$$

Moreover, if $y \in \mathbb{R}^N$, then it follows from (4.9.6) and (4.9.9) that

$$\|g(u)(\cdot - y) - g(u)(\cdot)\|_{L^p} \le C\|u\|^\alpha_{L^\sigma}\|u(\cdot - y) - u(\cdot)\|_{L^r}.$$

Therefore, the inequality (4.9.7) follows from Remark 1.4.4(iii). Inequality (4.9.8) follows from (4.9.7), (4.9.10), and Remark 1.4.4(ii). □

PROPOSITION 4.9.5. *Let* $g \in C(\mathbb{C}, \mathbb{C})$. *Assume that* $g(0) = 0$ *and that* g *is globally Lipschitz continuous. Let* $0 < s < 1$, $1 \le r, q \le \infty$. *It follows that*

$$\|g(u)\|_{\dot{B}^s_{r,q}} \le C\|u\|_{\dot{B}^s_{r,q}} \tag{4.9.11}$$

and

$$\|g(u)\|_{B^s_{r,q}} \le C\|u\|_{B^s_{r,q}} \tag{4.9.12}$$

for all $u \in B^s_{r,q}(\mathbb{R}^N)$.

PROOF. The proof is similar to the proof of Proposition 4.9.4. □

PROOF OF THEOREM 4.9.1. We only consider positive times, the study of negative times being similar. We recall that (4.1.1) is equivalent to (4.1.2) by Remark 4.9.2. Decomposing $g = g_1 + g_2$ as in Remark 4.9.2, we write the equation (4.1.2) in the form

$$u = \mathcal{H}(u),$$

where

$$\mathcal{H}(u)(t) = \mathcal{T}(t)\varphi + \mathcal{G}(u)(t)$$

and

$$\begin{aligned}\mathcal{G}(u)(t) &= i\int_0^t \mathcal{T}(t-s)g(u(s))ds \\ &= i\int_0^t \mathcal{T}(t-s)g_1(u(s))ds + i\int_0^t \mathcal{T}(t-s)g_2(u(s))ds.\end{aligned}$$

We mostly use a fixed point argument, and we begin with some useful estimates. We observe that by (4.9.12) with $r = q = 2$,

$$\|g_1(u)\|_{H^s} \le C\|u\|_{H^s} \quad \text{for all } u \in H^s(\mathbb{R}^N). \tag{4.9.13}$$

Moreover, it follows from (4.9.8) with $p = \rho'$, $r = \rho$, and $q = 2$ that

$$\|g_2(u)\|_{B^s_{\rho',2}} \le C\|u\|^\alpha_{L^{\frac{N(\alpha+2)}{N-2s}}}\|u\|_{B^s_{\rho,2}}.$$

Since $B^s_{\rho,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{N(\alpha+2)}{N-2s}}(\mathbb{R}^N)$ by (4.9.5), we deduce that

$$\|g_2(u)\|_{B^s_{\rho',2}} \le C\|u\|^{\alpha+1}_{B^s_{\rho,2}} \tag{4.9.14}$$

for all $u \in B^s_{\rho,2}(\mathbb{R}^N)$. Next, it is clear that

$$\|g_1(u) - g_1(v)\|_{L^2} \le C\|u - v\|_{L^2}, \tag{4.9.15}$$

and it follows easily from (4.9.4), Hölder's inequality, and (4.9.5) that

$$\|g_2(u) - g_2(v)\|_{L^{\rho'}} \le C\big(\|u\|^{\alpha}_{B^s_{\rho,2}} + \|v\|^{\alpha}_{B^s_{\rho,2}}\big)\|u - v\|_{L^\rho} . \tag{4.9.16}$$

We now proceed in five steps.

STEP 1. Uniqueness. Let $I = (0,T)$ with $T > 0$ and consider two solutions $u, v \in L^\infty(I, H^s(\mathbb{R}^N)) \cap L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N))$ of (4.1.2). We deduce from (4.9.15) that

$$\|g_1(u) - g_1(v)\|_{L^1(I,L^2)} \le C\|u - v\|_{L^1(I,L^2)}. \tag{4.9.17}$$

Next, it follows from (4.9.16) that

$$\begin{aligned}&\|g_2(u) - g_2(v)\|_{L^{\gamma'}(I,L^{\rho'})} \le \\ &\qquad C\big(\|u\|^{\alpha}_{L^\gamma(I,B^s_{\rho,2})} + \|v\|^{\alpha}_{L^\gamma(I,B^s_{\rho,2})}\big)\|u - v\|_{L^p(I,L^\rho)} ,\end{aligned} \tag{4.9.18}$$

where

$$\frac{1}{p} = \frac{4 - \alpha(N - 2s)}{4} + \frac{1}{\gamma}$$

so that $p < \gamma$. Therefore, we deduce from Lemma 4.2.4 that $u = v$.

STEP 2. Proof of property (i). Let $I = (0,T)$ with $T > 0$ and consider a solution $u \in L^\infty(I, H^s(\mathbb{R}^N)) \cap L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N))$ of (4.1.2). We deduce from (4.9.13), (4.9.14), and Hölder's inequality in time that

$$\|g_1(u)\|_{L^1(I,H^s)} \le CT\|u\|_{L^\infty(I,H^s)} \tag{4.9.19}$$

and

$$\|g_2(u)\|_{L^{\gamma'}(I,B^s_{\rho',2})} \le CT^{\frac{4-\alpha(N-2s)}{4}}\|u\|^{\alpha+1}_{L^\gamma(I,B^s_{\rho,2})} . \tag{4.9.20}$$

We now apply the Strichartz estimates in Besov spaces (Remark 2.3.8 and Corollary 2.3.9) and we deduce from (4.9.19)–(4.9.20) that if (q,r) is any admissible pair, then $u = \mathcal{H}(u) \in L^q(I, B^s_{r,2}(\mathbb{R}^N)) \cap C([0,T], H^s(\mathbb{R}^N))$ and that there exists a constant C independent of I such that

$$\begin{aligned}&\|\mathcal{H}(u)\|_{L^q(I,B^s_{r,2})} \le \\ &\qquad C\|\varphi\|_{H^s} + CT\|u\|_{L^\infty(I,H^s)} + CT^{\frac{4-\alpha(N-2s)}{4}}\|u\|^{\alpha+1}_{L^\gamma(I,B^s_{\rho,2})} ,\end{aligned} \tag{4.9.21}$$

and property (i) follows.

STEP 3. Existence. We apply a fixed point argument in the set

$$\begin{aligned}E = \big\{&u \in L^\infty(I, H^s(\mathbb{R}^N)) \cap L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N)); \\ &\|u\|_{L^\infty(I,H^s)} \le M \text{ and } \|u\|_{L^\gamma(I,B^s_{\rho,2})} \le M\big\} ,\end{aligned} \tag{4.9.22}$$

where $I = (0,T)$ and $M, T > 0$ are to be chosen later. (E, d) is a complete metric space, where the distance d is defined by

$$\mathrm{d}(u,v) = \|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^\gamma(I,L^\rho)} .$$

(See Step 1 of the proof of Theorem 4.4.1.) It follows from (4.9.21)–(4.9.22) that there exists a constant C_0 independent of T such that

$$\begin{aligned} \|\mathcal{T}(\cdot)\varphi\|_{L^\infty(I,H^s)} + \|\mathcal{T}(\cdot)\varphi\|_{L^\gamma(I,B^s_{\rho,2})} &\le C_0\|\varphi\|_{H^s} , \\ \|\mathcal{G}(u)\|_{L^\infty(I,H^s)} + \|\mathcal{G}(u)\|_{L^\gamma(I,B^s_{\rho,2})} &\le C_0(T + T^{\frac{4-\alpha(N-2s)}{4}} M^\alpha)M \end{aligned} \tag{4.9.23}$$

for all $\varphi \in H^s(\mathbb{R}^N)$ and $u \in E$. Therefore, if we let

$$M = 2C_0\|\varphi\|_{H^s} , \tag{4.9.24}$$

and we then choose T sufficiently small so that

$$C_0(T + T^{\frac{4-\alpha(N-2s)}{4}} M^\alpha) \le \frac{1}{2} , \tag{4.9.25}$$

it follows that $\mathcal{H} : E \to E$. Next, we deduce from (4.9.17)–(4.9.18) that

$$\|g_1(u) - g_1(v)\|_{L^1(I,L^2)} \le CT\|u - v\|_{L^\infty(I,L^2)}$$

and

$$\|g_2(u) - g_2(v)\|_{L^{\gamma'}((0,T),L^{\rho'})} \le CT^{\frac{4-\alpha(N-2s)}{4}} M^\alpha \|u - v\|_{L^\gamma((0,T),L^\rho)}$$

for all $u, v \in E$. Applying Strichartz's estimates, it follows that there exists a constant C_1 such that

$$\mathrm{d}(\mathcal{H}(u), \mathcal{H}(v)) \le C_1(T + T^{\frac{4-\alpha(N-2s)}{4}} M^\alpha)\mathrm{d}(u,v) \quad \text{for all } u, v \in E . \tag{4.9.26}$$

Choosing now T possibly smaller so that

$$C_1(T + T^{\frac{4-\alpha(N-2s)}{4}} M^\alpha) < 1 , \tag{4.9.27}$$

we see that $\mathcal{H}$ is a strict contraction on E, and thus has a unique fixed point which is a solution of (4.1.2) on I.

STEP 4. The maximal solution and the blowup alternative. We proceed as in the proof of Theorem 3.3.9: using uniqueness, we define the maximal solution; and since the local solution is constructed by the fixed point argument on an interval depending on $\|\varphi\|_{H^s}$ (by (4.9.24)–(4.9.27)), we deduce the blowup alternative.

STEP 5. Continuous dependence. This is an easy consequence of the estimates of Step 3 above. Indeed, let $\varphi_n \to \varphi$ in $H^s(\mathbb{R}^N)$ and set $M = 4C_0$ where C_0 is as in (4.9.23). Since $\|\varphi_n\|_{H^s} \le 2\|\varphi\|_{H^s}$ for $n \ge n_0$ sufficiently large, we see that $M \ge 2C_0\|\varphi_n\|_{H^s}$ for $n \ge n_0$. It follows easily that if $T = T(\|\varphi\|_{H^s})$ satisfies (4.9.25)–(4.9.27), then the solutions u_n constructed by the argument of Step 3 all belong to the same set E (defined by (4.9.22) with $T = T(\|\varphi\|_{H^s})$) for $n \ge n_0$. Estimate (4.9.26) (together with Strichartz's estimates for the term $\mathcal{T}(\cdot)\varphi_n$) implies that $\mathrm{d}(u_n, u) \le C\|\varphi_n - \varphi\|_{L^2} + (1/2)\mathrm{d}(u_n, u)$, i.e., $\mathrm{d}(u_n, u) \le 2C\|\varphi_n - \varphi\|_{L^2}$. It follows that $u_n \to u$ in $L^\infty(I, L^2(\mathbb{R}^N)) \cap L^\gamma(I, L^\rho(\mathbb{R}^N))$ and a further use of Strichartz's estimates shows the convergence in $L^q(I, L^r(\mathbb{R}^N))$ for every admissible pair (q, r). Finally, the convergence in $L^\infty(I, H^{s-\varepsilon}(\mathbb{R}^N))$ follows from the convergence in $L^\infty(I, L^2(\mathbb{R}^N))$, the boundedness in $L^\infty(I, H^s(\mathbb{R}^N))$, and the elementary interpolation estimate $\|u\|_{H^{s-\varepsilon}} \le \|u\|_{H^s}^{\frac{s-\varepsilon}{s}} \|u\|_{L^2}^{\frac{\varepsilon}{s}}$. □

REMARK 4.9.6. Here are some further comments on Theorem 4.9.1 and its proof.

(i) The choice of the admissible pair (γ, ρ) given by (4.9.3) is (partially) arbitrary. It is not difficult to see that other choices are possible. The present choice leads to relatively simple calculations and is also a valid choice for the case $s > 1$ and for the critical case $\alpha = 4/(N - 2s)$ (see below).

(ii) If $|g(u) - g(v)| \le C(|u|^\alpha + |v|^\alpha)|u - v|$, one can do the fixed point argument in the set $E = \{u \in L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N)); \|u\|_{L^\gamma(I, B^s_{\rho,2})} \le M\}$, with the distance d defined by $\mathrm{d}(u, v) = \|u - v\|_{L^\gamma(I, L^\rho)}$.

(iii) It is not difficult to show that one can replace the set E defined by (4.9.22) by the following set

$$E' = \big\{u \in L^\infty(I, H^s(\mathbb{R}^N)) \cap L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N));\\ \|u\|_{L^\infty(I, \dot{H}^s)} \le M \text{ and } \|u\|_{L^\gamma(I, \dot{B}^s_{\rho,2})} \le M\big\}.$$

This requires two modifications in the proof. One needs the Sobolev inequality $\|u\|_{L^{\frac{N(\alpha+2)}{N-2s}}} \le C\|u\|_{\dot{B}^s_{\rho,2}}$ (see [28]). One also needs to show that (E', d) is complete. This amounts to showing a property of the type if $u_n \to u$ in $L^\gamma(I, L^\rho(\mathbb{R}^N))$ and $\|u_n\|_{L^\gamma(I, \dot{B}^s_{\rho,2})} \le M$, then $u \in L^\gamma(I, B^s_{\rho,2})(\mathbb{R}^N))$ and $\|u\|_{L^\gamma(I, \dot{B}^s_{\rho,2})} \le M$. This follows easily by using Theorem 1.2.5 together with the expression of $\|u\|_{\dot{B}^s_{\rho,2}}$ in terms of the Littlewood-Paley decomposition.

(iv) Note that the continuous dependence statement is weaker than usual. In particular, we do not know if the mappings $\varphi \mapsto T_{\min}(\varphi), T_{\max}(\varphi)$ are lower semicontinuous $H^s(\mathbb{R}^N) \to (0, \infty]$. We do not know either if we can let $\varepsilon = 0$, i.e., if continuous dependence holds in $L^\infty((-T, T), H^s(\mathbb{R}^N))$. On the other hand, note that the proof does not fully use the assumption $\varphi_n \to \varphi$ in $H^s(\mathbb{R}^N)$: it uses exactly that $\|\varphi_n\|_{H^s} \le 2\|\varphi\|_{H^s}$ for n large and that $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N)$.

Below is an analogue of Theorem 4.9.1 for the "critical case" $\alpha = 4/(N - 2s)$.

THEOREM 4.9.7. *Let $0 < s < \min\{1, N/2\}$. Let $g \in C(\mathbb{C}, \mathbb{C})$ with $g(0) = 0$ satisfy* (4.9.2) *with*

$$\alpha = \frac{4}{N - 2s},$$

and let (γ, ρ) be the admissible pair defined by (4.9.3). *For every $\varphi \in H^s(\mathbb{R}^N)$, there exist $T_{\max}, T_{\min} \in (0, \infty]$ and a unique, maximal solution u of the problem* (4.1.1) *in $C((-T_{\min}, T_{\max}), H^s(\mathbb{R}^N)) \cap L^\gamma_{\mathrm{loc}}((-T_{\min}, T_{\max}), B^s_{\rho,2}(\mathbb{R}^N))$. Moreover, the following properties hold:*

(i) *$u \in L^q_{\mathrm{loc}}((-T_{\min}, T_{\max}), B^s_{r,2}(\mathbb{R}^N))$ for every admissible pair (q, r).*

(ii) *If $T_{\max} < \infty$, then $\|u\|_{L^\infty((0,T_{\max}), H^s)} + \|u\|_{L^\gamma((0,T_{\max}), B^s_{\rho,2})} = \infty$. A similar statement holds if $T_{\min} < \infty$.*

(iii) *u depends continuously on φ in the sense of Theorem* 4.9.1(iii).

PROOF. We use the same notation and follow the same steps as in the proof of Theorem 4.9.1.

STEP 1. Uniqueness. Let $I = (0,T)$ with $T > 0$ and consider two solutions $u, v \in L^\infty(I, H^s(\mathbb{R}^N)) \cap L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N))$ of (4.1.2). Uniqueness being a local property, we need only show that $u = v$ if T is sufficiently small (see Step 2 of the proof of Theorem 4.6.1). We observe that (4.9.18) becomes

$$\|g_2(u)-g_2(v)\|_{L^{\gamma'}(I,L^{\rho'})} \le C\big(\|u\|^\alpha_{L^\gamma(I,B^s_{\rho,2})}+\|v\|^\alpha_{L^\gamma(I,B^s_{\rho,2})}\big)\|u-v\|_{L^\gamma(I,L^\rho)} \,. \tag{4.9.28}$$

Using (4.9.17), (4.9.28), and Strichartz's estimates, we deduce that there exists C independent of T such that

$$\begin{aligned}&\|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^\gamma(I,L^\rho)} \le \\ &\qquad CT\|u - v\|_{L^\infty(I,L^2)} + C\big(\|u\|^\alpha_{L^\gamma(I,B^s_{\rho,2})} + \|v\|^\alpha_{L^\gamma(I,B^s_{\rho,2})}\big)\|u - v\|_{L^\gamma(I,L^\rho)} \,.\end{aligned}$$

Since $\|u\|_{L^\gamma(I,B^s_{\rho,2})} \to 0$ as $T \downarrow 0$ and similarly for v, we see that if $T > 0$ is small enough, then

$$\|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^\gamma(I,L^\rho)} \le \frac{1}{2}\big(\|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^\gamma(I,L^\rho)}\big)$$

so that $u = v$.

STEP 2. Proof of property (i). It suffices to show that if $I \ni 0$ is a bounded interval and $u \in L^\infty(I, H^s(\mathbb{R}^N)) \cap L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N))$ is a solution of (4.1.1), then $u \in L^q(I, B^s_{r,2}(\mathbb{R}^N)) \cap C(\overline{I}, H^s(\mathbb{R}^N))$ for every admissible pair (q, r). This is proved as in Step 2 of the proof of Theorem 4.9.1.

STEP 3. Existence. We apply a fixed point argument in the set E defined by

$$\begin{aligned}E = \{&u \in L^\infty(I, H^s(\mathbb{R}^N)) \cap L^\gamma(I, B^s_{\rho,2}(\mathbb{R}^N));\\ &\|u\|_{L^\infty(I,H^s)} \le M_1 \text{ and } \|u\|_{L^\gamma(I,B^s_{\rho,2})} \le M_2\}\,,\end{aligned} \tag{4.9.29}$$

where $I = (0,T)$ and $M_1, M_2, T > 0$ are to be chosen later. (E, d) is a complete metric space, where the distance d is defined by

$$\mathrm{d}(u, v) = \|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^\gamma(I,L^\rho)} \,.$$

(See Step 1 of the proof of Theorem 4.4.1.) The proof of (4.9.21) yields

$$\|\mathcal{H}(u)\|_{L^q(I,B^s_{r,2})} \le C\|\mathfrak{T}(\cdot)\varphi\|_{L^q(I,B^s_{r,2})} + CT\|u\|_{L^\infty(I,H^s)} + C\|u\|^{\alpha+1}_{L^\gamma(I,B^s_{\rho,2})} \,.$$

In particular, given any $u \in E$,

$$\|\mathcal{H}(u)\|_{L^\infty(I,H^s)} \le C_0\|\varphi\|_{H^s} + C_0TM_1 + C_0M_2^{\alpha+1} \tag{4.9.30}$$

and

$$\|\mathcal{H}(u)\|_{L^\gamma(I,B^s_{\rho,2})} \le C_0\|\mathfrak{T}(\cdot)\varphi\|_{L^\gamma(I,B^s_{\rho,2})} + C_0TM_1 + C_0M_2^{\alpha+1} \tag{4.9.31}$$

for some constant C_0 independent of T. Similarly, one shows (see the proof of (4.9.26)) that there exists a constant C_1 independent of T such that

$$\|\mathcal{H}(u) - \mathcal{H}(v)\|_{L^\infty(I,L^2)} + \|\mathcal{H}(u) - \mathcal{H}(v)\|_{L^\gamma(I,L^\rho)} \le C_1(T + M_2^\alpha)\mathrm{d}(u, v) \tag{4.9.32}$$

for all $u, v \in I$. We now choose T, M_1, M_2 as follows. We first let

$$M_1 = 3C_0\|\varphi\|_{H^s},$$

then we choose $M_2 > 0$ sufficiently small so that

$$C_0 M_2^{\alpha+1} \le C_0\|\varphi\|_{H^s}, \quad C_0 M_2^{\alpha+1} \le \frac{M_2}{3}, \quad C_1 M_2^{\alpha} \le \frac{1}{4}.$$

Finally, we choose $T > 0$ sufficiently small so that

$$C_0 T M_1 \le C_0\|\varphi\|_{H^s}, \quad C_0 T M_1 \le \frac{M_2}{3}, \quad C_1 T \le \frac{1}{4},$$

and

$$C_0\|\mathcal{T}(\cdot)\varphi\|_{L^\gamma(I,B^s_{\rho,2})} \le \frac{M_2}{3}.$$

(This last condition can be achieved because $\gamma < \infty$, thus $\|\mathcal{T}(\cdot)\varphi\|_{L^\gamma((0,T),B^s_{\rho,2})} \downarrow 0$ as $T \downarrow 0$.) We then deduce from (4.9.30)–(4.9.31) that $\|\mathcal{H}(u)\|_{L^\infty(I,H^s)} \le M_1$ and $\|\mathcal{H}(u)\|_{L^\gamma(I,B^s_{\rho,2})} \le M_2$, i.e., $\mathcal{H} : E \to E$. Furthermore, we deduce from (4.9.32) that $\mathcal{H}$ is a strict contraction on E, and thus has a unique fixed point which is a solution of (4.1.2) on I.

STEP 4. The maximal solution and the blowup alternative. Using uniqueness, we define the maximal solution (as in the proof of Theorem 3.3.9). Assume $T_{\max} < \infty$. If $\|u\|_{L^\infty((0,T_{\max}),H^s)} + \|u\|_{L^\gamma((0,T_{\max}),B^s_{\rho,2})} < \infty$, then we deduce from Step 2 that $u \in C([0,T_{\max}], H^s(\mathbb{R}^N))$. By Step 3, we then can construct a solution v of (4.1.2), with φ replaced by $u(T_{\max})$, on some interval $[0,\varepsilon]$ with $\varepsilon > 0$. It follows that $\widetilde{u}$ defined on $[0, T_{\max}+\varepsilon]$ by $\widetilde{u}(t) = u(t)$ for $0 \le t \le T_{\max}$ and $\widetilde{u}(t) = v(t-T_{\max})$ for $T_{\max} \le t \le T_{\max}+\varepsilon$ is a solution of (4.1.2) on $[0, T_{\max}+\varepsilon]$. This contradicts the definition of $T_{\max}$. Thus $\|u\|_{L^\infty((0,T_{\max}),H^s)} + \|u\|_{L^\gamma((0,T_{\max}),B^s_{\rho,2})} = \infty$.

STEP 5. Continuous dependence. This is done as in the proof of Theorem 4.9.1, Step 5. □

REMARK 4.9.8. The observations of Remark 4.9.6 apply as well to Theorem 4.9.7 and its proof. We now focus in particular on the case where $|g(u)-g(v)| \le C(|u|^\alpha + |v|^\alpha)|u-v|$. Then one can do the fixed point argument in the set $E = \{u \in L^\gamma(I, \dot{B}^s_{\rho,2}(\mathbb{R}^N)); \|u\|_{L^\gamma(I,\dot{B}^s_{\rho,2})} \le M\}$, with the distance d defined by $\mathrm{d}(u,v) = \|u-v\|_{L^\gamma(I,L^\rho)}$ (see Remark 4.9.6(ii) and (iii)). In this case, instead of (4.9.31)–(4.9.32), one obtains

$$\|\mathcal{H}(u)\|_{L^\gamma(I,\dot{B}^s_{\rho,2})} \le C_0\|\mathcal{T}(\cdot)\varphi\|_{L^\gamma(I,\dot{B}^s_{\rho,2})} + C_0 M^{\alpha+1}$$

and

$$\|\mathcal{H}(u) - \mathcal{H}(v)\|_{L^\gamma(I,L^\rho)} \le C_1 M^\alpha\, \mathrm{d}(u,v).$$

Since $\|\mathcal{T}(\cdot)\varphi\|_{L^\gamma(\mathbb{R},\dot{B}^s_{\rho,2})} \le C\|\varphi\|_{\dot{H}^s}$, we obtain by letting $I = \mathbb{R}$

$$\|\mathcal{H}(u)\|_{L^\gamma(\mathbb{R},\dot{B}^s_{\rho,2})} \le C_3\|\varphi\|_{\dot{H}^s} + C_0 M^{\alpha+1},$$

$$\|\mathcal{H}(u) - \mathcal{H}(v)\|_{L^\gamma(\mathbb{R},L^\rho)} \le C_1 M^\alpha\, \mathrm{d}(u,v).$$

Therefore, if we first choose $M > 0$ small enough so that $C_0 M^{\alpha+1} \le M/2$ and $C_1 M^\alpha \le 1/2$ and then assume that $\|\varphi\|_{\dot{H}^s}$ is sufficiently small so that $C_3\|\varphi\|_{\dot{H}^s} \le M/2$, we see that $\mathcal{H}$ is a strict contraction on E. In this case, we obtain (under the assumption that $\|\varphi\|_{\dot H^s}$ is small) a global solution u of (4.1.1). Moreover, this solution belongs (by construction) to $L^\infty(\mathbb{R}, H^s(\mathbb{R}^N)) \cap L^\gamma(\mathbb{R}, B^s_{\rho,2}(\mathbb{R}^N))$. See [70] for details.

We now comment on the case $s > 1$. The restriction $s < 1$ in Theorems 4.9.1 and 4.9.7 is motivated only by the nonlinear estimates of Propositions 4.9.4 and 4.9.5. The rest of the proof is not subject to the condition $s < 1$. It turns out that estimates of the type (4.9.8) for $s > 1$ are true but require more regularity assumptions on g. The corresponding existence results of H^s solutions therefore hold provided g is sufficiently smooth. See Cazenave and Weissler [70] and Kato [206]. See also Pecher [295], where the author uses time derivatives to obtain H^s solutions with minimal regularity assumptions on the nonlinearity. For completeness, we state below two typical results.

THEOREM 4.9.9. *Assume $N \geq 3$ and $1 < s < N/2$. Let $g(u) = \lambda |u|^\alpha u$ with $\lambda \in \mathbb{C}$ and*

$$0 \leq \alpha < \frac{4}{N-2s}.$$

If α is not an even integer, suppose further that

$$[s] < \alpha. \tag{4.9.33}$$

Given $\varphi \in H^s(\mathbb{R}^N)$, there exist $T_{\max}, T_{\min} \in (0,\infty]$ and a unique, maximal solution $u \in C((-T_{\min}, T_{\max}), H^s(\mathbb{R}^N))$ of problem (4.1.1). *Moreover, the following properties hold:*

(i) *$u \in L^q_{\text{loc}}((-T_{\min}, T_{\max}), B^s_{r,2}(\mathbb{R}^N))$ for every admissible pair (q,r).*

(ii) (Blowup alternative) *If $T_{\max} < \infty$ (respectively, if $T_{\min} < \infty$), then $\|u(t)\|_{H^s} \to \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow -T_{\min}$).*

(iii) *u depends continuously on φ in the sense of Theorem* 4.9.1(iii).

PROOF. We refer to Cazenave and Weissler [70]. Note that uniqueness of H^s solutions follows from Remark 4.2.12. □

THEOREM 4.9.10. *Assume $N \geq 3$. Let $1 < s < N/2$ and $g(u) = \lambda |u|^{\frac{4}{N-2s}} u$ with $\lambda \in \mathbb{C}$. If α is not an even integer, suppose further that* (4.9.33) *holds. It follows that for every $\varphi \in H^s(\mathbb{R}^N)$, there exist $0 < T_{\max}, T_{\min} < \infty$ and a unique, maximal solution $u \in C((-T_{\min}, T_{\max}), H^s(\mathbb{R}^N))$ of problem* (4.1.1). *Moreover, the following properties hold:*

(i) *$u \in L^q_{\text{loc}}((-T_{\min}, T_{\max}), B^s_{r,2}(\mathbb{R}^N))$ for every admissible pair (q,r).*

(ii) *If $T_{\max} < \infty$, then $\|u\|_{L^\gamma((0,T_{\max}), B^s_{\rho,2})} = \infty$, where (γ, ρ) is the admissible pair defined by* (4.9.3). *A similar statement holds if $T_{\min} < \infty$.*

(iii) *u depends continuously on φ in the sense of Theorem* 4.9.1(iii).

PROOF. We refer to Cazenave and Weissler [70]. Note that uniqueness of H^s solutions follows from Proposition 4.2.13. □

REMARK 4.9.11. Here are some comments on the assumption (4.9.33) in Theorems 4.9.9 and 4.9.10.

(i) The fact that assumption (4.9.33) is not needed when α is an even integer is essentially due to the fact that $g \in C^\infty(\mathbb{C}, \mathbb{C})$ (in the real sense) in that case. See Cazenave and Weissler [70] and Kato [206].

(ii) In the framework of Theorem 4.9.9, Pecher [295] has improved assumption (4.9.33) by using time derivatives in the construction of the solution. More precisely, (4.9.33) is not needed if $N = 3, 4$, or, more generally, if $s < 2$. If $2 \leq s < 4$, then it can be replaced by the weaker condition $s < \alpha + 2$; and if $s \geq 4$, then it can be replaced by the weaker condition $s < \alpha + 3$.

REMARK 4.9.12. In the limiting case $s = N/2$, the embedding $H^{\frac{N}{2}}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for all $2 \leq p < \infty$ makes it possible to obtain local existence for (sufficiently regular) nonlinearities with arbitrary polynomial growth. See Kato [206]. In particular, there is local existence in the model case $g(u) = \lambda |u|^\alpha u$ for any $\alpha > 0$ such that $[s] < \alpha$. Using Trudinger's inequality, one can also consider nonlinearities of exponential growth. See Nakamura and Ozawa [256].

4.10. H^m Solutions, $m > N/2$

In this section we study the local existence of "smooth" solutions in $H^m(\mathbb{R}^N)$ for $m > N/2$. In principle, one can consider arbitrary real m (see Kato [206]), but we will only consider integers. The estimates are then simpler. The main point in considering $m > N/2$ is that $H^m(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. The consequence is that we need regularity of the nonlinearity but we do not need any control on its growth. We follow the method of Ginibre and Velo [135] and for simplicity, we only consider local nonlinearities. The main result is the following.

THEOREM 4.10.1. *Let $m > N/2$ be an integer and let $g \in C^m(\mathbb{C}, \mathbb{C})$ (in the real sense) satisfy $g(0) = 0$. For every $\varphi \in H^m(\mathbb{R}^N)$, there exist $T_{\max}, T_{\min} > 0$ and a unique, maximal solution $u \in C((-T_{\min}, T_{\max}), H^m(\mathbb{R}^N))$ of (4.1.1). Moreover, the following properties hold:*

(i) (Blowup alternative) *If $T_{\max} < \infty$ (respectively, $T_{\min} < \infty$), then $\|u(t)\|_{H^m} \to \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow -T_{\min}$). Moreover, $\limsup \|u(t)\|_{L^\infty} = \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow -T_{\min}$).*

(ii) *u depends continuously on φ in the following sense. The functions $T_{\max}$ and $T_{\min}$ are lower semicontinuous $H^m(\mathbb{R}^N) \to (0, \infty]$. Moreover, if $\varphi_n \to \varphi$ in $H^m(\mathbb{R}^N)$ and if u_n is the maximal solution of (4.1.1) with the initial value φ_n, then $u_n \to u$ in $L^\infty((-S, T), H^m(\mathbb{R}^N))$ for every $p < \infty$ and every interval $[-S, T] \subset (-T_{\min}, T_{\max})$.*

(iii) *If $(g(w), iw)_{L^2} = 0$ for all $w \in H^m(\mathbb{R}^N)$, then there is conservation of charge; i.e., $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in (-T_{\min}, T_{\max})$.*

(iv) *If there exists $G \in C^1(H^m(\mathbb{R}^N), \mathbb{R})$ such that $g = G'$, then there is conservation of energy; i.e., $E(u(t)) = E(\varphi)$ for all $t \in (-T_{\min}, T_{\max})$, where E is defined by (3.3.9).*

The proof of Theorem 4.10.1 relies on the following technical lemma.

LEMMA 4.10.2. *Let $m > N/2$ be an integer and let $g \in C^m(\mathbb{C}, \mathbb{C})$ satisfy $g(0) = 0$. It follows that the mapping $u \mapsto g(u)$ is continuous and bounded $H^m(\mathbb{R}^N) \to$*

$H^m(\mathbb{R}^N)$. More precisely, given any $M > 0$, there exists $C(M)$ such that

$$\|g(u)\|_{H^m} \le C(M)\|u\|_{H^m}\,, \tag{4.10.1}$$

$$\|g(u) - g(v)\|_{L^2} \le C(M)\|u - v\|_{L^2}\,, \tag{4.10.2}$$

for all $u, v \in H^m(\mathbb{R}^N)$ such that $\|u\|_{L^\infty}, \|v\|_{L^\infty} \le M$. Moreover,

$$\|g(u) - g(v)\|_{H^m} \le C(M)\big[\|u - v\|_{H^m} + \varepsilon_M(\|u - v\|_{L^2})\big] \tag{4.10.3}$$

for all $u, v \in H^m(\mathbb{R}^N)$ such that $\|u\|_{H^m}, \|v\|_{H^m} \le M$, where $\varepsilon_M(s) \to 0$ as $s \downarrow 0$.

PROOF. Let $M > 0$ and let

$$K(M) = \sup_{|u| \le M} |g'(u)| + \dots + |g^{(m)}(u)| < \infty\,. \tag{4.10.4}$$

It follows that, if $|u|, |v| \le M$, then

$$|g(u) - g(v)| \le K(M)|u - v| \tag{4.10.5}$$

and in particular

$$|g(u)| \le K(M)|u|\,. \tag{4.10.6}$$

Consider now $u \in C_c^\infty(\mathbb{R}^N)$. Given a multi-index α with $|\alpha| = m$, it is not difficult to show that $D^\alpha g(u)$ is a sum of terms of the form

$$g^{(k)}(u) \prod_{j=1}^{k} D^{\beta_j} u, \tag{4.10.7}$$

where k is an integer, $k \in \{1, \dots, |\alpha|\}$ and the β_j's are multi-indices such that $\alpha = \beta_1 + \dots + \beta_k$ and $|\beta_j| \ge 1$. Let $p_j = 2m/|\beta_j|$, so that

$$\sum_{j=1}^{k} \frac{1}{p_j} = \frac{1}{2}\,.$$

It follows from Hölder's inequality that

$$\Big\| \prod_{j=1}^{k} D^{\beta_j} u \Big\|_{L^2} \le \prod_{j=1}^{k} \|D^{\beta_j} u\|_{L^{p_j}}\,.$$

On the other hand, it follows from Gagliardo-Nirenberg's inequality that

$$\|D^{\beta_j} u\|_{L^{p_j}} \le C\|u\|_{H^m}^{\frac{|\beta_j|}{m}} \|u\|_{L^\infty}^{1 - \frac{|\beta_j|}{m}}\,,$$

and so

$$\Big\| \prod_{j=1}^{k} D^{\beta_j} u \Big\|_{L^2} \le C\|u\|_{H^m} \|u\|_{L^\infty}^{k-1}\,.$$

Applying now (4.10.7) and (4.10.4), we deduce that

$$\|D^\alpha g(u)\|_{L^2} \le CK(M)\|u\|_{H^m}\,.$$

Finally, we deduce from (4.10.6) that

$$\|g(u)\|_{L^2} \le K(M)\|u\|_{L^2}\,,$$

so that (4.10.1) follows from the two estimates above.

Let now $u \in H^m(\mathbb{R}^N)$ with $\|u\|_{L^\infty} \le M$ and let $(u_n)_{n\ge 0} \subset C_c^\infty(\mathbb{R}^N)$ satisfy $u_n \to u$ in $H^m(\mathbb{R}^N)$. Since $H^m(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, we see that $\|u_n\|_{L^\infty} \le 2M$ and $\|u_n\|_{H^m} \le 2\|u\|_{H^m}$ for n large. In particular, we deduce from (4.10.1), which we already established for u_n, that

$$\|g(u_n)\|_{H^m} \le 2C(2M)\|u\|_{H^m} \quad \text{for } n \text{ large.} \tag{4.10.8}$$

In particular, $(g(u_n))_{n\ge 0}$ is bounded in H^m. Since $g(u_n) \to g(u)$ in $L^2(\mathbb{R}^N)$ by (4.10.5), it follows that $g(u_n) \rightharpoonup g(u)$ in $H^m(\mathbb{R}^N)$. Applying (4.10.8), we deduce (4.10.1) (with $C(M)$ replaced by $2C(2M)$). Inequality (4.10.2) is an immediate consequence of (4.10.5).

We finally prove (4.10.3). First note that, given $u \in H^m(\mathbb{R}^N)$ and $(u_n)_{n\ge 0} \subset C_c^\infty(\mathbb{R}^N)$ as above, we may assume (after possibly extracting a subsequence) that $D^\alpha u_n \to D^\alpha u$ a.e. for all $|\alpha| \le m$. Thus we see that formula (4.10.7) holds a.e. for every $u \in H^m(\mathbb{R}^N)$. Let now $M > 0$ and $u, v \in H^m(\mathbb{R}^N)$ with $\|u\|_{H^m}, \|v\|_{H^m} \le M$. Given a multi-index α with $|\alpha| = m$, we deduce that $D^\alpha[g(u) - g(v)]$ is a sum of terms of the form

$$g^{(k)}(u) \prod_{j=1}^{k} D^{\beta_j} u - g^{(k)}(v) \prod_{j=1}^{k} D^{\beta_j} v ,$$

where k and the β_j's are as in (4.10.7). Each of the above terms can itself be decomposed as a sum of terms where the first one is

$$[g^{(k)}(u) - g^{(k)}(v)] \prod_{j=1}^{k} D^{\beta_j} u . \tag{4.10.9}$$

The other ones have the form

$$g^{(k)}(v) \prod_{j=1}^{k} D^{\beta_j} w_j ,$$

where all the w_j's are equal to u or v, except one which is equal to $u - v$. Let now $p_j = 2m/|\beta_j|$. We see that

$$\Big\| g^{(k)}(v) \prod_{j=1}^{k} D^{\beta_j} w_j \Big\|_{L^2} \le \|g^{(k)}(v)\|_{L^\infty} \prod_{j=1}^{k} \|D^{\beta_j} w_j\|_{L^{p_j}} \le C(M)\|u - v\|_{H^m} ,$$

where the last inequality follows from the embeddings $H^m(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ and $H^m(\mathbb{R}^N) \hookrightarrow L^{p_j}(\mathbb{R}^N)$. If $k \le m - 1$, then $g^{(k)}$ is Lipschitz on bounded sets, so that the terms in (4.10.9) are estimated as above by $C(M)\|u - v\|_{H^m}$. It remains to estimate the terms (4.10.9) when $k = m$. This last term is estimated as above by

$$\|g^{(m)}(u) - g^{(m)}(v)\|_{L^\infty} \|u\|_{H^m}^m . \tag{4.10.10}$$

Since $g^{(m)}$ is continuous, hence uniformly continuous on bounded sets, and $\|u\|_{H^m}, \|v\|_{H^m} \le M$, we see that $\|g^{(m)}(u) - g^{(m)}(v)\|_{L^\infty} \le \delta_M(\|u - v\|_{L^\infty})$, with $\delta_M(s) \to 0$ as $s \downarrow 0$. Since $\|u - v\|_{L^\infty} \le C\|u - v\|_{H^m}^{\frac{N}{2m}} \|u - v\|_{L^2}^{\frac{2m-N}{2m}}$, we may replace $\delta_M(\|u - v\|_{L^\infty})$ by $\delta_M(\|u - v\|_{L^2})$, so that (4.10.3) now follows from (4.10.10). □

Proof of Theorem 4.10.1. We first note that by Lemmas 4.2.8 and 4.10.2, problems (4.1.1) and (4.1.2) are equivalent. We then proceed in four steps.

STEP 1. Existence. We construct solutions by a fixed-point argument. Given $M, T > 0$ to be chosen later, we set $I = (-T, T)$ and we consider

$$E = \{u \in L^\infty(I, H^m(\mathbb{R}^N)) : \|u\|_{L^\infty(I,H^m)} \le M\}.$$

It follows that (E, d) is a complete metric space, where the distance d is defined by

$$\mathrm{d}(u, v) = \|u - v\|_{L^\infty(I,L^2)}.$$

(This is established by the argument of Step 1 of the proof of Theorem 4.4.1.) We now consider $\mathcal{H}$ defined by

$$\mathcal{H}(u)(t) = \mathfrak{T}(t)\varphi + \mathcal{G}(u)(t),$$

where

$$\mathcal{G}(u)(t) = i\int_0^t \mathfrak{T}(t-s)g(u(s))ds$$

for all $u \in E$ and all $t \in I$. We note that if $u \in L^\infty(I, H^m(\mathbb{R}^N))$, then $g(u) \in L^\infty(I, H^m(\mathbb{R}^N))$ by Lemma 4.10.2, so that $\mathcal{G}(u) \in C(\overline{I}, H^m(\mathbb{R}^N))$. Since $\mathfrak{T}(t)$ is an isometry on $H^m(\mathbb{R}^N)$, it follows from (4.10.3) that for every $u \in E$ and $t \in I$,

$$\|\mathcal{H}(u)(t)\|_{H^m} \le \|\varphi\|_{H^m} + T\|g(u)\|_{L^\infty(I,H^m)} \le \|\varphi\|_{H^m} + TC(M)M.$$

Furthermore, it follows from (4.10.2) that, if $u, v \in E$, then

$$\|\mathcal{H}(u)(t) - \mathcal{H}(v)(t)\|_{L^2} \le TC(M)\|u - v\|_{L^\infty(I,L^2)}.$$

Therefore, we see that if

$$M = 2\|\varphi\|_{H^m} \quad \text{and} \quad TC(M) \le \frac{1}{2}, \tag{4.10.11}$$

then $\mathcal{H}$ is a strict contraction of (E, d) and thus has a fixed-point which is a solution of (4.1.3), hence of (4.1.1).

STEP 2. Uniqueness, the maximal solution, and the blowup alternative. We get uniqueness from Proposition 4.2.9. We then proceed as in the proof of Theorem 3.3.9: using uniqueness, we define the maximal solution; and since the local solution is constructed by the fixed point argument on an interval depending on $\|\varphi\|_{H^m}$ (by (4.10.11)), we deduce the blowup alternative on $\|u(t)\|_{H^m}$. We then show that if $T_{\max} < \infty$, then $\limsup \|u(t)\|_{L^\infty} = \infty$ as $t \uparrow T_{\max}$. Indeed, suppose by contradiction that $\limsup \|u(t)\|_{L^\infty} < \infty$. Since $u \in C([0, T_{\max}), H^m(\mathbb{R}^N))$ and $H^m(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, it follows that

$$M = \sup_{0 \le t < T_{\max}} \|u(t)\|_{L^\infty} < \infty.$$

Applying now (4.10.1), we deduce from (4.1.2) that

$$\|u(t)\|_{H^m} \le \|\varphi\|_{H^m} + C(M)\int_0^t \|u(s)\|_{H^m}\,ds.$$

Applying Gronwall's lemma, we obtain that $\|u(t)\|_{H^m} \le \|\varphi\|_{H^m} e^{T_{\max}C(M)}$ for all $0 \le t < T_{\max}$, which contradicts the blowup of $\|u(t)\|_{H^m}$ at $T_{\max}$.

STEP 3. Continuous dependence. Let $\varphi \in H^m(\mathbb{R}^N)$ and consider $(\varphi_n)_{n\ge 0} \subset H^m(\mathbb{R}^N)$ such that $\varphi_n \to \varphi$ in $H^m(\mathbb{R}^N)$ as $n \to \infty$. Let u_n be the maximal solution of (4.1.1) corresponding to the initial value φ_n. We claim that there exists $T > 0$

depending on $\|\varphi\|_{H^m}$ such that u_n is defined on $[-T,T]$ for n large enough and $u_n \to u$ in $C([-T,T],H^m(\mathbb{R}^N))$ as $n \to \infty$. The result follows by iterating this property in order to cover any compact subset of $(-T_{\min},T_{\max})$.

We now prove the claim. Since $\|\varphi_n\|_{H^m} \le 2\|\varphi\|_{H^m}$ for n sufficiently large, we deduce from the estimates of Step 1 that there exists $T = T(\|\varphi\|_{H^m})$ such that u and u_n are defined on $[-T,T]$ for $n \ge n_0$ and

$$\|u\|_{L^\infty((-T,T),H^m)} + \sup_{n\ge n_0} \|u_n\|_{L^\infty((-T,T),H^m)} \le 4\|\varphi\|_{H^m}\,. \tag{4.10.12}$$

Note that $u_n(t) - u(t) = \mathcal{T}(t)(\varphi_n - \varphi) + \mathcal{G}(u_n)(t) - \mathcal{G}(u)(t)$. Therefore, it follows from (4.10.2), (4.10.12) and the embedding $H^m(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ that there exists C such that

$$\|u(t) - u_n(t)\|_{L^2} \le \|\varphi - \varphi_n\|_{L^2} + C\left|\int_0^t \|u(s) - u_n(s)\|_{L^2}\,ds\right|$$

for all $t \in (-T,T)$. We then deduce from Gronwall's lemma that

$$\|u(t) - u_n(t)\|_{L^2} \le \|\varphi - \varphi_n\|_{L^2} e^{TC} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{4.10.13}$$

We then deduce from (4.10.3) and (4.10.13) that there exists C and $\varepsilon_n \downarrow 0$ such that

$$\|u(t) - u_n(t)\|_{H^m} \le \varepsilon_n + \|\varphi - \varphi_n\|_{H^m} + C\left|\int_0^t \|u(s) - u_n(s)\|_{H^m}\,ds\right|$$

for all $t \in (-T,T)$. The claim now follows from Gronwall's lemma.

STEP 4. The conservation laws. Note that the case $m = 1$ has already been studied (Theorem 4.4.6 and Remark 4.4.8), so that we may assume $m \ge 2$. The conservation laws (properties (iii) and (iv)) then follow by multiplying the equation by $\overline{u}$ and $\overline{u}_t$, respectively. See Steps 3 and 4 of the proof of Theorem 4.8.1. □

REMARK 4.10.3. Let $g(u) = \lambda|u|^\alpha u$ with $\alpha > 0$ and $\lambda \in \mathbb{C}$. If α is an even integer, then $g \in C^\infty(\mathbb{C},\mathbb{C})$. Therefore, we may apply Theorem 4.10.1 for any $m > N/2$. If α is an odd integer, then $g \in C^m(\mathbb{C},\mathbb{C})$ only for $m \le [\alpha]$, so that we may apply Theorem 4.10.1 only in the case $[\alpha] > N/2$ and for $N/2 < m \le [\alpha]$. If α is not an integer, then $g \in C^m(\mathbb{C},\mathbb{C})$ only for $m \le [\alpha]+1$, so that we may apply Theorem 4.10.1 only in the case $[\alpha]+1 > N/2$ and for $N/2 < m \le [\alpha]+1$.

4.11. Cauchy Problem for a Nonautonomous Schrödinger Equation

In this section we study the Cauchy problem for equation (7.5.5) below, starting from any point $t \in [0,1]$. In fact, we consider the more general Cauchy problem (see [72])

$$\begin{cases} iv_t + \Delta v + h(t)|v|^\alpha v = 0 \\ v(0) = \psi, \end{cases} \tag{4.11.1}$$

where $h \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$. We study the equation (4.11.1) in the equivalent form

$$v(t) = \mathcal{T}(t)\psi + i\int_0^t \mathcal{T}(t-s)h(s)|v(s)|^\alpha v(s)ds\,. \tag{4.11.2}$$

We have the following existence and uniqueness result.

THEOREM 4.11.1. *Assume* $0 < \alpha < 4/(N-2)$ $(0 < \alpha < \infty$ *if* $N = 1)$. *Let* $\theta = 4/[4 - \alpha(N-2)]$ $(\theta = 1$ *if* $N = 1$, $\theta > 1$ *and* $(2-\alpha)\theta \leq 1$ *if* $N = 2)$, *and consider a real-valued function* $h \in L^\theta_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$. *It follows that for every* $\psi \in H^1(\mathbb{R}^N)$, *there exist* $T_{\max}, T_{\min} > 0$ *and a unique, maximal solution* $v \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N)) \cap W^{1,\theta}_{\mathrm{loc}}((-T_{\min}, T_{\max}), H^{-1}(\mathbb{R}^N))$ *of equation* (4.11.2). *The solution* v *is maximal in the sense that if* $T_{\max} < \infty$ *(respectively,* $T_{\min} < \infty$*), then* $\|u(t)\|_{H^1} \to \infty$ *as* $t \uparrow T_{\max}$ *(respectively,* $t \downarrow -T_{\min}$*). In addition, the solution* v *has the following properties*:

(i) *If* $T_{\max} < \infty$, *then* $\liminf_{t \uparrow T_{\max}} \{\|v(t)\|^\alpha_{H^1} \|h\|_{L^\theta(t, T_{\max})}\} > 0$.
(ii) *If* $T_{\min} < \infty$, *then* $\liminf_{t \downarrow -T_{\min}} \{\|v(t)\|^\alpha_{H^1} \|h\|_{L^\theta(-T_{\min}, t)}\} > 0$.
(iii) $u \in L^q_{\mathrm{loc}}((-T_{\min}, T_{\max}), W^{1,r}(\mathbb{R}^N))$ *for every admissible pair* (q, r).
(iv) *There exists* $\delta > 0$, *depending only on* N, α, *and* θ *such that if*

$$\|\psi\|^{\alpha\theta}_{H^1} \int_{-\tau}^{\tau} |h(s)|^\theta \, ds \leq \delta \, ,$$

then $[-\tau, \tau] \subset (-T_{\min}, T_{\max})$ *and* $\|v\|_{L^q((-\tau,\tau), W^{1,r})} \leq K\|\psi\|_{H^1}$ *for every admissible pair* (q, r), *where* K *depends only on* N, α, θ, *and* q. *In addition, if* ψ' *is another initial value satisfying the above condition and if* v' *is the corresponding solution of* (4.11.2), *then* $\|v - v'\|_{L^\infty((-\tau,\tau), L^2)} \leq K\|\psi - \psi'\|_{L^2}$.
(v) *If* $|\cdot|\psi \in L^2(\mathbb{R}^N)$, *then* $|\cdot|v \in C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N))$.

PROOF. We apply the method of Section 4.4. We suppose first that $N \geq 3$, then we indicate the modifications needed to handle the cases $N = 2$ and $N = 1$. Let $2^* = 2N/(N-2)$ and define r by

$$1 - \frac{2}{r} = \frac{\alpha}{2^*} \, . \tag{4.11.3}$$

Since $(N-2)\alpha < 4$, we have $2 < r < 2^*$. Therefore, there exists q such that (q, r) is an admissible pair. A simple calculation shows that

$$\frac{1}{q'} = \frac{1}{\theta} + \frac{1}{q} \, . \tag{4.11.4}$$

By Strichartz's estimates, there exists K such that

$$\|\mathcal{T}(\cdot)\psi\|_{L^\infty(\mathbb{R}, H^1)} + \|\mathcal{T}(\cdot)\psi\|_{L^q(\mathbb{R}, W^{1,r})} \leq K\|\psi\|_{H^1}$$

for every $\psi \in H^1(\mathbb{R}^N)$. Given $M > 0$ and $0 \leq T_1, T_2$ such that $T_1 + T_2 > 0$, let

$$E = \{v \in L^\infty((-T_1, T_2), H^1(\mathbb{R}^N)) \cap L^q((-T_1, T_2), W^{1,r}(\mathbb{R}^N)),$$
$$\|v\|_{L^\infty((-T_1,T_2), H^1)} + \|v\|_{L^q((-T_1,T_2), W^{1,r})} \leq (K+1)M\} \, .$$

Endowed with the metric $\mathrm{d}(u, v) = \|v - u\|_{L^q((-T_1,T_2), L^r)}$, (E, d) is a complete metric space. Given $v \in E$, it follows from (4.11.3), (4.11.4), and Sobolev's and Hölder's inequalities that $h|v|^\alpha v \in L^{q'}((-T_1, T_2), W^{1,r'}(\mathbb{R}^N))$ and that

$$\begin{aligned} &\|h|v|^\alpha v\|_{L^{q'}((-T_1,T_2), W^{1,r'})} \\ &\leq C\|h\|_{L^\theta(-T_1,T_2)} \|v\|^\alpha_{L^\infty((-T_1,T_2), L^{2^*})} \|v\|_{L^q((-T_1,T_2), W^{1,r})} \\ &\leq C_1 \|h\|_{L^\theta(-T_1,T_2)} (K+1)^{\alpha+1} M^{\alpha+1} \, . \end{aligned} \tag{4.11.5}$$

Furthermore, given $u, v \in E$, one has as well

$$\begin{aligned}\|h(|v|^\alpha v - |u|^\alpha u)\|_{L^{q'}((-T_1,T_2),L^{r'})} \le \\ C\|h\|_{L^\theta(-T_1,T_2)}\big(\|v\|^\alpha_{L^\infty((-T_1,T_2),H^1)} \\ + \|u\|^\alpha_{L^\infty((-T_1,T_2),H^1)}\big)\|v-u\|_{L^q((-T_1,T_2),L^r)},\end{aligned} \tag{4.11.6}$$

and so

$$\|h(|v|^\alpha v - |u|^\alpha u)\|_{L^{q'}((-T_1,T_2),L^{r'})} \le C_2\|h\|_{L^\theta(-T_1,T_2)}(K+1)^\alpha M^\alpha \mathrm{d}(u,v). \tag{4.11.7}$$

Given $v \in E$ and $\psi \in H^1(\mathbb{R}^N)$ such that $\|\psi\|_{H^1} \le M$, let $\mathcal{H}(v)$ be defined by

$$\mathcal{H}(v)(t) = \mathcal{T}(t)\psi + i\int_0^t \mathcal{T}(t-s)h(s)|v|^\alpha v(s)ds \quad \text{for } t \in (-T_1,T_2).$$

It follows from Strichartz's estimates and (4.11.5) that

$$\mathcal{H}(v) \in C([-T_1,T_2],H^1(\mathbb{R}^N)) \cap L^q((-T_1,T_2),W^{1,r}(\mathbb{R}^N))$$

and

$$\begin{aligned}\|\mathcal{H}(v)\|_{L^\infty((-T_1,T_2),H^1)} + \|\mathcal{H}(v)\|_{L^q((-T_1,T_2),W^{1,r})} \le \\ KM + C_3(K+1)^{\alpha+1}M^{\alpha+1}\|h\|_{L^\theta(-T_1,T_2)}.\end{aligned}$$

Therefore, if T_1+T_2 is small enough so that

$$C_3(K+1)^{\alpha+1}M^\alpha\|h\|_{L^\theta(-T_1,T_2)} \le 1,$$

then $\mathcal{H}(v) \in E$. Furthermore, Strichartz's estimates and (4.11.7) imply that

$$\mathrm{d}(\mathcal{H}(v),\mathcal{H}(u)) \le C_4(K+1)^\alpha M^\alpha\|h\|_{L^\theta(-T_1,T_2)}\mathrm{d}(u,v).$$

Consequently, if T_1+T_2 is small enough so that

$$K_1M^\alpha\|h\|_{L^\theta(-T_1,T_2)} \le \frac{1}{2}, \tag{4.11.8}$$

where $K_1 = (K+1)^{\alpha+1}\max\{C_3,C_4\}$, then $\mathcal{H}$ is Lipschitz continuous $E \to E$ with Lipschitz constant $1/2$. Therefore, $\mathcal{H}$ has a unique fixed point $v \in E$, which satisfies equation (4.11.2). In addition, the first part of property (iv) follows from (4.11.8), (4.11.5), and Strichartz's estimates, and the second part from Strichartz's estimates. Uniqueness in the class $C([-T_1,T_2],H^1(\mathbb{R}^N))$ follows from (4.11.6) and Strichartz's estimates. (Note that uniqueness is a local property and needs only be established for T_1+T_2 small enough; see Step 2 of the proof of Theorem 4.6.1.) Now, by uniqueness, v can be extended to a maximal interval $(-T_{\min},T_{\max})$, and property (iii) follows from property (iv). Suppose that $T_{\max} < \infty$. Applying the above local existence result to $v(t)$, $t < T_{\max}$ with $T_1 = 0$, we see from (4.11.8) that if

$$K_1\|v(t)\|^\alpha_{H^1}\|h\|_{L^\theta(t,T_{\max})} \le \frac{1}{2},$$

then v can be continued up to and beyond $T_{\max}$, which is a contradiction. Therefore,

$$K_1\|v(t)\|^\alpha_{H^1}\|h\|_{L^\theta(t,T_{\max})} > \frac{1}{2},$$

which proves property (i). Property (ii) is proved by the same argument. Finally, since v satisfies equation (4.11.1) in $L^\theta_{\mathrm{loc}}((-T_{\min},T_{\max}),H^{-1}(\mathbb{R}^N))$ and h is real valued, property (v) is proved by standard arguments. For example, multiply the above equation by $|x|^2 e^{-\varepsilon |x|^2} \overline{v}$, take the imaginary part and integrate over $\mathbb{R}^N$, then let $\varepsilon \downarrow 0$ (see Lemma 6.5.2 below).

If $N = 2$, the proof is the same as in the case $N \geq 3$, except that we set $r = 2\theta$ and use the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ with $p = \alpha\theta/(\theta - 1)$.

If $N = 1$, the argument is slightly simpler. We let

$$E = \left\{ v \in L^\infty((-T_1, T_2), H^1(\mathbb{R})) : \|v\|_{L^\infty((-T_1,T_2),H^1)} \leq 2M \right\}$$

equipped with the metric $d(u, v) = \|v - u\|_{L^\infty((-T_1,T_2),L^2)}$, and use the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. □

We now study the continuous dependence of the solutions on the initial value. The result is the following.

THEOREM 4.11.2. *Under the assumptions of Theorem* 4.11.1, *suppose there exists* $\theta_1 > \theta$ *such that* $h \in L^{\theta_1}_{\mathrm{loc}}(\mathbb{R})$. *The solution* v *of* (4.11.2) *given by Theorem* 4.11.1 *depends continuously on* ψ *in the following way.*

(i) *The mappings* $\psi \mapsto T_{\max}$ *and* $\psi \mapsto T_{\min}$ *are lower semicontinuous* $H^1(\mathbb{R}^N) \to (0, \infty]$.

(ii) *If* $\psi_n \underset{n\to\infty}{\longrightarrow} \psi$ *in* $H^1(\mathbb{R}^N)$ *and if* v_n *denotes the solution of* (4.11.2) *with initial value* ψ_n, *then* $v_n \to v$ *in* $C([-T_1, T_2], H^1(\mathbb{R}^N))$ *for any interval* $[-T_1, T_2] \subset (-T_{\min}, T_{\max})$. *If, in addition,* $|\cdot|\psi_n \to |\cdot|\psi$ *in* $L^2(\mathbb{R}^N)$, *then* $|\cdot|v_n \to |\cdot|v$ *in* $C([T_1, T_2], L^2(\mathbb{R}^N))$.

PROOF. We apply the method of Section 4.4. We proceed in two steps.

STEP 1. We show that for every $M > 0$, there exists $\tau > 0$ such that if $\psi \in H^1(\mathbb{R}^N)$ satisfies $\|\psi\|_{H^1} < M$, then $[-\tau, \tau] \subset (-T_{\min}, T_{\max})$, and v has the following continuity properties.

(a) If $\|\psi\|_{H^1} < M$, $\psi_n \underset{n\to\infty}{\longrightarrow} \psi$ in $H^1(\mathbb{R}^N)$, and if v_n denotes the solution of (4.11.2) with initial value ψ_n, then $v_n \to v$ in $C([-\tau, \tau], H^1(\mathbb{R}^N))$.

(b) If, in addition, $|\cdot|\psi_n \to |\cdot|\psi$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$, then $|\cdot|v_n \to |\cdot|v$ in $C([-\tau, \tau], L^2(\mathbb{R}^N))$ as $n \to \infty$.

We only prove the result in the case $N \geq 3$ (see the proof of Theorem 4.11.1 for the necessary modifications in the cases $N = 1, 2$). Given $M > 0$, we choose τ so that the inequality in property (iv) of Theorem 4.11.1 is met whenever $\|\psi\|_{H^1} < M$. In particular, if $\|\psi\|_{H^1} < M$, then $[-\tau, \tau] \subset (-T_{\min}, T_{\max})$. Next, observe that $1/\theta > (4 - \alpha N)/4$, and so we may assume without loss of generality that $1/\theta_1 > (4 - \alpha N)/4$. Therefore, if we define σ by

$$\frac{1}{\sigma} = \frac{2}{N}\left(1 - \frac{1}{\theta_1}\right),$$

then $2 < \alpha\sigma < 2N/(N - 2)$. Let now ρ be defined by

$$1 - \frac{2}{\rho} = \frac{1}{\sigma}.$$

Since $\sigma > N/2$, we see that $2 < \rho < 2N/(N-2)$. Finally, let γ be such that (γ, ρ) is an admissible pair. It follows easily from Hölder's inequality that for every $-\infty < a < b < \infty$,

$$\|hwz\|_{L^{\gamma'}((a,b),L^{\rho'})} \le \left(\int_a^b |h(s)|^{\theta_1} \|w(s)\|_{L^\sigma}^{\theta_1} \right)^{1/\theta_1} \|z\|_{L^\gamma((a,b),L^\rho)} . \tag{4.11.9}$$

Consider now ψ such that $\|\psi\|_{H^1} \le M$, and let ψ_n be as in (a). Let v, v_n be the corresponding solutions of (4.11.2). We deduce from Strichartz's estimates that there exists C, depending only on γ such that

$$\begin{aligned} &\|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} + \|v - v_n\|_{L^\infty((-\tau,\tau),H^1)} \le \\ &\qquad C\|\psi - \psi_n\|_{H^1} + \|h(|v|^\alpha v - |v_n|^\alpha v_n)\|_{L^{\gamma'}((-\tau,\tau),W^{1,\rho'})} . \end{aligned} \tag{4.11.10}$$

On the other hand, a straightforward calculation shows that

$$|\nabla(|v|^\alpha v - |v_n|^\alpha v_n)| \le C|v_n|^\alpha |\nabla v - \nabla v_n| + \phi(v, v_n)|\nabla v| , \tag{4.11.11}$$

where C depends on α, and the function $\phi(x, y)$ is bounded by $C(|x|^\alpha + |y|^\alpha)$ and satisfies $\phi(x,y) \underset{y\to x}{\longrightarrow} 0$. Therefore, applying (4.11.9), (4.11.10), and (4.11.11), we get

$$\begin{aligned} &\|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} + \|v - v_n\|_{L^\infty((-\tau,\tau),H^1)} \\ &\quad \le C\|\psi - \psi_n\|_{H^1} \\ &\qquad + \|h\|_{L^{\theta_1}(-\tau,\tau)} \|v_n\|_{L^\infty((-\tau,\tau),L^{\alpha\sigma})} \|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} \\ &\qquad + C\left(\int_a^b |h(s)|^{\theta_1} \|\phi(v, v_n)\|_{L^\sigma}^{\theta_1} \right)^{1/\theta_1} \|v\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} . \end{aligned} \tag{4.11.12}$$

Note that by property (iv) of Theorem 4.11.1, v_n is bounded in $H^1(\mathbb{R}^N)$, hence in $L^{\alpha\sigma}(\mathbb{R}^N)$, with the bound for $t \in [-\tau, \tau]$, depending only on $\|\psi_n\|_{H^1}$, hence (for large values of n) only on M. Also, the bound on $\|v\|_{L^\gamma((-\tau,\tau),W^{1,\rho})}$ depends only on M. Therefore, it follows from (4.11.12) that

$$\begin{aligned} &\|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} + \|v - v_n\|_{L^\infty((-\tau,\tau),H^1)} \\ &\quad \le C\|\psi - \psi_n\|_{H^1} + C\|h\|_{L^{\theta_1}(-\tau,\tau)} \|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} \\ &\quad + C\left(\int_a^b |h(s)|^{\theta_1} \|\phi(v, v_n)\|_{L^\sigma}^{\theta_1} \right)^{1/\theta_1} , \end{aligned}$$

where the constant C depends only on M. Therefore, if we consider τ possibly smaller so that $C\|h\|_{L^{\theta_1}(-\tau,\tau)} \le 1/2$ (note that τ still depends on M), we have

$$\begin{aligned} &\|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} + \|v - v_n\|_{L^\infty((-\tau,\tau),H^1)} \le \\ &\qquad C\|\psi - \psi_n\|_{H^1} + C\left(\int_a^b |h(s)|^{\theta_1} \|\phi(v, v_n)\|_{L^\sigma}^{\theta_1} \right)^{1/\theta_1} . \end{aligned}$$

Therefore, property (a) follows, provided we show that

$$\left(\int_a^b |h(s)|^{\theta_1} \|\phi(v, v_n)\|_{L^\sigma}^{\theta_1} \right) \underset{n\to\infty}{\longrightarrow} 0 .$$

By the dominated convergence theorem, it suffices to verify that

$$\|\phi(v, v_n)\|_{L^\sigma} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for all } t \in [-\tau, \tau].$$

To see this, we argue by contradiction. We assume that there exist t and a subsequence, which we still denote by $v_n(t)$ such that $\|\phi(v(t), v_n(t))\|_{L^\sigma} \geq \mu > 0$. Note that $v_n(t) \to v(t)$ in $L^2(\mathbb{R}^N)$ and $v_n(t)$ is bounded in $H^1(\mathbb{R}^N)$ by property (iv) of Theorem 4.11.1. Therefore, by Sobolev's and Hölder's inequalities, $v_n(t) \to v(t)$ in $L^{\alpha\sigma}(\mathbb{R}^N)$. It follows that there exist a subsequence, which we still denote by $v_n(t)$, and a function $f \in L^{\alpha\sigma}(\mathbb{R}^N)$ such that $v_n(t) \to v(t)$ a.e. in $\mathbb{R}^N$ and $|v_n(t)| \leq f$ a.e. in $\mathbb{R}^N$. Applying the dominated convergence theorem, we deduce that $\|\phi(v(t), v_n(t))\|_{L^\sigma} \to 0$, which is a contradiction. Hence property (a) is proven.

Property (b) follows from property (a). (See Corollary 6.5.3 below.)

STEP 2. Let $\psi \in H^1(\mathbb{R}^N)$, let v be the maximal solution of (4.11.2) given by Theorem 4.11.1, and let $[-T_1, T_2] \subset (-T_{\min}, T_{\max})$. Set

$$M = 2 \sup_{-T_1 \leq t \leq T_2} \|v(t)\|_{H^1},$$

and consider $\tau > 0$ given by Step 1. By applying Step 1 m times, where $(m-1)\tau < T_1 + T_2 \leq m\tau$, we see that if $\|\psi - \widetilde{\psi}\|_{H^1}$ is small enough, then the solution of (4.11.2) with initial value $\widetilde{\psi}$ exists on $[-T_1, T_2]$. Property (i) follows. Property (ii) follows easily from the same argument. □

4.12. Comments

The results of this chapter are mostly based on the Strichartz estimates. Thus we may expect that the results of the previous sections have a counterpart for the abstract equation

$$\begin{cases} iu_t + Au + g(u) = 0 \\ u(0) = x, \end{cases} \tag{4.12.1}$$

whenever $\mathcal{T}(t) = e^{itA}$ satisfies Strichartz-type estimates. Theorem 2.7.1 gives a sufficient condition for such estimates, which we recall below. Let Ω be a domain of $\mathbb{R}^N$ and let $X = L^2(\Omega)$. Let A be a $\mathbb{C}$-linear, self-adjoint ≤ 0 operator on X with domain $D(A)$. Let X_A be the completion of $D(A)$ for the norm $\|x\|_{X_A}^2 = \|x\|_X^2 - (Ax, x)_X$, $X_A^\star = (X_A)^\star$, and $\overline{A}$ be the extension of A to $(D(A))^\star$. Finally, let $\mathcal{T}(t)$ be the group of isometries generated on $(D(A))^\star$, $X_A^\star$, X, X_A, or $D(A)$ by the skew-adjoint operator iA. If, in addition,

$$\|\mathcal{T}(t)\varphi\|_{L^\infty} \leq C|t|^{-\frac{N}{2}} \|\varphi\|_{L^1} \quad \text{for all } \varphi \in \mathcal{D}(\Omega), \tag{4.12.2}$$

then $\mathcal{T}(t)$ satisfies the Strichartz estimates (see Theorem 2.7.1). As a first application, we have the following analogue of Theorem 4.3.1.

THEOREM 4.12.1. *Let A be as in the statement of Theorem 3.7.1, and assume that $\mathcal{T}(t) = e^{itA}$ satisfies estimate (4.12.2). Let $g \in C(X_A, X_A^\star)$, and assume that there exist $g_1, \dots, g_k \in C(X_A, X_A^\star)$ such that*

$$g = g_1 + \dots + g_k,$$

where each of the g_j's satisfies the assumptions (3.7.3)–(3.7.6) for some exponents r_j, ρ_j. Finally, let

$$G = G_1 + \dots + G_k,$$

and set $E(v) = (1/2)(\|v\|_{X_A}^2 - \|v\|_X^2) - G(v)$ *for all* $v \in X_A$. *It follows that the initial value problem* (4.12.1) *is locally well posed in* X_A. *Moreover, there is conservation of charge and energy; i.e.,*

$$\|u(t)\|_{L^2} = \|x\|_{L^2}, \quad E(u(t)) = E(x), \quad \text{for all } t \in (-T_{\min}, T_{\max}),$$

where u *is the solution of* (4.12.1) *with the initial value* $x \in X_A$. *(Here, the notion of local well-posedness is as in Section* 3.1).

PROOF. By Theorem 3.7.1, we need only show uniqueness. This is proved like Proposition 4.2.3 by using the Strichartz estimates of Theorem 2.7.1. □

REMARK 4.12.2. If we assume further that for every $A > 0$, there exist $\varepsilon(A) > 0$ and $K(A) < \infty$ such that

$$G(u) \leq K(A) + \frac{1 - \alpha(A)}{2} \|u\|_A^2$$

for all $u \in H_0^1(\Omega)$ such that $\|u\|_{L^2} \leq A$, then all solutions given by Theorem 3.7.1 or Theorem 4.12.1 are global (see Section 3.4).

We now give an analogue of Theorem 4.8.1. Most objects used in the statement and proof of Theorem 4.8.1 have an obvious analogue in the abstract setting. It is clear that Δ should be replaced by A and $H^2(\mathbb{R}^N)$ by $D(A)$. As for the analogue of $H^s(\mathbb{R}^N)$ with $0 \leq s < 2$, it is clear from the proof of Theorem 4.8.1 that the essential property we need is the interpolation estimate (4.8.20). In fact, we do not fully use (4.8.20), we only need an estimate of the type $\|u\|_{H^s} \leq \varepsilon\|u\|_{H^2} + C_\varepsilon\|u\|_{L^2}$. Thus we may assume that there exists a space $D(A) \hookrightarrow Y \hookrightarrow X$ such that for every $\varepsilon > 0$ there exists a constant C_ε with

$$\|u\|_Y \leq \varepsilon\|u\|_{D(A)} + \|u\|_X \quad \text{for all } u \in D(A). \tag{4.12.3}$$

Taking $Y = D(A^{\frac{s}{2}})$ is a possible choice. Depending on the applications, good choices may also be an L^p space or even an H^s space. Following the proof of Theorem 4.8.1, it is not difficult to establish the following result.

THEOREM 4.12.3. *Let* A *be as in the statement of Theorem* 3.7.1, *and assume that* $\mathcal{T}(t) = e^{itA}$ *satisfies estimate* (4.12.2). *Assume there exists a Banach space* $D(A) \hookrightarrow Y \hookrightarrow X$ *such that for every* $\varepsilon > 0$ *there exists a constant* C_ε *for which* (4.12.3) *holds. Let* $g = g_1 + \cdots + g_k$ *with* $g_j : D(A) \to X$, *and assume there exist exponents* $2 \leq r_j, \rho_j < 2N/(N-2)$ $(2 \leq r_j, \rho_j < \infty$ *if* $N = 1)$ *such that* $g_j \in C(Y, X)$ *is bounded on bounded sets and*

$$\|g_j(u) - g_j(v)\|_{L^{\rho_j'}} \leq L(M)\|u - v\|_{L^{r_j}}$$

for all $u, v \in D(A)$ *such that* $\|u\|_Y, \|v\|_Y \leq M$. *For every* $x \in D(A)$, *there exist* $T_{\max}, T_{\min} > 0$ *and a unique, maximal solution* $u \in C((-T_{\min}, T_{\max}), D(A)) \cap C^1((-T_{\min}, T_{\max}), X)$ *of* (4.12.1). *Moreover, there is the blowup alternative; i.e., if* $T_{\max} < \infty$ *(respectively,* $T_{\min} < \infty$*), then* $\|u(t)\|_{D(A)} \to \infty$ *as* $t \uparrow T_{\max}$ *(respectively, as* $t \downarrow -T_{\min}$*).*

REMARK 4.12.4. Note that one can also show, as in Theorem 4.8.1, a form of continuous dependence as well as the conservation laws (whenever the relevant conditions on g are satisfied).

REMARK 4.12.5. Note that the results of Section 4.6, i.e., the existence of L^2 solutions, have an obvious counterpart in the setting of Theorems 4.12.1 and 4.12.3. This is not the case for those of Sections 4.4 (Kato's method), 4.9 (H^s solutions, $s < N/2$), and 4.10 (H^m solutions, $m > N/2$). Indeed, those results are obtained by differentiating the equation, in one way or another, with respect to space. It is not clear, in general, what a good analogue of the space differentiation would be.

REMARK 4.12.6. We established in the previous sections local existence results for (4.1.1) in spaces of the type $H^s(\mathbb{R}^N)$ with $s \geq 0$. One may wonder if there is any local well-posedness result in H^s spaces with $s < 0$, as is the case for KdV, for example. This is a delicate question. For nonlinearities of the type $g(u) = \lambda|u|^\alpha u$, the answer is no; see Birnir et al. [32], Christ, Colliander, and Tao [80], and Kenig, Ponce, and Vega [216]. On the other hand, the answer is yes if, for example, $g(u) = \lambda u^2$. See Kenig, Ponce, and Vega [214]. More generally, one can investigate the minimal value of s for which the initial value problem (4.1.1) is (locally or globally) well-posed in H^s. This question has been (and is still being) studied by many authors. See, for example, Bourgain [38], Kenig, Ponce, and Vega [214], Staffilani [318], and Tao [335]. Note also that the nonlinear Schrödinger equation (4.1.1) can be solved in certain spaces that are not based on L^2, like Lorenz spaces $L^{p,\infty}$ (see Cazenave, Vega, and Vilela [67]) or Besov spaces $\dot{B}^s_{2,\infty}$ (see Planchon [298, 299]).

CHAPTER 5

Regularity and the Smoothing Effect

In this chapter we consider the nonlinear Schrödinger equation (4.1.1) in $\mathbb{R}^N$, We address the problems of regularity of solutions and the C^∞ smoothing effect.

The problem of regularity of solutions can be formulated as follows. Suppose $\varphi \in H^s(\mathbb{R}^N)$ for some $s \geq 0$ and suppose the nonlinearity g is such that there is a local existence theory of H^s solutions (see Chapter 4). It follows that there is a maximal solution $u \in C((-T_{\text{min}}, T_{\text{max}}), H^s(\mathbb{R}^N))$ of (4.1.1). Suppose now that φ is smoother than just $H^s(\mathbb{R}^N)$, say $\varphi \in H^{s_1}(\mathbb{R}^N)$ for some $s_1 > s$. The question is then: does u belong to $C((-T_{\text{min}}, T_{\text{max}}), H^{s_1}(\mathbb{R}^N))$? In fact, one can simplify the problem by assuming that there is also a local existence theory of H^{s_1} solutions. It follows that there is a maximal solution $u_1 \in C((-T^1_{\text{min}}, T^1_{\text{max}}), H^{s_1}(\mathbb{R}^N))$ of (4.1.1). Of course, an H^{s_1} solution is in particular an H^s solution. By uniqueness of H^s solutions, we deduce that $u = u_1$ on the larger interval where the two solutions are defined. We then see that $(-T^1_{\text{min}}, T^1_{\text{max}}) \subset (-T_{\text{min}}, T_{\text{max}})$ because, again, an H^{s_1} solution is an H^s solution. The question then becomes: does $T_{\text{min}} = T^1_{\text{min}}$ and $T_{\text{max}} = T^1_{\text{max}}$? In other words, can u blow up in $H^{s_1}(\mathbb{R}^N)$ before it (possibly) blows up in $H^s(\mathbb{R}^N)$? At this level of generality, there is no complete answer to this question. (It seems that there is no counterexample either.) We give, however, partial answers to this question in Sections 5.1–5.5.

The problem of the C^∞ smoothing effect is the following: Let $\varphi \in H^s(\mathbb{R}^N)$ and let $u \in C(I, H^s(\mathbb{R}^N))$ be a solution of (4.1.1). Under what conditions on φ and g is the solution u in $C^\infty((I \setminus \{0\}) \times \mathbb{R}^N)$? In other words, when are the properties of Section 2.5 preserved for the nonlinear equation? We study this question in Section 5.6.

Finally, we observe that the results of this chapter are stated for one equation, but similar results obviously hold for systems of the same form. See Remark 3.3.12 for an appropriate setting.

5.1. H^s Regularity, $0 < s < \min\{1, N/2\}$

In this section we consider local nonlinearities, so that we may apply the H^s theory of Section 4.9. In this case, there is regularity at the H^{s_1} level for any $s < s_1 < \min\{1, N/2\}$, as the following result shows.

THEOREM 5.1.1. *Let $0 \leq s < \min\{1, N/2\}$. Let $g \in C(\mathbb{C}, \mathbb{C})$ satisfy $g(0) = 0$ and*

$$|g(u) - g(v)| \leq C(1 + |u|^\alpha + |v|^\alpha)|u - v| \quad \textit{for all } u, v \in \mathbb{C}$$

with

$$0 \leq \alpha < \frac{4}{N - 2s}.$$

Let (γ,ρ) be the admissible pair defined by (4.9.3). *Let $\varphi \in H^s(\mathbb{R}^N)$ and let $u \in C((-T_{\min},T_{\max}),H^s(\mathbb{R}^N))\cap L^\gamma_{\text{loc}}((-T_{\min},T_{\max}),B^s_{\rho,2}(\mathbb{R}^N))$ be the maximal H^s solution of* (4.1.1) *given by Theorem* 4.9.1 (*case $s>0$*) *or Theorem* 4.6.1 (*case $s=0$*). *If $\varphi \in H^{s_1}(\mathbb{R}^N)$ for some $s<s_1<\min\{1,N/2\}$, then for every admissible pair (q,r), $u\in C((-T_{\min},T_{\max}),H^{s_1}(\mathbb{R}^N))\cap L^q_{\text{loc}}((-T_{\min},T_{\max}),B^{s_1}_{r,2}(\mathbb{R}^N))$.*

PROOF. We consider $t\geq 0$, the argument for $t\leq 0$ being the same. We know that u is an H^{s_1} solution (in the sense of Theorem 4.9.1) on some maximal interval $[0,T)$ with $T\leq T_{\max}$, and we need to show that $T=T_{\max}$ (see the discussion at the beginning of this chapter). We argue by contradiction, and we suppose $T<T_{\max}$. In particular, $T<\infty$ so that

$$\|u(t)\|_{H^{s_1}} \underset{t\uparrow T}{\longrightarrow} \infty . \tag{5.1.1}$$

Moreover, since $T<T_{\max}$,

$$\|u\|_{L^\gamma((0,T),B^s_{\rho,2})} + \sup_{0\leq t\leq T}\|u(t)\|_{H^s} < \infty . \tag{5.1.2}$$

We now reproduce some estimates from the proof of Theorem 4.9.1. We decompose $g=g_1+g_2$ as in Remark 4.9.2 and we deduce from Propositions 4.9.4 and 4.9.5 that

$$\|g_1(u)\|_{B^{s_1}_{2,2}} \leq C\|u\|_{B^{s_1}_{2,2}} \tag{5.1.3}$$

and

$$\|g_2(u)\|_{B^{s_1}_{\rho',2}} \leq C\|u\|^\alpha_{L^{\frac{N(\alpha+2)}{N-2s}}}\|u\|_{B^{s_1}_{\rho,2}} \leq C\|u\|^\alpha_{B^s_{\rho,2}}\|u\|_{B^{s_1}_{\rho,2}} , \tag{5.1.4}$$

where the last inequality follows from the embedding $B^s_{\rho,2}(\mathbb{R}^N)\hookrightarrow L^{\frac{N(\alpha+2)}{N-2s}}(\mathbb{R}^N)$. It then follows from (5.1.4), Hölder's inequality and (5.1.2) that for any interval $I\subset(0,T)$,

$$\|u\|_{L^\gamma(I,B^{s_1}_{\rho,2})} \leq C\|u\|^\alpha_{L^\gamma(I,B^s_{\rho,2})}\|u\|_{L^p(I,B^{s_1}_{\rho,2})} \leq C\|u\|_{L^p(I,B^{s_1}_{\rho,2})} , \tag{5.1.5}$$

where $1\leq p<\gamma$ is given by

$$\frac{1}{p}=\frac{1}{\gamma}+\frac{4-\alpha(N-2s)}{4} .$$

We now apply Strichartz's inequalities in Besov spaces and we deduce from equation (4.1.2) and from (5.1.3) and (5.1.5) that (see the proof of Theorem 4.9.1)

$$\|u\|_{L^\infty(I,H^{s_1})}+\|u\|_{L^\gamma(I,B^{s_1}_{\rho,2})} \leq C\|\varphi\|_{H^{s_1}}+C\|u\|_{L^1(I,H^{s_1})}+C\|u\|_{L^p(I,B^{s_1}_{\rho,2})} \tag{5.1.6}$$

for every interval $0\in I\subset(0,T)$. We now let $0<\varepsilon<T$ and we consider I of the form $(0,\tau)$ with $\varepsilon<\tau<T$. We have

$$\begin{aligned}\|u\|_{L^1(I,H^{s_1})} &\leq \|u\|_{L^1((0,\tau-\varepsilon),H^{s_1})}+\|u\|_{L^1((\tau-\varepsilon,\tau),H^{s_1})}\\ &\leq \|u\|_{L^1((0,T-\varepsilon),H^{s_1})}+\varepsilon\|u\|_{L^\infty((\tau-\varepsilon,\tau),H^{s_1})}\\ &\leq C_\varepsilon+\varepsilon\|u\|_{L^\infty(I,H^{s_1})} .\end{aligned}$$

Similarly,

$$\|u\|_{L^p(I,B^{s_1}_{\rho,2})} \leq C_\varepsilon+\varepsilon^{\frac{4-\alpha(N-2s)}{4}}\|u\|_{L^\gamma(I,B^{s_1}_{\rho,2})} .$$

It then follows from (5.1.6) that

$$\|u\|_{L^\infty(I,H^{s_1})} + \|u\|_{L^\gamma(I,B^{s_1}_{\rho,2})} \le C + C_\varepsilon + \varepsilon C\|u\|_{L^\infty(I,H^{s_1})} + \varepsilon^{\frac{4-\alpha(N-2s)}{4}} C\|u\|_{L^\gamma(I,B^{s_1}_{\rho,2})},$$

where the various constants are independent of $\tau < T$. We therefore may fix ε small enough so that the last two terms in the right-hand side are absorbed by the left-hand side. It follows that

$$\|u\|_{L^\infty(I,H^{s_1})} + \|u\|_{L^\gamma(I,B^{s_1}_{\rho,2})} \le C,$$

where C is independent of $\tau < T$. Letting $\tau \uparrow T$, we obtain a contradiction with (5.1.1). □

5.2. H^1 Regularity

In this section we establish H^1 regularity. This can be done for local nonlinearities, by starting from the H^s solutions of Section 4.9. For more general nonlinearities, this can be done starting from the L^2 solutions of Section 4.6. We begin with the first case.

THEOREM 5.2.1. *Let $0 \le s < \min\{1, N/2\}$. Let $g \in C(\mathbb{C}, \mathbb{C})$ satisfy $g(0) = 0$ and*

$$|g(u) - g(v)| \le C(1 + |u|^\alpha + |v|^\alpha)|u - v| \quad \text{for all } u, v \in \mathbb{C}$$

with

$$0 \le \alpha < \frac{4}{N - 2s}.$$

Let (γ, ρ) be the admissible pair defined by (4.9.3). Let $\varphi \in H^s(\mathbb{R}^N)$ and let $u \in C((-T_{\min}, T_{\max}), H^s(\mathbb{R}^N)) \cap L^\gamma_{\text{loc}}((-T_{\min}, T_{\max}), B^s_{\rho,2}(\mathbb{R}^N))$ be the maximal H^s solution of (4.1.1) given by Theorem 4.9.1 (case $s > 0$) or Theorem 4.6.1 (case $s = 0$)). If $\varphi \in H^1(\mathbb{R}^N)$, then $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N))$.

PROOF. The proof is very similar to the proof of Theorem 5.2.1, except that we use the Sobolev spaces $H^1(\mathbb{R}^N)$ and $W^{1,r}(\mathbb{R}^N)$ instead of $H^{s_1}(\mathbb{R}^N)$ and $B^{s_1}_{r,2}(\mathbb{R}^N)$, and the inequalities

$$\|g_1(u)\|_{H^1} \le C\|u\|_{H^1}$$

and

$$\|g_2(u)\|_{W^{1,\rho'}} \le C\|u\|^\alpha_{L^{\frac{N(\alpha+2)}{N-2s}}} \|u\|_{W^{1,\rho}} \le C\|u\|^\alpha_{B^s_{\rho,2}} \|u\|_{W^{1,\rho}}$$

instead of (5.1.3) and (5.1.4). □

We now consider more general nonlinearities and study the H^1 regularity of the L^2 solutions of Section 4.6. Since there are two slightly different results for the local existence in H^1 (Sections 4.3 and 4.4), there are two possible regularity results, depending on what set of assumptions on g we choose. For simplicity, we only establish one such result.

We recall the assumptions of Theorem 4.6.4. Let

$$2 \le r < \frac{2N}{N-2} \quad (2 \le r \le \infty \text{ if } N = 1), \tag{5.2.1}$$

and let $g : L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \to L^{r'}(\mathbb{R}^N)$. Assume that there exists $\alpha > 0$ such that for every $M > 0$, there exist $K(M) < \infty$ such that

$$\|g(v) - g(u)\|_{L^{r'}} \le K(M)(\|u\|_{L^r}^{\alpha} + \|v\|_{L^r}^{\alpha})\|v - u\|_{L^r} \tag{5.2.2}$$

for all $u, v \in L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ such that $\|u\|_{L^2}, \|v\|_{L^2} \le M$. Set

$$\frac{2}{q} = N\Big(\frac{1}{2} - \frac{1}{r}\Big)$$

so that (q, r) is an admissible pair and assume

$$\alpha + 2 < q\,. \tag{5.2.3}$$

We have the following result.

THEOREM 5.2.2. *Let $g = g_1 + \cdots + g_k$ be as in Theorem 4.6.4; i.e., each of the g_j's satisfies (5.2.1)–(5.2.3) for some r_j, α_j, q_j. Assume, in addition, that*

$$\|\nabla g_j(u)\|_{L^{r_j'}} \le K(M)\|u\|_{L^{r_j}}^{\alpha_j}\|\nabla u\|_{L^{r_j}} \tag{5.2.4}$$

for all $u \in H^1(\mathbb{R}^N)$ such that $\|u\|_{L^2} \le M$. Set $r = \max\{r_1, \dots, r_k\}$ and $q = \min\{q_1, \dots, q_k\}$. Let $\varphi \in L^2(\mathbb{R}^N)$ and let $u \in C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N)) \cap L^q_{\mathrm{loc}}((-T_{\min}, T_{\max}), L^r(\mathbb{R}^N))$ be the maximal solution of problem (4.1.1) given by Theorem 4.6.4. If $\varphi \in H^1(\mathbb{R}^N)$, then $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N))$.

PROOF. We first observe that g satisfies the assumptions of Theorem 4.4.6, so that there is local existence in $H^1(\mathbb{R}^N)$. We consider $t \ge 0$, the argument for $t \le 0$ being the same. We know that u is an H^1 solution (in the sense of Theorem 4.4.6) on some maximal interval $[0, T)$ with $T \le T_{\max}$, and we need to show that $T = T_{\max}$ (see the discussion at the beginning of this chapter). We argue by contradiction, and we suppose $T < T_{\max}$. In particular, $T < \infty$ so that

$$\|u(t)\|_{H^1} \underset{t\uparrow T}{\longrightarrow} \infty\,. \tag{5.2.5}$$

Moreover, since $T < T_{\max}$,

$$\sup_{0\le t\le T} \|u(t)\|_{L^2} + \sup_{1\le j\le k} \|u\|_{L^{q_j}((0,T),L^{r_j})} < \infty\,. \tag{5.2.6}$$

We now reproduce some estimates from the proof of Theorem 4.4.6. It follows from (5.2.2) and (5.2.4) that

$$\|g_j(u)\|_{W^{1,r_j'}} \le \|g(0)\|_{L^{r_j'}} + CK(M)\|u\|_{L^{r_j}}^{\alpha_j}\|u\|_{W^{1,r_j}}\,.$$

Applying (5.2.6), we deduce that

$$\|g_j(u)\|_{W^{1,r_j'}} \le C + C\|u\|_{L^{r_j}}^{\alpha_j}\|u\|_{W^{1,r_j}} \quad \text{for all } 0 \le t < T.$$

It then follows from Hölder's inequality that

$$\|g_j(u)\|_{L^{q_j'}(I,W^{1,r_j'})} \le CT^{\frac{1}{q_j'}} + C\|u\|_{L^{q_j}(I,L^{r_j})}^{\alpha_j}\|u\|_{L^{p_j}(I,W^{1,r_j})}$$

for any interval $I \in [0, T)$, where

$$\frac{1}{p_j} = \frac{1}{q_j} + 1 - \frac{\alpha_j + 2}{q_j} \tag{5.2.7}$$

so that $p_j < q_j$ by (5.2.3). Using again (5.2.6), we see that

$$\|g_j(u)\|_{L^{q'_j}(I,W^{1,r'_j})} \le C + C\|u\|_{L^{p_j}(I,W^{1,r_j})}. \tag{5.2.8}$$

We now let $0 < \varepsilon < T$ and we consider I of the form $(0,\tau)$ with $\varepsilon < \tau < T$. We have

$$\begin{aligned}\|u\|_{L^{p_j}(I,W^{1,r_j})} &\le \|u\|_{L^{p_j}((0,\tau-\varepsilon),W^{1,r_j})} + \|u\|_{L^{p_j}((\tau-\varepsilon,\tau),W^{1,r_j})}\\ &\le \|u\|_{L^{p_j}((0,T-\varepsilon),W^{1,r_j})} + \varepsilon^{1-\frac{\alpha_j+2}{q_j}}\|u\|_{L^{q_j}((\tau-\varepsilon,\tau),W^{1,r_j})}\end{aligned}$$

by (5.2.7). We next observe that u is an H^1 solution on $(0,T)$ so that by Theorem 4.4.6, $u \in L^{p_j}((0,T-\varepsilon),W^{1,r_j}(\mathbb{R}^N))$ for every $\varepsilon > 0$. It then follows from (5.2.8) and the above estimate that

$$\|g_j(u)\|_{L^{q'_j}(I,W^{1,r'_j})} \le C_\varepsilon + \varepsilon^{1-\frac{\alpha_j+2}{q_j}} C\|u\|_{L^{q_j}(I,W^{1,r_j})}, \tag{5.2.9}$$

where the constants are independent of $\tau < T$. We now apply Strichartz's inequalities and we deduce from equation (4.1.2) and from (5.2.9) that

$$\|u\|_{L^\infty(I,H^1)} + \sum_{j=1}^k \|u\|_{L^{q_j}(I,W^{1,r_j})} \le C_\varepsilon + C\sum_{j=1}^k \varepsilon^{1-\frac{\alpha_j+2}{q_j}}\|u\|_{L^{q_j}(I,W^{1,r_j})}.$$

We therefore may fix ε small enough so that the sum in the right-hand side is absorbed by the left-hand side. It follows that

$$\|u\|_{L^\infty(I,H^1)} + \sum_{j=1}^k \|u\|_{L^{q_j}(I,W^{1,r_j})} \le C,$$

where C is independent of $\tau < T$. Letting $\tau \uparrow T$, we obtain a contradiction with (5.2.5). □

REMARK 5.2.3. Theorem 5.2.2 applies to the model case

$$g(u) = Vu + f(u(\cdot)) + (W \star |u|^2)u$$

under the following assumptions. The functions V and $\nabla V \in L^\delta(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\delta \ge 1$, $\delta > N/2$, and $W \in L^\sigma(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\sigma \ge 1$, $\sigma > N/2$. We have $f(0) = 0$ and

$$|f(z_1) - f(z_2)| \le C(1 + |z_1| + |z_2|)^\beta |z_1 - z_2| \quad \text{for some } 0 \le \beta < \frac{4}{N}.$$

The fact that the assumptions are satisfied is easily verified; see Corollary 4.6.5 and Remark 4.4.8 for the details.

5.3. H^2 Regularity

In this section we study the H^2 regularity of solutions. Since we already established the H^1 regularity in the preceding section, one possibility is to start from an H^1 solution. However, we then need the assumptions of either Theorem 4.3.1 or Theorem 4.4.6. This imposes either the Hamiltonian structure or $W^{1,p}$ regularity of g. On the other hand, if we start from an L^2 solution, then we do not need such assumptions on g (see Theorem 4.6.4). Since the assumptions on g for local existence in H^2 require neither the Hamiltonian structure nor the $W^{1,p}$ regularity (see Theorem 4.8.1), it may be more economical for L^2 solutions to jump directly to the H^2 regularity. For this reason, we present two regularity results, one for H^1 solutions and the other for L^2 solutions.

THEOREM 5.3.1. *Let $g = g_1 + \cdots + g_k$ satisfy the assumptions of either Theorem* 4.3.1 *or Theorem* 4.4.6. *Assume further that there exists* $0 \le s < 2$ *such that, for all* $1 \le j \le k$,

$$\|g_j(u)\|_{L^2} \le C(M)(1 + \|u\|_{H^s}) \tag{5.3.1}$$

for all $u \in H^s(\mathbb{R}^N)$ such that $\|u\|_{H^1} \le M$. Let $\varphi \in H^1(\mathbb{R}^N)$ and let $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N))$ be the maximal solution of the problem (4.1.1) *given by Theorem* 4.3.1 *or Theorem* 4.4.6. *If $\varphi \in H^2(\mathbb{R}^N)$, then it follows that $u \in C((-T_{\min}, T_{\max}), H^2(\mathbb{R}^N))$.*

PROOF. We recall that if g satisfies the assumptions of either Theorem 4.3.1 or Theorem 4.4.6 as well as (5.3.1), then g satisfies the assumptions of Theorem 4.8.1, so there is local existence in $H^2(\mathbb{R}^N)$. We know that u is an H^2 solution (in the sense of Theorem 4.8.1) on some maximal interval $[0, T)$ with $T \le T_{\max}$, and we need to show that $T = T_{\max}$ (see the discussion at the beginning of this chapter). We argue by contradiction, and we suppose $T < T_{\max}$. In particular, $T < \infty$, so that

$$\|u(t)\|_{H^2} \underset{t\uparrow T}{\longrightarrow} \infty. \tag{5.3.2}$$

Moreover, since $T < T_{\max}$,

$$\sup_{0 \le t \le T} \|u(t)\|_{H^1} < \infty \tag{5.3.3}$$

and, since u is an H^2 solution on $[0, T)$,

$$\|u\|_{L^\infty((0,\tau),H^2)} + \|u_t\|_{L^a((0,\tau),L^b)} < \infty \tag{5.3.4}$$

for all admissible pairs (a, b) and all $0 < \tau < T$. We now reproduce some estimates from the proof of Theorem 4.8.1. For simplicity, we suppose $k = 1$, the case $k \ge 2$ being treated as in the proof of Theorem 5.2.1 above. We first observe that by (5.3.1), $g(\varphi) \in L^2(\mathbb{R}^N)$. Next, we recall that (by (3.3.7) or (4.4.21)) there exist $2 \le r, \rho < 2N/(N-2)$ ($2 \le r, \rho \le \infty$ if $N = 1$) such that

$$\|g(u) - g(v)\|_{L^{\rho'}} \le C(M)\|u - v\|_{L^r} \tag{5.3.5}$$

for all $u, v \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1}, \|v\|_{H^1} \le M$. We consider γ and q such that (γ, ρ) and (q, r) are admissible pairs. It follows from estimate (4.8.4) and from (5.3.3) that

$$\left\| \frac{d}{dt} g(u) \right\|_{L^{\gamma'}(J, L^{\rho'})} \le C \|u_t\|_{L^{\gamma'}(J, L^r)} \tag{5.3.6}$$

for every interval $I \subset [0, T)$. Applying now Lemma 4.8.2 and (4.8.7), we obtain that

$$\|u_t\|_{L^q(J, L^r)} \le C\|\varphi\|_{H^2} + C\|g(\varphi)\|_{L^2} + C\|u_t\|_{L^{\gamma'}(J, L^r)}$$

for every interval $0 \in I \subset [0, T)$. Using (5.3.4) and the fact that $\gamma' < q$, we deduce easily that (see the proofs of Theorems 5.1.1 or 5.2.1 above for the details) $u_t \in L^q((0, T), L^r(\mathbb{R}^N))$. Applying again (5.3.6) and Lemmas 4.8.2 and 4.8.5, it follows that

$$u_t \in L^\infty((0, T), L^2(\mathbb{R}^N)) . \tag{5.3.7}$$

Next, we easily deduce from (5.3.1), the interpolation inequality (4.8.20) and estimate (5.3.3) that there exists C such that

$$\|g(u(t))\|_{L^2} \le C + \frac{1}{2}\|u(t)\|_{H^2} \quad \text{for all } 0 \le t < T. \tag{5.3.8}$$

Finally, we use equation (4.1.1) and estimates (5.3.3), (5.3.7), and (5.3.8) to obtain

$$\|u(t)\|_{H^2} \le C + \frac{1}{2}\|u(t)\|_{H^2} \quad \text{for all } 0 \le t < T.$$

Thus $\|u(t)\|_{H^2} \le 2C$ for $0 \le t < T$, which yields a contradiction with (5.3.2). □

REMARK 5.3.2. We note that we did not use all the assumptions of Theorem 4.3.1 or Theorem 4.4.6. In fact, we need only assume that $g \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$ (so that the equation makes sense) and that each of the g_j's satisfies (5.3.1) and inequality (5.3.5) for some exponents $2 \le r_j, \rho_j < 2N/(N-2)$ ($2 \le r_j, \rho_j \le \infty$ if $N = 1$). Under these assumptions, it follows that if $\varphi \in H^2(\mathbb{R}^N)$ and if $u \in C(I, H^1(\mathbb{R}^N))$ is a solution of (4.1.1) on some interval $I \ni 0$, then $u \in C(I, H^2(\mathbb{R}^N))$. The argument is exactly the same. In practice, though, the existence of an H^1 solution is obtained by either Theorem 4.3.1 or Theorem 4.4.6.

REMARK 5.3.3. Here are two examples of applications of Theorem 5.3.1. Consider first

$$g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u ,$$

where V, f, and W are as follows. The function V is a real-valued potential, $V \in L^\delta(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\delta \ge 1$, $\delta > N/2$. W is an even, real-valued potential, $W \in L^\sigma(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\sigma \ge 1$, $\sigma > N/4$. The function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}$ and satisfies (3.2.7), (3.2.8), and (3.2.17). We extend f to $\mathbb{R}^N \times \mathbb{C}$ by (3.2.10). It follows that g satisfies the assumptions of Theorem 4.3.1 (see Example 3.2.11), and similar estimates show that (5.3.1) holds (for s sufficiently close to 2). In particular, we may let $f(x, u) = \lambda |u|^\alpha u$ with $\lambda \in \mathbb{R}$ and $0 \le \alpha < 4/(N-2)$ ($0 \le \alpha < \infty$ if $N = 1, 2$). Consider next

$$g(u) = Vu + f(u(\cdot)) + (W \star |u|^2)u ,$$

where $V, \nabla V \in L^{\delta}(\mathbb{R}^N) + L^{\infty}(\mathbb{R}^N)$ for some $\delta \geq 1$, $\delta > N/2$, $W \in L^{\sigma}(\mathbb{R}^N) + L^{\infty}(\mathbb{R}^N)$ for some $\sigma \geq 1$, $\sigma > N/4$, and f is as in Theorem 4.4.1 (for example, $f(z) = \lambda|z|^{\alpha}z$ with $\lambda \in \mathbb{C}$ and $(N-2)\alpha < 4$). It follows that g satisfies the assumptions of Theorem 4.4.6 (see Remark 4.4.8) and that (5.3.1) holds (for s sufficiently close to 2).

We now study the H^2 regularity of L^2 solutions.

THEOREM 5.3.4. *Let $g = g_1 + \cdots + g_k$ be as in Theorem 4.6.4; i.e., each of the g_j's satisfies (5.2.1)–(5.2.3) for some r_j, α_j, q_j. Set $r = \max\{r_1, \ldots, r_k\}$ and $q = \min\{q_1, \ldots, q_k\}$. Assume, in addition, that there exists $0 \leq s < 2$ such that*

$$\|g(u)\|_{L^2} \leq K(M)(1 + \|u\|_{H^s}) \quad \text{for all } u \in H^2(\mathbb{R}^N) \tag{5.3.9}$$

such that $\|u\|_{L^2} \leq M$. Let $\varphi \in L^2(\mathbb{R}^N)$ and let $u \in C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N)) \cap L^q_{\text{loc}}((-T_{\min}, T_{\max}), L^r(\mathbb{R}^N))$ be the maximal solution of the problem (4.1.1) given by Theorem 4.6.4. If $\varphi \in H^2(\mathbb{R}^N)$, then $u \in C((-T_{\min}, T_{\max}), H^2(\mathbb{R}^N))$.

PROOF. We first observe that by (5.2.2) and (5.3.9), there is local existence in $H^2(\mathbb{R}^N)$ by Theorem 4.8.1. The proof is then analogous to the proof of Theorem 5.3.1 (see also the proof of Theorem 5.2.2). □

REMARK 5.3.5. Theorem 5.3.4 applies to the model case

$$g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u$$

under the following assumptions. The function $V \in L^{\delta}(\mathbb{R}^N) + L^{\infty}(\mathbb{R}^N)$ for some $\delta \geq 1$, $\delta > N/2$, and $W \in L^{\sigma}(\mathbb{R}^N) + L^{\infty}(\mathbb{R}^N)$ for some $\sigma \geq 1$, $\sigma > N/2$. We have $f(x, 0) = 0$ for all $x \in \mathbb{R}^N$ and

$$|f(x, z_1) - f(x, z_2)| \leq C(1 + |z_1| + |z_2|)^{\beta}|z_1 - z_2| \quad \text{for some } 0 \leq \beta < \frac{4}{N}.$$

The fact that the assumptions are satisfied is easily verified, see Corollaries 4.6.5 and 4.8.6 for the details.

5.4. H^m Regularity, $m > N/2$

In this section we consider a nonlinearity g that satisfies the assumptions of Theorem 4.10.1 for some $m > N/2$ and we study the H^{m_1} regularity of the solutions for $m_1 > m$. This is a particularly simple case as the following result shows.

THEOREM 5.4.1. *Let $m > N/2$ be an integer and let $g \in C^m(\mathbb{C}, \mathbb{C})$ (in the real sense) with $g(0) = 0$. Let $\varphi \in H^m(\mathbb{R}^N)$ and let $u \in C((-T_{\min}, T_{\max}), H^m(\mathbb{R}^N))$ be the maximal solution of (4.1.1) given by Theorem 4.10.1. If $\varphi \in H^{m_1}(\mathbb{R}^N)$ for some $m_1 > m$ and if $g \in C^{m_1}(\mathbb{C}, \mathbb{C})$, then $u \in C^{m_1}((-T_{\min}, T_{\max}), H^{m_1}(\mathbb{R}^N))$.*

PROOF. We consider $t \geq 0$, the argument for $t \leq 0$ being the same. We know that u is an H^{m_1} solution on some maximal interval $[0, T)$ with $T \leq T_{\max}$, and we need to show that $T = T_{\max}$ (see the discussion at the beginning of this chapter). Consider $\tau < T_{\max}$. It follows that

$$\sup_{0 \leq t \leq \tau} \|u(t)\|_{H^{m_1}} < \infty$$

so that

$$\sup_{0\le t\le \tau} \|u(t)\|_{L^\infty} < \infty .$$

Applying property (i) of Theorem 4.10.1 (at the level m_1), we deduce that $T > \tau$. Thus $T = T_{\max}$. □

5.5. Arbitrary Regularity

So far, we have established regularity up to the level $H^2(\mathbb{R}^N)$ for H^s solutions, $0 \le s \le 1$, and regularity of arbitrary level for H^m solutions with $m > N/2$. In higher dimensions, there is of course a gap between H^2 and H^m with $m > N/2$. It seems that there is no general result concerning regularity at higher order. See Ginibre and Velo [135] and Kato [206, Corollary 4.3] for some partial results in that direction. Here is a result concerning a very particular nonlinearity.

Let $g(u) = \lambda|u|^\alpha u$ with $\lambda \in \mathbb{C}$ and α an even integer. In particular, $g \in C^\infty(\mathbb{C},\mathbb{C})$. By Theorems 4.6.1 (case $s = 0$), 4.9.1 (case $0 < s < \min\{1, N/2\}$), 4.4.1 (case $s = 1$) or 4.9.9 (case $1 < s < N/2$), there is local existence in $H^s(\mathbb{R}^N)$ for (4.1.1) when $0 \le s < N/2$ and $s > N/2 - 2/\alpha$. Moreover, for every admissible pair (q, r), $u \in L^q_{\text{loc}}((-T_{\min}, T_{\max}), H^{s,r}(\mathbb{R}^N))$ (see the above-mentioned theorems and Remark 4.4.3 for the case $s = 1$). We have the following regularity result.

THEOREM 5.5.1. *Let $g(u) = \lambda|u|^\alpha u$ with $\lambda \in \mathbb{C}$ and α an even integer. Let $0 \le s < N/2$ satisfy*

$$s > \frac{N}{2} - \frac{2}{\alpha}. \tag{5.5.1}$$

Finally, let $\varphi \in H^s(\mathbb{R}^N)$ and let u be the corresponding maximal H^s solution of (4.1.1), $u \in C((-T_{\min}, T_{\max}), H^s(\mathbb{R}^N)) \cap L^q_{\text{loc}}((-T_{\min}, T_{\max}), H^{s,r}(\mathbb{R}^N))$ for every admissible pair (q, r) (see above). If $\varphi \in H^m(\mathbb{R}^N)$ for some $m > N/2$, then $u \in C((-T_{\min}, T_{\max}), H^m(\mathbb{R}^N))$.

PROOF. Suppose first $s \le 1$. If $s < 1$ and $m = 1$, then regularity follows from Theorem 5.2.1 and if $m = 2$, regularity follows from Theorem 5.3.1. If $m \ge 3$, then in particular u is an H^2 solution by Theorem 5.3.1. If $N \le 2$, then regularity follows from Theorem 5.4.1, and if $N \ge 3$ we are reduced to the case $s > 1$ and $N \ge 3$, which we study below.

We define $r_0 > 2$ by

$$\frac{1}{r_0} = \frac{1}{2} - \frac{2}{N\alpha}. \tag{5.5.2}$$

We observe that $\alpha \ge 2$ so that

$$2 < r_0 \le \frac{2N}{N-2} \tag{5.5.3}$$

(with equality if $\alpha = 2$). Moreover, it follows from (5.5.1) and (5.5.2) that

$$r_0 s > N. \tag{5.5.4}$$

We deduce from (5.5.3)–(5.5.4) that there exists r such that

$$2 < r < \frac{2N}{N-2}, \quad rs > N.$$

In particular, there exists q such that (q, r) is an admissible pair. We also note that by (5.5.2),

$$\frac{1}{r} < \frac{1}{2} - \frac{2}{N\alpha}$$

so that $q > \alpha$. It follows that $u \in L^\alpha_{\mathrm{loc}}((-T_{\min}, T_{\max}), H^{s,r}(\mathbb{R}^N))$. Since $H^{s,r}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ because $rs > N$, we obtain

$$u \in L^\alpha_{\mathrm{loc}}\big((-T_{\min}, T_{\max}), L^\infty(\mathbb{R}^N)\big)\,. \tag{5.5.5}$$

We now observe that, due to the particular structure of $g(u)$, the proof of Lemma 4.10.2 yields the estimate

$$\|g(v)\|_{H^m} \le C\|v\|^\alpha_{L^\infty}\|v\|_{H^m} \quad \text{for all } v \in H^m(\mathbb{R}^N). \tag{5.5.6}$$

We consider $t \ge 0$, the argument for $t \le 0$ being the same. We know that u is an H^m solution on some maximal interval $[0, T)$ with $T \le T_{\max}$, and we need to show that $T = T_{\max}$ (see the discussion at the beginning of this chapter). We use equation (4.1.1), the property that $\mathcal{T}(t)$ is an isometry in $H^m(\mathbb{R}^N)$, and estimate (5.5.6) to obtain

$$\|u(t)\|_{H^m} \le \|\varphi\|_{H^m} + C\int_0^t \|u(s)\|^\alpha_{L^\infty}\|u(s)\|_{H^m}\,ds \quad \text{for all } 0 \le t < T.$$

Applying Gronwall's lemma, we deduce

$$\|u(t)\|_{H^m} \le \|\varphi\|_{H^m} \exp\left(C\int_0^t \|u(s)\|^\alpha_{L^\infty}\,ds\right) \quad \text{for all } 0 \le t < T.$$

If $T < T_{\max}$, then (5.5.5) yields $\limsup_{t\uparrow T}\|u(t)\|_{H^m} < \infty$, a contradiction with the blowup alternative in $H^m(\mathbb{R}^N)$. □

5.6. The C^∞ Smoothing Effect

In this section we present a result of Hayashi, Nakamitsu, and Tsutsumi [177, 178, 179] describing a C^∞ smoothing effect similar to the one observed for the linear equation (see Section 2.5). More precisely, under suitable assumptions on the nonlinearity, if the initial value φ decays fast enough as $|x| \to \infty$, then the corresponding solution of (4.1.1) is smooth in both t and x for $t \ne 0$, even if φ is not smooth. There are several results in that direction, depending on what are the assumptions on $g(u)$ and on the initial data. Some of these results, however, are fairly complicated technically. Therefore, and for the sake of simplicity, we only give a simple, typical result in order to illustrate the idea, and we refer to the papers of Hayashi, Nakamitsu, and Tsutsumi [177, 178, 179] for a more complete study.

THEOREM 5.6.1. *Assume that $N = 1$. Let $T > 0$, let $\varphi \in H^1(\mathbb{R})$, and let u be a strong H^1-solution of*

$$\begin{cases} iu_t + u_{xx} + |u|^2u = 0 \\ u(0) = \varphi \end{cases} \quad \text{on } [0, T]. \tag{5.6.1}$$

If φ has compact support, then $u \in C^\infty((0, T) \times \mathbb{R})$.

PROOF. Let us first do a formal calculation, in order to make clear the idea, which is quite simple. We use the operators P_α defined in Section 2.5. More precisely for any positive integer ℓ, let

$$u^\ell(t,x) = (x + 2it\partial_x)^\ell u(t,x) \,. \tag{5.6.2}$$

We deduce from formula (2.5.4) that

$$u^\ell(t,x) = (2it)^\ell e^{i\frac{x^2}{4t}} \partial_x^\ell \big(e^{-i\frac{x^2}{4t}} u(t,x)\big) \,. \tag{5.6.3}$$

It follows from (5.6.2), (2.5.5), (5.6.1), and (5.6.3) that

$$iu_t^\ell + u_{xx}^\ell + (2it)^\ell e^{i\frac{x^2}{4t}} \partial_x^\ell \big(|e^{-i\frac{x^2}{4t}} u|^2 e^{-i\frac{x^2}{4t}} u\big) = 0 \,. \tag{5.6.4}$$

Note that $|v|^2 v = v\overline{v}v$, and so

$$\partial_x^\ell |v|^2 v = \sum_{n+j+k=\ell} \partial_x^n v \partial_x^j \overline{v} \partial_x^k v \,.$$

Therefore, setting

$$v(t,x) = e^{-i\frac{x^2}{4t}} u(t,x) \,, \tag{5.6.5}$$

we deduce from (5.6.4) that

$$iu_t^\ell + u_{xx}^\ell + (2it)^\ell e^{i\frac{x^2}{4t}} \sum_{n+j+k=\ell} \partial_x^n v \partial_x^j \overline{v} \partial_x^k v = 0 \,.$$

Since $u^\ell(0) = x^\ell \varphi$, we see that

$$u^\ell(t) = \mathcal{T}(t)(x^\ell\varphi) + i \int_0^t \mathcal{T}(t-s)\Big((2is)^\ell e^{i\frac{x^2}{4s}} \sum_{n+j+k=\ell} \partial_x^n v(s) \partial_x^j \overline{v(s)} \partial_x^k v(s) \Big) ds \,,$$

and so

$$\|u^\ell(t)\|_{L^2} \le \|x^\ell\varphi\|_{L^2} + \int_0^t (2s)^\ell \| \sum_{n+j+k=\ell} \partial_x^n v(s) \partial_x^j \overline{v(s)} \partial_x^k v(s) \|_{L^2} \, ds \,. \tag{5.6.6}$$

Next, by Hölder's inequality,

$$\Big\| \sum_{n+j+k=\ell} \partial_x^n v(s) \partial_x^j \overline{v(s)} \partial_x^k v(s)) \Big\|_{L^2} \le \sum_{n+j+k=\ell} \|\partial_x^n v(s)\|_{L^{\frac{2\ell}{n}}} \|\partial_x^j v(s)\|_{L^{\frac{2\ell}{j}}} \|\partial_x^k v(s)\|_{L^{\frac{2\ell}{k}}} \,.$$

Furthermore, it follows from Gagliardo-Nirenberg's inequality that

$$\big\|\partial_x^j v(s)\big\|_{L^{\frac{2\ell}{j}}} \le C \big\|\partial_x^\ell v(s)\big\|_{L^2}^{\frac{j}{\ell}} \|v(s)\|_{L^\infty}^{\frac{\ell-j}{\ell}} \quad \text{for every } j \in \{0,\dots,\ell\} \,.$$

Therefore,

$$\sum_{n+j+k=\ell} \left\|\partial_x^n v(s)\right\|_{L^{\frac{2\ell}{n}}} \left\|\partial_x^j v(s)\right\|_{L^{\frac{2\ell}{j}}} \left\|\partial_x^k v(s)\right\|_{L^{\frac{2\ell}{k}}} \le C\left\|\partial_x^\ell v(s)\right\|_{L^2} \|v(s)\|_{L^\infty}^2$$
$$\le \frac{C}{s^\ell}\|u^\ell(s)\|_{L^2}\|u(s)\|_{L^\infty}^2$$
$$\le \frac{C}{s^\ell}\|u^\ell(s)\|_{L^2}\,.$$

(Note that u is bounded in $H^1(\mathbb{R})$, hence in $L^\infty(\mathbb{R})$.) Applying (5.6.6), we deduce from Gronwall's inequality that

$$\|u^\ell(t)\|_{L^2} \le C\|x^\ell\varphi\|_{L^2} \quad \text{for all } t\in[0,T)\,, \tag{5.6.7}$$

where C depends only on T, ℓ, and $\|u\|_{L^\infty((0,T),H^1)}$. In particular, given $0<\varepsilon<T$, $v\in L^\infty((\varepsilon,T),H^\ell(\mathbb{R}))$ for every positive integer ℓ. Since the mapping $v\mapsto |v|^2v$ is continuous $H^\ell\to H^\ell$ (see above), it follows from (5.6.4) that $u_t^\ell\in L^\infty((\varepsilon,T),L^2_{\text{loc}}(\mathbb{R}))$, and so $v_t\in L^\infty((\varepsilon,T),H^\ell_{\text{loc}}(\mathbb{R}))$ for every positive integer ℓ. In particular, $v\in C([\varepsilon,T),H^\ell_{\text{loc}}(\mathbb{R}))$, for every positive integer ℓ. Applying again (5.6.4), we deduce that $v\in C^1([\varepsilon,T),H^\ell_{\text{loc}}(\mathbb{R}))$ for every positive integer ℓ. Differentiating the equation k times with respect to t, we obtain eventually, with the same argument, that $v\in C^k([\varepsilon,T),H^\ell_{\text{loc}}(\mathbb{R}))$ for all positive integers ℓ and k. Therefore, $v\in C^\infty([\varepsilon,T]\times\mathbb{R})$, which means that $u\in C^\infty([\varepsilon,T]\times\mathbb{R})$. The result follows, since $\varepsilon>0$ is arbitrary.

Now, we want to make that argument rigorous. In order to do that, we need the following result.

LEMMA 5.6.2. *Suppose φ and u are as above. If, in addition, $\varphi\in\mathcal{S}(\mathbb{R})$, then $u\in C^\infty([0,T),\mathcal{S}(\mathbb{R}))$.*

PROOF. We proceed in three steps.

STEP 1. $u\in L^\infty((0,T),H^\ell(\mathbb{R}))$ for every positive integer ℓ. This follows from Theorem 5.5.1.

STEP 2. $x^pu\in L^\infty((0,T),H^\ell(\mathbb{R}))$ for all nonnegative integers p and ℓ. We argue by induction on p. We have already established the result for $p=0$ (Step 1). Assuming it is true up to some $p\ge 0$, let us show that it is true for $p+1$. Set $u^k(t)=\partial_x^k u(t)$. Given a positive integer ℓ,

$$iu_t^\ell + u_{xx}^\ell + \sum_{n+j+k=\ell} u^n\overline{u^j}u^k = 0. \tag{5.6.8}$$

Taking the L^2 scalar product of (5.6.8) with $ie^{-2\varepsilon x^2}x^{2p+2}u^\ell$, where $\varepsilon\in(0,1)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e^{-\varepsilon x^2}x^{p+1}u^\ell(t)\|_{L^2}^2 &= \operatorname{Im}\int u_{xx}^\ell e^{-2\varepsilon x^2}x^{2p+2}\overline{u^\ell}\\ &\quad + \operatorname{Im}\int\Big(e^{-2\varepsilon x^2}x^{2p+2}\overline{u^\ell}\sum_{n+j+k=\ell}u^n\overline{u^j}u^k\Big)\\ &= \alpha+\beta\,.\end{aligned} \tag{5.6.9}$$

We integrate the term α by parts, and we note that $\operatorname{Im} u_x^\ell \overline{u_x^\ell} = 0$. It follows that

$$(5.6.10)\qquad \begin{aligned}\alpha &= -\operatorname{Im} \int \left(((2p+2) - 4\varepsilon x^2) e^{-\varepsilon x^2} x^p u^{\ell+1} e^{-\varepsilon x^2} x^{p+1} \overline{u^\ell}\right) \\ &\le C(p) \|x^p u^{\ell+1}\|_{L^2} \, \|e^{-\varepsilon x^2} x^{p+1} u^\ell\|_{L^2} \le C(p,\ell) \|e^{-\varepsilon x^2} x^{p+1} u^\ell\|_{L^2}\end{aligned}$$

by the induction assumption. On the other hand,

$$(5.6.11)\qquad \begin{aligned}\beta &\le \|e^{-\varepsilon x^2} x^{p+1} u^\ell\|_{L^2} \sum_{n+j+k=\ell} \|e^{-\varepsilon x^2} x^{p+1} u^n \overline{u^j} u^k\|_{L^2} \\ &\le C(\ell) \|e^{-\varepsilon x^2} x^{p+1} u^\ell\|_{L^2} \sum_{k=0}^{\ell} \|e^{-\varepsilon x^2} x^{p+1} u^k\|_{L^2} ,\end{aligned}$$

since u^j is bounded in L^∞ for every j, by Step 1. It follows from (5.6.9), (5.6.10), and (5.6.11) that

$$\frac{1}{2}\frac{d}{dt} \|e^{-\varepsilon x^2} x^{p+1} u^\ell(t)\|_{L^2}^2 \le$$
$$C(p,\ell) \|e^{-\varepsilon x^2} x^{p+1} u^\ell\|_{L^2} + C(\ell) \|e^{-\varepsilon x^2} x^{p+1} u^\ell\|_{L^2} \sum_{k=0}^{\ell} \|e^{-\varepsilon x^2} x^{p+1} u^k\|_{L^2}$$

for every nonnegative integer ℓ. Therefore,

$$\frac{1}{2}\frac{d}{dt} \sum_{k=0}^{\ell} \|e^{-\varepsilon x^2} x^{p+1} u^k(t)\|_{L^2}^2 \le$$
$$C(p,\ell) \sum_{k=0}^{\ell} \|e^{-\varepsilon x^2} x^{p+1} u^k\|_{L^2} + C(\ell) \sum_{k=0}^{\ell} \|e^{-\varepsilon x^2} x^{p+1} u^k\|_{L^2}^2 .$$

Hence the result follows by integrating the above differential inequality and letting $\varepsilon \downarrow 0$.

STEP 3. Applying Step 2 and equation (5.6.8), we see that, for all nonnegative integers ℓ and p, $x^p u_t^\ell \in L^\infty((0,T), L^2)$; in particular, $x^p u^\ell \in C([0,T], L^2)$. Considering again (5.6.8), we deduce that $x^p u^\ell \in C^1([0,T], L^2)$ for all ℓ and p. Iterating that argument, we obtain that $x^p u^\ell \in C^\infty([0,T], L^2)$ for all nonnegative integers ℓ and p. This completes the proof. □

END OF THE PROOF OF THEOREM 5.6.1. Consider a sequence $\varphi_k \in \mathcal{S}(\mathbb{R})$ such that $\varphi_k \to \varphi$ in $H^1(\mathbb{R})$ as $k \to \infty$, and such that $\|x^p \varphi_k\|_{L^2} \le 2\|x^p \varphi\|_{L^2}$ for all positive integers k and p. Let u_k be the solution of (5.6.1) with initial value φ_k given by Theorem 3.5.1. It follows from Theorem 3.5.1 that $u_k \to u$ in $C([0,T], H^1(\mathbb{R}))$ as $k \to \infty$. On the other hand, by Lemma 5.6.2, the calculations of the formal argument above are rigorous for the solutions u_k. Therefore, estimate (5.6.7) holds for the solution u_k and is uniform in k. Thus $\|u^\ell(t)\|_{L^2} \le C(\ell)$ for all $t \in [0,T)$ and all $\ell \ge 1$. One concludes as in the formal argument that we described before. □

REMARK 5.6.3. Note that we have shown in fact that the function v defined by (5.6.5) belongs to $C^\infty((0,T), H^m(\mathbb{R}^N))$ for all $m \ge 0$, whenever $(1+x^2)^{\frac{m}{2}} \varphi \in L^2(\mathbb{R})$ for every positive integer m. Evidently, there are also partial results if we only assume that $(1+x^2)^{\frac{m_0}{2}} \varphi \in L^2(\mathbb{R})$ for some given positive integer m_0.

REMARK 5.6.4. Note that we did not really use that $u \in C([0,T), H^1(\mathbb{R}))$. What we used precisely is that $u \in L^2((0,T), L^\infty(\mathbb{R}))$ and that the solution depends continuously on φ in $L^2((0,T), L^\infty(\mathbb{R}))$. This may be used to show a C^∞ smoothing effect for initial data in $L^2(\mathbb{R})$ (see Corollary 5.7.5 below).

REMARK 5.6.5. In Theorem 5.6.1, we chose the nonlinearity $g(u) = |u|^2 u$ to simplify the calculation of $\partial_x^m g(u)$. With exactly the same method, one can establish the same result for $g(u) = \lambda |u|^{2k} u$, where k is a nonnegative integer and $\lambda \in \mathbb{R}$. More generally, the result holds when $g(u) = f(|u|^2)u$, where $f : [0,\infty) \to \mathbb{R}$ is in C^∞, but the calculations are technically a little bit more complicated.

5.7. Comments

Theorem 5.3.1 can be generalized in the framework of Theorem 4.12.1. More precisely, we have the following result.

THEOREM 5.7.1. *Let A and $g = g_1 + \dots + g_k$ be as in the statement of Theorem 4.12.1. Assume further that there exists a Banach space $D(A) \hookrightarrow Y \hookrightarrow X$ such that, for every $\varepsilon > 0$, there exists a constant C_ε for which $\|u\|_Y \le \varepsilon \|u\|_{D(A)} + \|u\|_X$ for all $u \in D(A)$. Let $\varphi \in X_A$, and consider the maximal solution $u \in C((-T_{\min}, T_{\max}), X_A)$ of the problem (3.7.7) given by Theorem 4.12.1. If $\varphi \in D(A)$, it follows that $u \in C((-T_{\min}, T_{\max}), D(A)) \cap C^1((-T_{\min}, T_{\max}), L^2(\Omega))$.*

PROOF. Local existence in $D(A)$ follows from Theorem 4.12.3. The proof of Theorem 5.3.1 is then easily adapted (with the estimates from the proof of Theorem 4.12.3). □

REMARK 5.7.2. Let Ω be a smooth, open subset of $\mathbb{R}^2$, and let g satisfy the assumptions of Theorem 3.6.1. Consider $\varphi \in H_0^1(\Omega)$, and let u be the maximal solution of (3.1.1) given by Theorem 3.6.1. If $\varphi \in H^2(\Omega)$, then (from Brezis and Gallouët [45]) $u \in C((-T_{\min}, T_{\max}), H^2(\Omega)) \cap C^1((-T_{\min}, T_{\max}), L^2(\Omega))$.

THEOREM 5.7.3. *Assume that $N = 2$ or $N = 3$, and let k be any positive integer if $N = 2$, and $k = 1$ if $N = 3$. Let $T > 0$, $\lambda \in \mathbb{R}$, let $\varphi \in H^1(\mathbb{R})$, and let $u \in C([0,T), H^1(\mathbb{R})) \cap C^1([0,T), H^{-1}(\mathbb{R}))$ satisfy the equation*

$$\begin{cases} iu_t + \Delta u + \lambda |u|^{2k} u = 0 \\ u(0) = \varphi. \end{cases}$$

If φ has compact support, then $u \in C^\infty((0,T) \times \mathbb{R})$.

PROOF. The proof is adapted from the proof of Theorem 5.6.1 (see also Remark 5.6.5). □

REMARK 5.7.4. The C^∞ smoothing effect of Theorem 5.6.1 holds as well (with an obvious adaptation of the proof) for the nonlinearity $g(u) = (W \star |u|^2)u$, where $W \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$. The proof again makes use of the property that the solution depends continuously on φ in $L^2((0,T), L^\infty(\mathbb{R}))$. (See Hayashi [162] and Hayashi and Ozawa [188]).

COROLLARY 5.7.5. *Assume $N = 1$. Let $\lambda \in \mathbb{R}$ and $W \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$. Consider $\varphi \in L^2(\mathbb{R})$ and let $u \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^4_{\text{loc}}(\mathbb{R}, L^\infty(\mathbb{R}))$ be the solution of the problem*

$$\begin{cases} iu_t + \Delta u + \lambda |u|^2 u + (W \star |u|^2)u = 0 \\ u(0) = \varphi \end{cases}$$

given by Corollary 4.6.5. *If φ has compact support, then $u \in C^\infty((\mathbb{R} \setminus \{0\}) \times \mathbb{R})$.*

PROOF. Note that $(4, \infty)$ is an admissible pair in dimension 1. Therefore, the result follows from Remarks 5.6.4 and 5.7.4. □

REMARK 5.7.6. A smoothing effect of analytic type was established for equations of the type $iu_t + \Delta u = F(u, \overline{u})$ in $\mathbb{R}^N$, where F is a polynomial in $(u, \overline{u})$ (for example, $F(u, \overline{u}) = |u|^{2m} u$, where m is a nonnegative integer). Under some decay and smoothness assumptions on the initial value $u(0)$ (that do not imply that $u(0)$ is analytic), it is shown that the corresponding solution is (real) analytic in space and/or in time (see A. de Bouard [97], Hayashi [161, 163], Hayashi and Kato [174, 175], and Hayashi and Saitoh [192, 193]).

CHAPTER 6

Global Existence and Finite-Time Blowup

Throughout this chapter we continue the study of equation (4.1.1) in the whole space $\mathbb{R}^N$. So far, we have studied the local properties of solutions of nonlinear Schrödinger equations: local existence, regularity, and the smoothing effect. In this chapter we begin the study of the global properties of the solutions. We establish criteria on the nonlinearity and/or the initial data to determine whether the solutions exist for all times, or blow up in finite time.

In Section 6.1 we apply the results of Section 3.4. These results are based on the conservation of charge and energy and yield global existence for all initial data or for small data only, depending on the nonlinearity.

In Sections 6.2, 6.3, and 6.4 we establish global existence under a certain assumption of smallness on the initial value, without assuming the Hamiltonian structure (i.e., without the conservation laws). The smallness condition can be just a quantity related to the H^1 norm of the initial value (Section 6.2), or depend on how the initial value "oscillates" as $|x| \to \infty$ (Section 6.3), or depend on how the initial value behaves like a homogeneous function as $|x| \to \infty$ (Section 6.4).

In Section 6.5 we obtain sufficient conditions on the nonlinearity and the initial value for finite-time blowup and we establish some lower estimates of the norms that blow up.

In Section 6.6 we consider the so-called "critical" or "pseudoconformal" case $g(u) = \lambda|u|^{\frac{4}{N}}u$. We first establish sharp existence results concerning the initial-value problem in $H^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$. Next, we describe some properties of the blowup solutions that are only known for the critical nonlinearity.

In Section 6.7 we still consider the so-called "critical" or "pseudoconformal" case $g(u) = \lambda|u|^{\frac{4}{N}}u$, and we apply the pseudoconformal transformation to derive some further information on the nature of the blowup.

6.1. Energy Estimates and Global Existence

In this section we give some sufficient conditions for global existence based on the conservation of charge and energy. For that reason, we consider solutions in the energy space $H^1(\mathbb{R}^N)$ and nonlinearities for which there is conservation of charge and energy.

THEOREM 6.1.1. *Let g be as in Theorem 4.3.1. Assume further that there exist $A > 0$, $C(A) > 0$, and $\varepsilon \in (0,1)$ such that*

$$G(u) \le \frac{1-\varepsilon}{2}\|u\|_{H^1}^2 + C(A) \quad \textit{for all } u \in H^1(\mathbb{R}^N) \tag{6.1.1}$$

such that $\|u\|_{L^2} \le A$. Consider $\varphi \in H_0^1(\Omega)$ and let $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N))$ be the corresponding maximal solution of (4.1.1) *given by Theorem* 4.3.1. *If $\|\varphi\|_{L^2} \le A$, then $T_{\min} = T_{\max} = \infty$. In addition $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^N))$.*

PROOF. This is an immediate consequence of Theorems 3.4.1 and 4.3.1. □

Below is an application of Theorem 6.1.1 to the model nonlinearity of Corollary 4.3.3.

COROLLARY 6.1.2. *Let $g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u$, where V, f, and W are as follows: V is a real-valued potential, $V \in L^\delta(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\delta \ge 1$, $\delta > N/2$; W is an even, real-valued potential, $W \in L^\sigma(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\sigma \ge 1$, $\sigma > N/4$, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}$ and satisfies* (3.2.7), (3.2.8), *and* (3.2.17). *The function f is extended to $\mathbb{R}^N \times \mathbb{C}$ by* (3.2.10). *Assume further that there exist $A \ge 0$ and $0 \le \nu < 4/N$ such that*

$$F(x,u) \le A|u|^2(1 + |u|^\nu)\,, \tag{6.1.2}$$

and that

$$W^+ \in L^\theta(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \tag{6.1.3}$$

for some $\theta \ge 1$, $\theta \ge N/2$ (and $\theta > 1$ if $N = 2$). It follows that for every $\varphi \in H^1(\mathbb{R}^N)$, the maximal strong H^1-solution u of (4.1.1) *given by Corollary* 4.3.3 *is global and* $\sup\{\|u(t)\|_{H^1} : t \in \mathbb{R}\} < \infty$.

PROOF. We claim that, with the notation of Corollary 4.3.3,

$$G(u) \le \frac{1}{4}\|u\|_{H^1}^2 + C(\|u\|_{L^2}) \quad \text{for all } u \in H^1(\mathbb{R}^N). \tag{6.1.4}$$

The result then follows from Theorem 6.1.1. To prove the claim, let $V = V_1 + V_2$, where $V_1 \in L^\infty$ and $V_2 \in L^\delta$, and let $W^+ = W_1 + W_2$, where $W_1 \in L^\infty$ and $W_2 \in L^\theta$. We have

$$\begin{aligned}4G(u) &\le 2\|V_1\|_{L^\infty}\|u\|_{L^2}^2 + 2\|V_2\|_{L^\delta}\|u\|^2_{L^{\frac{2\delta}{\delta-1}}} + 4A\|u\|_{L^2}^2 + 4A\|u\|_{L^{\nu+2}}^{\nu+2}\\ &\quad + \|W_1\|_{L^\infty}\|u\|_{L^2}^4 + \|W_2\|_{L^\theta}\|u\|^4_{L^{\frac{4\theta}{\theta-1}}}\\ &\le C(1 + \|u\|_{L^2}^4) + C\|u\|^2_{L^{\frac{2\delta}{\delta-1}}} + C\|u\|_{L^{\nu+2}}^{\nu+2} + C\|u\|^4_{L^{\frac{4\theta}{\theta-1}}}\,.\end{aligned}$$

On the other hand, it follows from Gagliardo-Nirenberg's inequality that

$$\begin{aligned}\|u\|^2_{L^{\frac{2\delta}{\delta-1}}} &\le C\|u\|_{H^1}^{\frac{N}{\delta}}\|u\|_{L^2}^{\frac{2\delta-N}{\delta}}\,,\\ \|u\|_{L^{\nu+2}}^{\nu+2} &\le C\|u\|_{H^1}^{\frac{N\nu}{2}}\|u\|_{L^2}^{\nu+2-\frac{N\nu}{2}}\,,\\ \|u\|^4_{L^{\frac{4\theta}{\theta-1}}} &\le C\|u\|_{H^1}^{\frac{N}{\theta}}\|u\|_{L^2}^{\frac{4\theta-N}{\theta}}\,.\end{aligned}$$

Since $\frac{N}{\delta}, \frac{n\nu}{2}, \frac{N}{\theta} < 2$, (6.1.4) follows from the inequality $ab \le \varepsilon a^r + C(\varepsilon)b^{r'}$. □

REMARK 6.1.3. In the case where F satisfies (6.1.2) with $\nu = 4/N$, then instead of (6.1.4), we obtain the following inequality:

$$G(u) \le \left(\frac{1}{2} + C\|u\|_{L^2}^{\frac{4}{N}}\right)\|u\|_{H^1}^2 + C(\|u\|_{L^2}) \quad \text{for all } u \in H^1(\mathbb{R}^N). \tag{6.1.5}$$

In this case, all solutions of (4.1.1) are global and uniformly bounded in H^1, provided $\|\varphi\|_{L^2}$ is small enough. This follows from (6.1.5) and Theorem 6.1.1.

We now give an example of application of Theorem 3.4.3, which shows global existence under a smallness assumption on the initial value.

THEOREM 6.1.4. *Let g be as in Theorem 4.3.1. Assume further that $G(0) = 0$ and that there exist $\varepsilon > 0$ and a nonnegative function $\eta \in C([0,\varepsilon), \mathbb{R}^+)$, with $\eta(0) = 0$, such that*

$$G(u) \le \frac{1-\varepsilon}{2}\|u\|_{H^1}^2 + \eta(\|u\|_{L^2}) \quad \textit{for all } u \in H^1(\mathbb{R}^N) \tag{6.1.6}$$

such that $\|u\|_{H^1} \le \varepsilon$. It follows that there exists $a > 0$ such that, for every $\varphi \in H^1(\mathbb{R}^N)$ with $\|\varphi\|_{H^1} \le a$, the maximal strong H^1-solution u of (4.1.1) given by Theorem 4.3.1 is global and $\sup\{\|u(t)\|_{H^1} : t \in \mathbb{R}\} \le \varepsilon$.

PROOF. The result follows from Theorem 3.4.3. □

COROLLARY 6.1.5. *Let $g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u$, where V, f, and W are as follows: V is a real-valued potential, $V \in L^\delta(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\delta \ge 1$, $\delta > N/2$; W is an even, real-valued potential, $W \in L^\sigma(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $\sigma \ge 1$, $\sigma > N/4$; and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}$ and satisfies (3.2.7), (3.2.8), and (3.2.17). The function f is extended to $\mathbb{R}^N \times \mathbb{C}$ by (3.2.10). It follows that there exists $a > 0$ such that, for every $\varphi \in H^1(\mathbb{R}^N)$ with $\|\varphi\|_{H^1} \le a$, the maximal strong H^1-solution u of (4.1.1) given by Corollary 4.3.3 is global and* $\sup\{\|u(t)\|_{H^1} : t \in \mathbb{R}\} < \infty$.

PROOF. With the notation of Corollary 4.3.3, we have

$$G(u) \le C\|u\|_{L^2}^2 + C\|u\|_{H^1}^{\alpha+2} + C\|u\|_{H^1}^4 + \|V_2\|_{L^\delta}\|u\|_{H^1}^2 .$$

Since in the splitting $V = V_1 + V_2$, $\|V_2\|_{L^\delta}$ can be made arbitrarily small, the result follows from Theorem 6.1.4. □

6.2. Global Existence for Small Data

In this section we do not assume conservation of charge or energy. We establish global existence for small initial values in $H^1(\mathbb{R}^N)$ for nonlinearities that vanish at a sufficient order at the origin. We rely on the method of Kato [206]. We essentially reproduce the estimates of the fixed-point arguments of Chapter 4, but we eliminate the dependence on t. Note that with this technique, we obtain not only the global existence but also a certain decay of the solution. For the sake of simplicity, we consider local nonlinearities only.

THEOREM 6.2.1. *Let $g \in C(\mathbb{C}, \mathbb{C})$ satisfy $g(0) = 0$. Assume there exist*

$$\frac{4}{N} \le \alpha_1 \le \alpha_2 < \frac{4}{N-2} \quad \left(\frac{4}{N} \le \alpha_1 \le \alpha_2 < \infty \textit{ if } N = 1\right) \tag{6.2.1}$$

such that

(6.2.2) $$|g(u)-g(v)| \le C(|u|^{\alpha_1}+|v|^{\alpha_1}+|u|^{\alpha_2}+|v|^{\alpha_2})|u-v| \quad \text{for all } u,v\in\mathbb{C}.$$

There exists $\varepsilon_0>0$ such that if $\varphi\in H^1(\mathbb{R}^N)$ satisfies $\|\varphi\|_{H^1}\le\varepsilon_0$, then the corresponding maximal H^1 solution u of (4.1.1) *given by Theorem* 4.4.6 *is global, i.e., $T_{\min}=T_{\max}=\infty$. Moreover, $u\in L^q(\mathbb{R},W^{1,r}(\mathbb{R}^N))$ for every admissible pair (q,r).*

PROOF. We may assume without loss of generality that $\alpha_1=\frac{4}{N}$. Moreover, arguing as in the proof of Theorem 4.4.1, we can write $g=g_1+g_2$ where $g_1(0)=g_2(0)=0$ and for $j=1,2$,

(6.2.3) $$|g_j(u)-g_j(v)|\le C(|u|^{\alpha_j}+|v|^{\alpha_j})|u-v| \quad \text{for all } u,v\in\mathbb{C}.$$

We consider the admissible pairs (γ_j,ρ_j), $j=1,2$, such that $\rho_j=\alpha_j+2$; in particular, $\gamma_1=\rho_1=2+4/N$. Given $0<t<T_{\max}$, we set

$$f(t)=\|u\|_{L^\infty((0,t),H^1)}+\|u\|_{L^{\gamma_1}((0,t),W^{1,\rho_1})}+\|u\|_{L^{\gamma_2}((0,t),W^{1,\rho_2})}\,.$$

Since $\gamma_j<\infty$ (and u is continuous with values in $H^1(\mathbb{R}^N)$), we see that

(6.2.4) $$f(t)\downarrow\|\varphi\|_{H^1} \quad \text{as } t\downarrow 0.$$

On the other hand, it follows from Strichartz's estimates that there exists C independent of t such that

(6.2.5) $$f(t)\le C\|\varphi\|_{H^1}+C\|g_1(u)\|_{L^{\gamma_1'}((0,t),W^{1,\rho_1'})}+C\|g_2(u)\|_{L^{\gamma_2'}((0,t),W^{1,\rho_2'})}\,.$$

Since $|g_j(u)|+|\nabla g_j(u)|\le C|u|^{\alpha_j}(|u|+|\nabla u|)$ by (6.2.3) and Remark 1.3.1(vii), it follows from Hölder's inequality in space and time that there exists C independent of T such that

(6.2.6) $$\begin{aligned}\|g_1(u)\|_{L^{\gamma_1'}((0,t),W^{1,\rho_1'})}&\le C\|u\|^{\alpha_1}_{L^{\gamma_1}((0,t),L^{\rho_1})}\|u\|_{L^{\gamma_1}((0,t),W^{1,\rho_1})}\\&\le Cf(t)^{\alpha_1+1}\,.\end{aligned}$$

Similarly, there exists C independent of T such that

$$\|g_2(u)\|_{L^{\gamma_2'}((0,t),W^{1,\rho_2'})}\le C\|u\|^{\alpha_2}_{L^{\mu}((0,t),L^{\rho_2})}\|u\|_{L^{\gamma_2}((0,t),W^{1,\rho_2})}\,,$$

where μ is given by

$$\frac{1}{\mu}=\frac{4-(N-2)\alpha_2}{2\alpha_2(\alpha_2+2)}\,.$$

We note that due to the assumption (6.2.1), $\gamma_2\le\mu<\infty$. Therefore, since $H^1\hookrightarrow L^{\rho_2}$ and $W^{1,\rho_2}\hookrightarrow L^{\rho_2}$, we see that $\|u\|_{L^\mu((0,t),L^{\rho_2})}\le f(t)$, and so

(6.2.7) $$\|g_2(u)\|_{L^{\gamma_2'}((0,t),W^{1,\rho_2'})}\le Cf(t)^{\alpha_2+1}\,.$$

It now follows from (6.2.5)–(6.2.7) that

$$f(t)\le C\|\varphi\|_{H^1}+Cf(t)^{\alpha_1+1}+Cf(t)^{\alpha_2+1} \quad \text{for all } 0<t<T_{\max}.$$

Applying (6.2.4), we deduce easily that if $\|\varphi\|_{H^1}\le\varepsilon_0$ where $\varepsilon_0>0$ is sufficiently small so that

$$(2C\varepsilon_0)^{\alpha_1+1}+(2C\varepsilon_0)^{\alpha_2+1}<1\,,$$

then $f(t) \le 2C\|\varphi\|_{H^1}$ for all $0 < t < T_{\max}$. Letting $t \uparrow T_{\max}$, we deduce in particular that $\|u\|_{L^\infty((0,T_{\max}),H^1)} < \infty$, so that $T_{\max} = \infty$ by the blowup alternative. Thus $f(t)$ is bounded as $t \to \infty$, so that

$$\|u\|_{L^\infty((0,\infty),H^1)} + \|u\|_{L^{\gamma_1}((0,\infty),W^{1,\rho_1})} + \|u\|_{L^{\gamma_2}((0,\infty),W^{1,\rho_2})} < \infty .$$

By Strichartz's estimates and the previous estimates of $g_j(u)$, this implies that $u \in L^q((0,\infty), W^{1,r}(\mathbb{R}^N))$ for every admissible pair (q,r). The estimate for $t < 0$ is obtained by the same argument. □

REMARK 6.2.2. Theorem 6.2.2 says that if $\|\varphi\|_{H^1}$ is small, then the corresponding H^1 solution u is global and decays as $t \to \infty$ in the sense that $u \in L^q(\mathbb{R}, W^{1,r}(\mathbb{R}^N))$ for all admissible pairs (q,r). Note that we already mentioned results of the above type. See in particular Remarks 4.5.4, 4.7.5, and 4.9.8 where it is assumed that $\|\nabla\varphi\|_{L^2}$, $\|\varphi\|_{L^2}$, and $\|\varphi\|_{\dot{H}^s}$, respectively, are small. These results, however, apply to (essentially) homogeneous nonlinearities. This is a major difference with Theorem 6.2.2 which applies, for example, to the case $g(u) = a|u|^{\alpha_1}u + b|u|^{\alpha_2}u$.

REMARK 6.2.3. The smallness condition on $\|\varphi\|_{H^1}$ can be improved, depending on the assumptions on g. Also, instead of considering H^1 solutions, one can consider more generally H^s solutions. See Section 5 of Kato [206] and Pecher [295]. Note that global existence results for small data hold under various assumptions on the nonlinearity and for smallness of the initial data in various spaces. See, for example, Hayashi and Naumkin [183], and Nakamura and Ozawa [257].

6.3. Global Existence for Oscillating Data

In this section we consider the model nonlinearity

$$g(u) = \lambda|u|^\alpha u, \tag{6.3.1}$$

where

$$\lambda \in \mathbb{C}, \quad 0 \le \alpha < \frac{4}{N-2} \quad (0 \le \alpha < \infty \text{ if } N = 1). \tag{6.3.2}$$

We show that the solutions of (4.1.1) are "positively global"; i.e., $T_{\max} = \infty$ if the initial value φ is sufficiently "oscillating" in a sense to be made precise below (see Theorem 6.3.4 and Remark 6.3.5). This result is based on a global existence result for small data whose proof makes use of a Strichartz inequality for nonadmissible pairs. We first introduce the number α_0, which we will also use in Chapter 7. We let

$$\alpha_0 = \alpha_0(N) = \frac{2 - N + \sqrt{N^2 + 12N + 4}}{2N}; \tag{6.3.3}$$

i.e., α_0 is the positive root of the polynomial $Nx^2 + (N-2)x - 4$.

REMARK 6.3.1. We note that $\alpha \ge 0$ satisfies $N\alpha^2 + (N-2)\alpha - 4 > 0$ if and only if $\alpha > \alpha_0$. We also note that

$$\frac{2}{N} \le \frac{4}{N+2} < \alpha_0 < \frac{4}{N} < \frac{4}{N-2} \quad \text{if } N \ge 2,$$
$$\frac{2}{N} < \alpha_0 < \frac{4}{N} \quad \text{if } N = 1.$$

We have the following global existence result.

THEOREM 6.3.2. *Let g satisfy (6.3.1)–(6.3.2). Suppose further that $\alpha > \alpha_0$, where α_0 is defined by (6.3.3), and let*

$$a = \frac{2\alpha(\alpha+2)}{4-\alpha(N-2)}. \tag{6.3.4}$$

There exists $\varepsilon > 0$ such that if $\varphi \in H^1(\mathbb{R}^N)$ and $\|\mathcal{T}(\cdot)\varphi\|_{L^a((0,\infty),L^{\alpha+2})} \le \varepsilon$, then the maximal H^1 solution u of (4.1.1) given by Theorem 4.4.1 is positively global, i.e., $T_{\max} = \infty$. Moreover, $u \in L^a((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$, and $u \in L^\gamma((0,\infty), W^{1,\rho}(\mathbb{R}^N))$ for every admissible pair (γ,ρ).

For the proof of Theorem 6.3.2, we will use the following lemma.

LEMMA 6.3.3. *Let $r = \alpha + 2$, let (q,r) be the corresponding admissible pair, and let a be given by (6.3.4). It follows that $a > q/2$ if and only if $\alpha > \alpha_0$, with α_0 defined by (6.3.3). For such values of α and a, and for $0 < T \le \infty$, we have the following estimates for $\mathcal{A}$ defined by*

$$\mathcal{A}(f)(t) = \int_0^t \mathcal{T}(t-s)f(s)ds \quad \text{for } 0 \le t < T.$$

(i) *If $u \in L^a((0,T), L^r(\mathbb{R}^N))$, then $\mathcal{A}(|u|^\alpha u) \in L^a((0,T), L^r(\mathbb{R}^N))$. Furthermore, there exists C independent of T such that*

$$\|\mathcal{A}(|u|^\alpha u)\|_{L^a((0,T),L^r)} \le C\|u\|^{\alpha+1}_{L^a((0,T),L^r)} \tag{6.3.5}$$

for every $u \in L^a((0,T), L^r(\mathbb{R}^N))$.

(ii) *If $u \in L^a((0,T), L^r(\mathbb{R}^N)) \cap L^q((0,T), W^{1,r}(\mathbb{R}^N))$ and if (γ,ρ) is any admissible pair, then $\mathcal{A}(|u|^\alpha u) \in L^\gamma((0,T), W^{1,\rho}(\mathbb{R}^N))$. Furthermore, there exists C independent of T such that*

$$\|\mathcal{A}(|u|^\alpha u)\|_{L^\gamma((0,T),W^{1,\rho})} \le C\|u\|^\alpha_{L^a((0,T),L^r)}\|u\|_{L^q((0,T),W^{1,r})} \tag{6.3.6}$$

for every $u \in L^a((0,T), L^r(\mathbb{R}^N)) \cap L^q((0,T), W^{1,r}(\mathbb{R}^N))$.

PROOF. The first part of the lemma is a simple calculation, which we omit. For assertions (i) and (ii) consider $\widetilde{a}$ defined by (2.4.2). Since $(\alpha+1)r' = r$, $(\alpha+1)\widetilde{a}' = a$, and

$$\frac{1}{q'} = \frac{1}{q} + \frac{\alpha}{a},$$

we see that

$$\||u|^\alpha u\|_{L^{\widetilde{a}'}((0,T),L^{r'})} = \|u\|^{\alpha+1}_{L^a((0,T),L^r)}$$

and (applying Hölder's inequality twice) that

$$\||u|^\alpha u\|_{L^{q'}((0,T),W^{1,r'})} \le C\|u\|^\alpha_{L^a((0,T),L^r)}\|u\|_{L^q((0,T),W^{1,r})}.$$

The results now follow from (2.4.3) and Strichartz's estimates, respectively. □

PROOF OF THEOREM 6.3.2. We use the notation of Lemma 6.3.3. Let $\varepsilon > 0$, let $\varphi \in H^1(\mathbb{R}^N)$ be such that

$$\|\mathcal{T}(\cdot)\varphi\|_{L^a((0,\infty),L^r)} \le \varepsilon,$$

and let u be the maximal solution of (4.1.1) defined on $[0, T_{\max})$ with $0 < T_{\max} \leq \infty$. It follows from equation (4.1.2) and from (6.3.5)–(6.3.6) that there exists K independent of T and φ such that

$$\|u\|_{L^a((0,T),L^r)} \leq \varepsilon + K\|u\|^{\alpha+1}_{L^a((0,T),L^r)}, \tag{6.3.7}$$

and

$$\|u\|_{L^q((0,T),W^{1,r})} \leq K\|\varphi\|_{H^1} + K\|u\|^{\alpha}_{L^a((0,T),L^r)}\|u\|_{L^q((0,T),W^{1,r})} \tag{6.3.8}$$

for every $T < T_{\max}$. (The term $K\|\varphi\|_{H^1}$ in (6.3.8) comes from Strichartz's estimates.) Assume that ε satisfies

$$2^{\alpha+1}K\varepsilon^{\alpha} < 1. \tag{6.3.9}$$

Let $f(t) = \|u\|_{L^a((0,t),L^r)}$. It follows that $f \in C([0, T_{\max}))$ and that $f(0) = 0$. Furthermore, it follows from (6.3.7) that $f(t) \leq \varepsilon + Kf(t)^{\alpha+1}$ for all $0 \leq t < T_{\max}$. Using (6.3.9), we deduce by a simple continuity argument that $f(t) \leq 2\varepsilon$ for all $0 \leq t < T_{\max}$, so that

$$\|u\|_{L^a((0,T_{\max}),L^r)} \leq 2\varepsilon. \tag{6.3.10}$$

Applying (6.3.8) and (6.3.10), we obtain

$$\|u\|_{L^q((0,T_{\max}),W^{1,r})} \leq 2K\|\varphi\|_{H^1}. \tag{6.3.11}$$

We now deduce from equation (4.1.2) and from Strichartz's estimates (for the linear term) and (6.3.6) that $u \in L^{\gamma}((0, T_{\max})W^{1,\rho}(\mathbb{R}^N))$ for every admissible pair (γ, ρ). In particular, it follows from the blowup alternative that $T_{\max} = \infty$. This completes the proof. □

The main result of this section is now the following (see [72]).

THEOREM 6.3.4. *Let g satisfy (6.3.1)–(6.3.2). Suppose further that $\alpha > \alpha_0$, where α_0 is defined by (6.3.3), and let a be defined by (6.3.4). Let $\varphi \in H^1(\mathbb{R}^N)$ satisfy $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$. Given $b \in \mathbb{R}$, set*

$$\varphi_b(x) = e^{i\frac{b|x|^2}{4}}\varphi(x), \tag{6.3.12}$$

and let $\widetilde{u}_b$ be the maximal H^1 solution of (4.1.1) with the initial value $\varphi_b \in H^1(\mathbb{R}^N)$. There exists $b_0 < \infty$ such that if $b \geq b_0$, then $T_{\max}(\varphi_b) = \infty$. Moreover, $\widetilde{u}_b \in L^a((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$, and $\widetilde{u}_b \in L^{\gamma}((0,\infty), W^{1,\rho}(\mathbb{R}^N))$ for every admissible pair (γ, ρ).

PROOF. Let $r = \alpha+2$ and let (q, r) be the corresponding admissible pair. A direct calculation, based on the explicit kernel of the Schrödinger group (see Lemma 2.2.4), shows that

$$[\mathcal{T}(t)\varphi_b](x) = e^{i\frac{b|x|^2}{4(1+bt)}}\Big[D_{\frac{1}{1+bt}}\mathcal{T}\Big(\frac{t}{1+bt}\Big)\varphi\Big](x),$$

where the dilation operator D_β, $\beta > 0$, is defined by $D_\beta w(x) = \beta^{\frac{N}{2}}w(\beta x)$. It easily follows that

$$\|\mathcal{T}(\cdot)\varphi_b\|^a_{L^a((0,\infty),L^r)} = \int_0^{1/b}(1 - b\tau)^{\frac{2(a-q)}{q}}\|\mathcal{T}(\tau)\varphi\|^a_{L^r}\,d\tau.$$

Since $\|\mathfrak{T}(\tau)\varphi\|_{L^r} \leq C\|\mathfrak{T}(\tau)\varphi\|_{H^1} \leq C\|\varphi\|_{H^1}$ and since

$$\frac{2(a-q)}{q} > -1,$$

by Lemma 6.3.3, we see that

$$\lim_{b\uparrow\infty} \|\mathfrak{T}(\cdot)\varphi_b\|_{L^a((0,\infty),L^r)} = 0.$$

The result now follows from Theorem 6.3.2. □

REMARK 6.3.5. We note that $|\varphi_b(x)| \equiv |\varphi(x)|$. In particular, Theorem 6.3.4 implies that there is (positively) global existence for initial values of arbitrarily large amplitude. The condition $b \geq b_0$ means that φ_b is sufficiently "oscillating" as $|x| \to \infty$.

REMARK 6.3.6. It is the condition $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$ that ensures $\varphi_b \in H^1(\mathbb{R}^N)$. For a general $\varphi \in H^1(\mathbb{R}^N)$, $\varphi_b \in L^2(\mathbb{R}^N) \cap H^1_{\text{loc}}(\mathbb{R}^N)$, but $\varphi_b \notin H^1(\mathbb{R}^N)$.

REMARK 6.3.7. Here are some comments on Theorem 6.3.4.

(i) Let $q = 4(\alpha+2)/N\alpha$, so that $(q, \alpha+2)$ is an admissible pair. One easily verifies that if $\alpha > 4/N$, then $q < a$, where a is given by (6.3.4), and that if $\alpha < 4/N$, then $q > a$. Next, if $u \in L^\gamma((0,\infty), W^{1,\rho}(\mathbb{R}^N))$ for every admissible pair (γ, ρ), we have in particular $u \in L^q((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$, and also $u \in L^\infty((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$ by Sobolev's embedding theorem. Therefore, $u \in L^a((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$ if $\alpha \geq 4/N$. On the other hand, if $\alpha < 4/N$, then the property $u \in L^a((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$ expresses a better decay at infinity.

(ii) If $\lambda < 0$, then all H^1 solutions of (4.1.1) are global (see Section 6.1). Therefore, Theorem 6.3.4 means that all the solutions $\widetilde{u}_b$ have a certain decay as $t \to \infty$ for b large enough.

(iii) If $\lambda > 0$ and $\alpha < 4/N$, then all H^1 solutions of (4.1.1) are global (see Section 6.1). Therefore, Theorem 6.3.4 means that $\widetilde{u}_b$ has a certain decay as $t \to \infty$ if b is large enough. Note that certain solutions do not decay, in particular the standing waves, i.e., solutions of the form $e^{i\omega t}\varphi(x)$ (see Chapter 8).

(iv) If $\lambda > 0$ and $\alpha \geq 4/N$, then (4.1.1) posesses solutions that blow up in finite time (see Section 6.5 below). Theorem 6.3.4 means that for any $\varphi \in H^1(\mathbb{R}^N)$ with $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$, the initial value φ_b gives rise to a solution which is (positively) global and which decays as $t \to \infty$ provided b is large enough.

REMARK 6.3.8. Assume g satisfies (6.3.1)–(6.3.2). Suppose further that $\alpha > \alpha_0$, where α_0 is defined by (6.3.3), and let a be defined by (6.3.4). Let $\varphi \in H^1(\mathbb{R}^N)$ satisfy $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$. Given $s \in \mathbb{R}$, let u^s be the maximal H^1 solution of (4.1.1) with the initial value $\psi_s = \mathfrak{T}(s)\varphi$. It follows that there exists $s_0 < \infty$ such that for every $s \geq s_0$, $T_{\max}(\psi_s) = \infty$. Moreover, $u^s \in L^a((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$, and $u^s \in L^\gamma((0,\infty), W^{1,\rho}(\mathbb{R}^N))$ for every admissible pair (γ, ρ). Indeed, since $\|\mathfrak{T}(\cdot)\varphi\|_{L^a((0,\infty),L^r)}$ is finite (See Corollary 2.5.4), we see that

$$\|\mathfrak{T}(\cdot)\psi_s\|_{L^a((0,\infty),L^r)} = \|\mathfrak{T}(\cdot)\varphi\|_{L^a((s,\infty),L^r)} \underset{s\to\infty}{\longrightarrow} 0,$$

and the result follows from Theorem 6.3.2.

6.4. Global Existence for Asymptotically Homogeneous Initial Data

In this section we consider the model nonlinearity

$$g(u) = \lambda |u|^\alpha u, \tag{6.4.1}$$

where

$$\lambda \in \mathbb{C}, \quad \alpha_0 < \alpha < \frac{4}{N-2} \quad (\alpha_0 < \alpha < \infty \text{ if } N = 1), \tag{6.4.2}$$

where α_0 is defined by (6.3.3). We first establish global existence of solutions for initial values that are sufficiently small in a certain sense. We then apply this result to initial values that are asymptotically homogeneous using the estimates of Section 2.6. The results of this section are based on Cazenave and Weissler [73, 75].

We first introduce some notation. Let

$$\beta = \frac{4-(N-2)\alpha}{2\alpha(\alpha+2)} \tag{6.4.3}$$

so that

$$\frac{N\alpha}{2(\alpha+2)} + \beta\alpha = 1. \tag{6.4.4}$$

Note that by (6.4.2),

$$0 < \beta < \frac{N\alpha}{2(\alpha+2)} < 1 \tag{6.4.5}$$

and

$$\beta(\alpha+1) < 1. \tag{6.4.6}$$

Next, given $0 < T \le \infty$, we define the spaces X_T and W_T by

$$X_T = \left\{ u \in L^\infty_{\mathrm{loc}}((0,T), L^{\alpha+2}(\mathbb{R}^N)) : \operatorname*{ess\,sup}_{0<t<T} t^\beta \|u(t)\|_{L^{\alpha+2}} < \infty \right\}, \tag{6.4.7}$$

$$W_T = \left\{ \varphi \in \mathcal{S}'(\mathbb{R}^N) : \sup_{0<t<T} t^\beta \|\mathcal{T}(t)\varphi\|_{L^{\alpha+2}} < \infty \right\}, \tag{6.4.8}$$

where we use the convention that if $\psi \in \mathcal{S}'(\mathbb{R}^N)$, then $\|\psi\|_{L^{\alpha+2}} = \infty$ if $\psi \notin L^{\alpha+2}(\mathbb{R}^N)$. We note that, given $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, the mapping $t \mapsto \mathcal{T}(t)\varphi$ is continuous $\mathbb{R} \to \mathcal{S}'(\mathbb{R}^N)$, so that the definition (6.4.8) makes sense. We also observe that X_T and W_T are Banach spaces when equipped with the norms

$$\|u\|_{X_T} = \operatorname*{ess\,sup}_{0<t<T} t^\beta \|u(t)\|_{L^{\alpha+2}}, \tag{6.4.9}$$

$$\|\varphi\|_{W_T} = \sup_{0<t<T} t^\beta \|\mathcal{T}(t)\varphi\|_{L^{\alpha+2}}. \tag{6.4.10}$$

This is quite clear for X_T by applying Theorem 1.2.5. For W_T, we argue as follows. Suppose $(\varphi_n)_{n\ge 0}$ is a Cauchy sequence. It follows that $(u_n)_{n\ge0}$ defined by $u_n(t) = \mathcal{T}(t)\varphi_n$ is a Cauchy sequence in X_T and thus has a limit $u \in X_T$. It now suffices to show that $u(t) = \mathcal{T}(t)\varphi$ for some $\varphi \in \mathcal{S}'(\mathbb{R}^N)$. Since $u_n \to u$ in X_T, there exists $t_0 \in (0,T)$ and a subsequence $(n_k)_{k\ge0}$ such that $u_{n_k}(t_0) \to u(t_0)$ in $L^{\alpha+2}(\mathbb{R}^N)$, hence in $\mathcal{S}'(\mathbb{R}^N)$. Therefore, $\varphi_{n_k} = \mathcal{T}(-t_0)u_{n_k}(t_0) \to \mathcal{T}(-t_0)u(t_0)$ in $\mathcal{S}'(\mathbb{R}^N)$. We let $\varphi = \mathcal{T}(-t_0)u(t_0)$. Since $\varphi_{n_k} \to \varphi$ in $\mathcal{S}'(\mathbb{R}^N)$ and $u_{n_k}(t) = \mathcal{T}(t)\varphi_{n_k}$, we see that $u(t) = \mathcal{T}(t)\varphi$.

We begin with the following existence result.

THEOREM 6.4.1. *Assume* (6.4.1)–(6.4.2) *and consider the spaces* X_T *and* W_T *defined by* (6.4.3)–(6.4.10). *There exists* $\varepsilon_0 > 0$ *such that the following properties hold:*

(i) *Let* $0 < T \le \infty$ *and* $\varphi \in W_T$. *If* $\|\varphi\|_{W_T} \le \varepsilon \le \varepsilon_0$, *then there exists a unique solution* $u \in X_T$ *of equation* (4.1.2) *such that* $\|u\|_{X_T} \le 2\varepsilon$.

(ii) *Let* $\varphi, \psi \in W_\infty$ *satisfy* $\|\varphi\|_{W_\infty}, \|\psi\|_{W_\infty} \le \varepsilon \le \varepsilon_0$ *and let* u, v *be the corresponding solutions of* (4.1.2) *in* X_∞ *such that* $\|u\|_{X_\infty}, \|v\|_{X_\infty} \le 2\varepsilon$. *If*

$$\sup_{t>0} t^\mu \|\mathcal{T}(t)(\varphi - \psi)\|_{L^{\alpha+2}} = A < \infty \quad \text{for some } \beta < \mu < \frac{N\alpha}{2(\alpha+2)}, \tag{6.4.11}$$

and if ε *is sufficiently small (depending on* μ*), then*

$$\sup_{t>0} t^\mu \|u(t) - v(t)\|_{L^{\alpha+2}} \le 2A. \tag{6.4.12}$$

In particular, $t^\beta \|u(t) - v(t)\|_{L^{\alpha+2}} \to 0$ *as* $t \to \infty$.

(iii) *Let* $\varphi \in W_\infty$ *satisfy* $\|\varphi\|_{W_\infty} \le \varepsilon \le \varepsilon_0$ *and let* u *be the corresponding solution of* (4.1.2) *in* X_∞ *such that* $\|u\|_{X_\infty} \le 2\varepsilon$. *If*

$$\sup_{t>0} t^\mu \|\mathcal{T}(t)\varphi\|_{L^{\alpha+2}} = A < \infty \quad \text{for some } \mu \text{ satisfying (6.4.11)}$$

and if ε *is sufficiently small (depending on* μ*), then*

$$\sup_{t>0} t^\mu \|u(t)\|_{L^{\alpha+2}} \le 2A \quad \text{and} \quad t^\mu \|u(t) - \mathcal{T}(t)\varphi\|_{L^{\alpha+2}} \to 0 \text{ as } t \to \infty.$$

REMARK 6.4.2. We note that the equation (4.1.2) makes sense for $\varphi \in W_T$ and $u \in X_T$. Indeed, $\mathcal{T}(t)\varphi$ is well defined and the integral

$$\mathcal{G}(u)(t) = i \int_0^t \mathcal{T}(t-s) g(u(s))\, ds \tag{6.4.13}$$

is too. To see this, we observe that

$$\|g(u(t))\|_{L^{\frac{\alpha+2}{\alpha+1}}} \le |\lambda| \|u(t)\|_{L^{\alpha+2}}^{\alpha+1} \le |\lambda| t^{-\beta(\alpha+1)} \|u\|_{X_T}^{\alpha+1}. \tag{6.4.14}$$

Thus the mapping $s \mapsto \mathcal{T}(t-s)g(u(s))$ belongs to $L^\infty_{\mathrm{loc}}((0,t), L^{\alpha+2}(\mathbb{R}^N))$ and

$$\|\mathcal{T}(t-s)g(u(s))\|_{L^{\alpha+2}} \le |\lambda| (t-s)^{-\frac{N\alpha}{2(\alpha+2)}} s^{-\beta(\alpha+1)} \|u\|_{X_T}^{\alpha+1}. \tag{6.4.15}$$

It follows from (6.4.5)–(6.4.6) that $\mathcal{G}(u)$ is well defined, and in fact $t \mapsto \mathcal{G}(u)(t)$ is continuous $(0,T) \to L^{\alpha+2}(\mathbb{R}^N)$. Note also that $t \mapsto \mathcal{T}(t)\varphi$ is continuous $[0,\infty) \to \mathcal{S}'(\mathbb{R}^N)$ and bounded $[\delta, \infty) \to L^{\alpha+2}(\mathbb{R}^N)$ for every $\delta > 0$. It follows that $\mathcal{T}(t)\varphi$ is weakly continuous $(0,\infty) \to L^{\alpha+2}(\mathbb{R}^N)$. Since $\mathcal{G}(u)$ is strongly continuous, we see that if $\varphi \in W_T$ and if $u \in X_T$ satisfies (4.1.2), then u is weakly continuous $(0,T) \to L^{\alpha+2}(\mathbb{R}^N)$.

PROOF OF THEOREM 6.4.1. We proceed in three steps.

STEP 1. Proof of (i). This follows from a quite simple fixed-point argument. Given $\varphi \in W_T$, set

$$\mathcal{H}(u)(t) = \mathcal{T}(t)\varphi + \mathcal{G}(u)(t) \quad \text{for all } u \in X_T \text{ and } t \in (0,T),$$

where $\mathcal{G}(u)$ is defined by (6.4.13). It follows from (6.4.15) that

$$\|\mathcal{H}(u)(t)\|_{L^{\alpha+2}} \le t^{-\beta}\|\varphi\|_{W_T} + |\lambda|\|u\|_{X_T}^{\alpha+1} \int_0^t (t-s)^{-\frac{N\alpha}{2(\alpha+2)}} s^{-\beta(\alpha+1)}\, ds\,.$$

Using (6.4.4), (6.4.5), and (6.4.6), we see that

$$\begin{aligned}\int_0^t (t-s)^{-\frac{N\alpha}{2(\alpha+2)}} s^{-\beta(\alpha+1)}\, ds &= t^{-\beta} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2(\alpha+2)}} \sigma^{-\beta(\alpha+1)}\, d\sigma \\ &= Ct^{-\beta}\,.\end{aligned}$$

It follows that $\mathcal{H}(u) \in X_T$ and that there exists K independent of T such that

$$\|\mathcal{H}(u)\|_{X_T} \le \|\varphi\|_{W_T} + K\|u\|_{X_T}^{\alpha+1} \quad \text{for all } u \in X_T.$$

Similarly, one shows that, by possibly choosing C larger,

$$\|\mathcal{H}(u) - \mathcal{H}(v)\|_{X_T} \le K\big(\|u\|_{X_T}^{\alpha} + \|v\|_{X_T}^{\alpha}\big)\|u-v\|_{X_T} \quad \text{for all } u,v \in X_T.$$

We deduce that if $\varepsilon_0 > 0$ satisfies $2K\varepsilon_0^\alpha < 1$, then for any $\varphi \in W_T$ with $\|\varphi\|_{W_T} \le \varepsilon \le \varepsilon_0$ and any $0 < T \le \infty$, $\mathcal{H}$ is a strict contraction on the ball of radius 2ε of X_T. Thus $\mathcal{H}$ has a unique fixed point, which solves (4.1.2).

STEP 2. Proof of (ii). Set

$$a(t) = \sup_{0<s<t} t^{\mu}\|u(s) - v(s)\|_{L^{\alpha+2}} \quad \text{for } t > 0.$$

We note that $\mu > \beta$ so that $a(t)$ is well defined. We deduce from equation (4.1.2) that

$$\begin{aligned} &t^{\mu}\|u(t) - v(t)\|_{L^{\alpha+2}} \\ &\le A + Ct^{\mu}\big(\|u\|_{X_T}^{\alpha} + \|v\|_{X_T}^{\alpha}\big)a(t) \int_0^t (t-s)^{-\frac{N\alpha}{2(\alpha+2)}} s^{-\alpha\beta} s^{-\mu}\, ds \\ &= A + C\big(\|u\|_{X_T}^{\alpha} + \|v\|_{X_T}^{\alpha}\big)a(t) \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2(\alpha+2)}} \sigma^{-\alpha\beta} \sigma^{-\mu}\, d\sigma \end{aligned}$$

for all $t > 0$, where the last identity follows from (6.4.4). Since

$$\alpha\beta + \mu < 1 \tag{6.4.16}$$

by (6.4.4) and (6.4.11), it follows that there exists C (depending on μ) such that

$$a(t) \le A + C\big(\|u\|_{X_T}^{\alpha} + \|v\|_{X_T}^{\alpha}\big)a(t) \le A + C(2\varepsilon)^{\alpha}a(t) \quad \text{for all } t > 0.$$

If $\varepsilon > 0$ is sufficiently small so that $C(2\varepsilon)^\alpha \le 1/2$, we deduce that $a(t) \le 2A$ for all $t > 0$ and (6.4.12) follows by letting $t \uparrow \infty$.

STEP 3. Proof of (iii). The first part of the statement follows from (ii) applied with $\psi = 0$. It remains to show that $t^{\mu}\|u(t) - \mathcal{T}(t)\varphi\|_{L^{\alpha+2}} \to 0$ as $t \to \infty$. We observe that

$$\|u(t) - \mathcal{T}(t)\varphi\|_{L^{\alpha+2}} \le |\lambda| \int_0^t (t-s)^{-\frac{N\alpha}{2(\alpha+2)}} \|u(s)\|_{L^{\alpha+2}}^{\alpha+1}\, ds$$

and that $\|u(t)\|_{L^{\alpha+2}} \le Ct^{-\nu}$ for every $\beta \le \nu \le \mu$. Assuming

$$\alpha\beta + \mu < (\alpha+1)\nu < 1, \tag{6.4.17}$$

which is possible by (6.4.6) and (6.4.16), we obtain

$$\begin{aligned}\|u(t) - \mathcal{T}(t)\varphi\|_{L^{\alpha+2}} &\le C\int_0^t (t-s)^{-\frac{N\alpha}{2(\alpha+2)}} s^{-(\alpha+1)\nu}\, ds \\ &= Ct^{1-\frac{N\alpha}{2(\alpha+2)}-(\alpha+1)\nu}.\end{aligned}$$

Applying now (6.4.4), we deduce that

$$t^{\mu}\|u(t) - \mathcal{T}(t)\varphi\|_{L^{\alpha+2}} \le Ct^{\mu+\alpha\beta-(\alpha+1)\nu} \underset{t\to\infty}{\longrightarrow} 0,$$

where the last property follows from (6.4.17). □

The relationship between the solutions in X_T constructed in Theorem 6.4.1 and finite energy solutions is given by the following lemma.

LEMMA 6.4.3. *Assume* (6.4.1)–(6.4.2) *and consider the spaces* X_T *and* W_T *defined by* (6.4.3)–(6.4.10). *Let* $\varepsilon_0 > 0$ *be given by Theorem* 6.4.1. *Let* $0 < T \le \infty$ *and* $\varphi \in W_T$ *satisfy* $\|\varphi\|_{W_T} \le \varepsilon_0$, *and let* $u \in X_T$ *be the unique solution of equation* (4.1.2) *such that* $\|u\|_{X_T} \le 2\varepsilon_0$, *given by Theorem* 6.4.1. *If* $\varphi \in H^1(\mathbb{R}^N)$, *then* $u \in C([0,T], H^1(\mathbb{R}^N))$.

PROOF. Let $u^1 \in C([0,T_{\max}), H^1(\mathbb{R}^N))$ be the maximal strong H^1 solution of (4.1.1) given by Theorem 4.4.1. We first observe that, since $H^1(\mathbb{R}^N) \hookrightarrow L^{\alpha+2}(\mathbb{R}^N)$,

$$\|u^1\|_{X_\tau} \le C\tau^{\beta}\|u\|_{L^\infty((0,\tau),H^1)} \underset{t\downarrow 0}{\longrightarrow} 0.$$

Thus there exists $0 < \tau < \min\{T, T_{\max}\}$ such that $\|u^1\|_{X_\tau} \le 2\varepsilon_0$. Also, $\|u\|_{X_\tau} \le \|u\|_{X_T} \le 2\varepsilon_0$. Using the uniqueness property in Theorem 6.4.1, we conclude that $u^1 = u$ on $(0,\tau)$. We now observe that for $0 < t < \min\{T, T_{\max}\}$,

$$u(t) - u^1(t) = i\int_0^t \mathcal{T}(t-s)[g(u(s)) - g(u^1(s))]ds,$$

so that (see above)

$$\begin{aligned}&\|u(t) - u^1(t)\|_{L^{\alpha+2}} \le \\ &\quad C\int_0^t (t-s)^{-\frac{N\alpha}{2(\alpha+2)}} \big(\|u(s)\|_{L^{\alpha+2}}^{\alpha} + \|u^1(s)\|_{L^{\alpha+2}}^{\alpha}\big)\|u(s) - u^1(s)\|_{L^{\alpha+2}}\, ds.\end{aligned}$$

We fix $0 < T' < \min\{T, T_{\max}\}$. On $(0,T')$, $\|u^1(s)\|_{L^{\alpha+2}}$ is bounded. Furthermore, $\|u(s)\|_{L^{\alpha+2}}^{\alpha} \le Cs^{-\beta\alpha}$. Thus, since $\|u(s) - u^1(s)\|_{L^{\alpha+2}} = 0$ for $s \le \tau$, we deduce from the above inequality that there exists C, depending on T', such that

$$\|u(t) - u^1(t)\|_{L^{\alpha+2}} \le C\int_0^t (t-s)^{-\frac{N\alpha}{2(\alpha+2)}} \|u(s) - u^1(s)\|_{L^{\alpha+2}}\, ds \quad \text{for all } 0 \le t \le T'.$$

It then follows from (a generalized form of) Gronwall's lemma that $u = u^1$ on $(0,T')$. Since $0 < T' < \min\{T, T_{\max}\}$ is arbitrary, we conclude that $u = u^1$ on $(0, \min\{T, T_{\max}\})$, and it remains to show that $T \le T_{\max}$. Assume by contradiction that $T_{\max} < T$. Since $u \in C([0, T_{\max}), H^1(\mathbb{R}^N))$, $\|u(t)\|_{L^{\alpha+2}}$ is bounded near 0; and since $u \in X_T$, $\|u(t)\|_{L^{\alpha+2}}$ is bounded away from 0. Thus

$$\sup_{0<t<T_{\max}} \|u(t)\|_{L^{\alpha+2}} < \infty .$$

It easily follows, by applying Remark 1.3.1 (vii) and Hölder's inequality, that there exists C such that

$$\|g(u(t))\|_{W^{1,\frac{\alpha+2}{\alpha+1}}} \le C\|u(t)\|_{W^{1,\alpha+2}} \quad \text{for all } 0 < t < T_{\max}. \tag{6.4.18}$$

We now let $0 \le \tau < T_{\max}$ and we observe that

$$u(t+\tau) = \mathcal{T}(t)u(\tau) + i\int_0^t \mathcal{T}(t-s)g(u(s+\tau))ds \quad \text{for } 0 < t < T_{\max} - \tau.$$

We now let $r = \alpha + 2$ and we let q be such that (q, r) is an admissible pair. Applying Strichartz's estimates and (6.4.18), we deduce that (see, e.g., the proof of Theorem 4.4.1)

$$\|u\|_{L^\infty((\tau,\theta),H^1)} + \|u\|_{L^q((\tau,\theta),W^{1,r})} \le C\|u(\tau)\|_{H^1} + C\|u\|_{L^{q'}((\tau,\theta),W^{1,r})} \tag{6.4.19}$$

with C independent of $\tau < \theta < T_{\max}$. Since

$$\|u\|_{L^{q'}((\tau,\theta),W^{1,r})} \le (\theta-\tau)^{\frac{q-2}{q}} \|u\|_{L^q((\tau,\theta),W^{1,r})} \le (T_{\max}-\tau)^{\frac{q-2}{q}} \|u\|_{L^q((\tau,\theta),W^{1,r})} ,$$

we see that if we fix τ sufficiently close to $T_{\max}$,

$$C\|u\|_{L^{q'}((\tau,\theta),W^{1,r})} \le \frac{1}{2}\|u\|_{L^q((\tau,\theta),W^{1,r})} ,$$

and it follows from (6.4.19) that

$$\|u\|_{L^\infty((\tau,\theta),H^1)} + \|u\|_{L^q((\tau,\theta),W^{1,r})} \le 2C\|u(\tau)\|_{H^1} .$$

Letting $\theta \uparrow T_{\max}$, we obtain $u \in L^\infty((\tau, T_{\max}), H^1(\mathbb{R}^N))$, which contradicts the blowup alternative of Theorem 4.4.1. □

REMARK 6.4.4. Lemma 6.4.3 is a regularity result. Under the same assumptions, one can show that if $\varphi \in H^s(\mathbb{R}^N)$ for some

$$\frac{N\alpha}{2(\alpha+2)} \le s < \min\left\{1, \frac{N}{2}\right\},$$

then $u \in C([0,T], H^s(\mathbb{R}^N))$ and u coincides with the H^s solution given by Theorem 4.9.1. The proof is similar (see the proof of Theorem 4.9.1). The assumption $s \ge N\alpha/2(\alpha+2)$ implies that $H^s(\mathbb{R}^N) \hookrightarrow L^{\alpha+2}(\mathbb{R}^N)$. Thus $\|u\|_{X_T} \to 0$ as $T \downarrow 0$ whenever u is an H^s solution. This is an essential step in the proof of Lemma 6.4.3. Note that the assumption $s \ge N\alpha/2(\alpha+2)$ also implies the condition $\alpha < 4/(N-2s)$ of Theorem 4.9.1.

COROLLARY 6.4.5. *Assume* (6.4.1)–(6.4.2) *and consider the spaces* X_T *and* W_T *defined by* (6.4.3)–(6.4.10). *Then there exists* $\varepsilon_0 > 0$ *with the following property. Let* $\varphi \in H^1(\mathbb{R}^N)$ *and let* $u \in C([0, T_{\max}), H^1(\mathbb{R}^N))$ *be the corresponding strong* H^1 *solution of* (4.1.1) *given by Theorem 4.4.1. If* $\|\varphi\|_{W_\infty} \le \varepsilon \le \varepsilon_0$, *then* $T_{\max} = \infty$, $u \in X_\infty$, *and* $\|u\|_{X_\infty} \le 2\varepsilon$.

PROOF. The result follows from Theorem 6.4.1 and Lemma 6.4.3. □

We now comment on the above results.

REMARK 6.4.6. The essential condition for global existence in Theorem 6.4.1 is $\|\varphi\|_{W_\infty} \le \varepsilon_0$. The main difficulty in exploiting this assumption is that the structure of W_∞ is not known. We give below some sufficient conditions for φ to belong to W_∞.

(i) $H^1(\mathbb{R}^N) \cap L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N) \hookrightarrow W_\infty$. Indeed,

$$\|\mathcal{T}(t)\varphi\|_{L^{\alpha+2}} \le C\|\mathcal{T}(t)\varphi\|_{H^1} \le C\|\varphi\|_{H^1}$$

and

$$\|\mathcal{T}(t)\varphi\|_{L^{\alpha+2}} \le |t|^{-\frac{N\alpha}{2(\alpha+2)}}\|\varphi\|_{L^{\frac{\alpha+2}{\alpha+1}}}$$

so that

$$(1+|t|)^{\frac{N\alpha}{2(\alpha+2)}}\|\mathcal{T}(t)\varphi\|_{L^{\alpha+2}} \le C\|\varphi\|_{H^1\cap L^{\frac{\alpha+2}{\alpha+1}}}.$$

We see in particular that if $\varphi \in H^1(\mathbb{R}^N) \cap L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$, then $\|\varphi\|_{W_T} \to 0$ as $T \downarrow 0$ and $\sup_{t>0} t^\mu \|\mathcal{T}(t)\varphi\|_{L^{\alpha+2}} < \infty$ for all $\mu \le N\alpha/2(\alpha+2)$.

(ii) Let $p \in \mathbb{C}$ with $\operatorname{Re} p = 2/\alpha$. It follows from Theorem 2.6.1 that $\psi(x) = |x|^{-p}$ satisfies $t^\beta\|\mathcal{T}(t)\psi\|_{L^{\alpha+2}} = c > 0$. In particular, $\psi \in W_\infty$. Note that the assumption (6.4.2) is essential.

(iii) Assume $\alpha < 4/N$ (in addition to (6.4.2)). Let $p \in \mathbb{C}$ with $\operatorname{Re} p = 2/\alpha$ and set $\psi(x) = |x|^{-p}$. We see that $\psi|_{\{|x|>1\}} \in H^1(\{|x|>1\})$ and that $\psi|_{\{|x|<1\}} \in L^{\frac{\alpha+2}{\alpha+1}}(\{|x|<1\})$. In particular, there exists $\varphi \in H^1(\mathbb{R}^N)$ such that $\varphi - \psi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$. For example, $\varphi = \eta\psi$ with $\eta \in C^\infty(\mathbb{R}^N)$ such that $\eta(x) = 0$ in a neighborhood of 0 and $\eta(x) = 1$ for $|x|$ large. Moreover, given any $\varphi \in H^1(\mathbb{R}^N)$ such that $\varphi - \psi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$, it follows from Corollary 2.6.7 that $\varphi \in W_\infty$ and that $\|\mathcal{T}(t)(\varphi-\psi)\|_{L^{\alpha+2}} \le Ct^{-\frac{N\alpha}{2(\alpha+2)}}$.

(iv) Let

$$\min\left\{\beta, \frac{N\alpha}{4(\alpha+2)}\right\} < \mu < \frac{N\alpha}{2(\alpha+2)}, \tag{6.4.20}$$

let $p \in \mathbb{C}$ satisfy

$$\operatorname{Re} p = 2\mu + \frac{N}{\alpha+2}, \tag{6.4.21}$$

and set $\psi(x) = |x|^{-p}$. It follows from Theorem 2.6.1 that $t^\mu\|\mathcal{T}(t)\psi\|_{L^{\alpha+2}} = c > 0$. Moreover, it follows from (6.4.20) that $\psi|_{\{|x|>1\}} \in H^1(\{|x|>1\})$ and that $\psi|_{\{|x|<1\}} \in L^{\frac{\alpha+2}{\alpha+1}}(\{|x|<1\})$. In particular, there exists $\varphi \in H^1(\mathbb{R}^N)$ such that $\varphi - \psi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$ (see (iii) above). In addition, given any $\varphi \in H^1(\mathbb{R}^N)$ such that $\varphi - \psi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$, it follows from Corollary 2.6.7 that $\varphi \in W_\infty$ and that $\|\mathcal{T}(t)\varphi\|_{L^{\alpha+2}} \le C(t^{-\frac{N\alpha}{2(\alpha+2)}} + t^{-\mu})$. Moreover, $t^\mu\|\mathcal{T}(t)\varphi\|_{L^{\alpha+2}} \to c$ as $t \to \infty$.

REMARK 6.4.7. Let $\psi(x) = \delta|x|^{-p}$ with $p \in \mathbb{C}$ such that $\operatorname{Re} p = 2/\alpha$ and $\delta \in \mathbb{C}$. It follows from Remark 6.4.6(ii) above that there exists a constant K such that $\|\psi\|_{W_\infty} \le K|\delta|$. In particular, if $|\delta| \le \varepsilon_0/K$, it follows from Theorem 6.4.1 that there exists a unique solution $v \in X_\infty$ of (4.1.2) (with the initial data ψ instead of φ) such that $\|v\|_{X_\infty} \le 2\varepsilon_0$. Such a solution is of particular interest since it is *self-similar.* We recall that if u is any solution of (4.1.1) (or (4.1.2)) on $(0,\infty) \times \mathbb{R}^N$ and if $\gamma > 0$, then u_γ defined by $u_\gamma(t,x) = \gamma^p u(\gamma^2 t, \gamma x)$ is also a solution. (The assumption $\operatorname{Re} p = 2/\alpha$ is essential.) A solution u is called *self-similar* if it is invariant under the transformation $u \mapsto u_\gamma$, i.e., if $u = u_\gamma$ for all $\gamma > 0$. We claim that v is self-similar. Indeed, it is easily verified that v_γ satisfies (4.1.2) with the initial value $\psi_\gamma(x) = \gamma^p \psi(\gamma x)$. Evidently, since ψ is homogeneous, $\psi_\gamma = \psi$. Moreover, it follows from a direct calculation that $\|v_\gamma\|_{X_\infty} = \|v\|_{X_\infty}$. Therefore, by the uniqueness property of Theorem 6.4.1, $v_\gamma = v$ for all $\gamma > 0$. We observe that v is weakly continuous $(0,\infty) \to L^{\alpha+2}(\mathbb{R}^N)$ by Remark 6.4.2, so that $f = u(1) \in L^{\alpha+2}(\mathbb{R}^N)$ is well defined. Applying the identity $v(t,x) = \gamma^p v(\gamma^2 t, \gamma x)$ with $\gamma = t^{-\frac{1}{2}}$, we see that

$$v(t,x) = t^{-\frac{p}{2}} f\Big(\frac{x}{\sqrt{t}}\Big);$$

i.e., the self-similar solution v is expressed in terms of its *profile* f. Note that self-similar solutions are not H^1 solutions in general; see [73, 75]. For a more detailed study of self-similar solutions, see Cazenave and Weissler [73, 74, 75], Furioli [119], Kavian and Weissler [209], Planchon [298], Ribaud and Youssfi [302], and Weissler [363].

REMARK 6.4.8. Here are some more applications of Theorem 6.4.1 and Corollary 6.4.5.

(i) Assume $\alpha < 4/N$ and fix μ satisfying (6.4.11). Let $p \in \mathbb{C}$ with $\operatorname{Re} p = 2/\alpha$ and let $\delta \in \mathbb{C}$. Set $\psi(x) = \delta|x|^{-p}$ and let $\varphi \in H^1(\mathbb{R}^N)$ be such that $\psi - \varphi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$. If $|\delta|$ and $\|\varphi - \psi\|_{L^{\frac{\alpha+2}{\alpha+1}}}$ are sufficiently small, then $\|\varphi\|_{W_\infty}, \|\psi\|_{W_\infty} \le \varepsilon$, where $\varepsilon > 0$ is as in part (ii) of Theorem 6.4.1 (see Remark 6.4.6(ii) and (iii)). If we denote by u and v the corresponding solutions of (4.1.2), then u is an H^1 solution by Lemma 6.4.3 and v is self-similar by Remark 6.4.7. Moreover, it follows from Theorem 6.4.1(ii) and Remark 6.4.6(iii) that $t^\mu \|u(t) - v(t)\|_{L^{\alpha+2}}$ is bounded uniformly in $t > 0$. Since $t^\beta \|v(t)\|_{L^{\alpha+2}} = c > 0$, we deduce that $t^\beta \|u(t)\|_{L^{\alpha+2}} \to c$ as $t \to \infty$. In particular, we know the exact rate of decay of $\|u(t)\|_{L^{\alpha+2}}$ as $t \to \infty$. Note also that $u(t)$ is asymptotic to the self-similar solution v as $t \to \infty$ (in the sense that $t^\beta \|u(t) - v(t)\|_{L^{\alpha+2}} \to 0$ as $t \to \infty$).

(ii) Let μ satisfy (6.4.20) and let $p \in \mathbb{C}$ satisfy (6.4.21). Set $\psi(x) = \delta|x|^{-p}$ and let $\varphi \in H^1(\mathbb{R}^N)$ be such that $\psi - \varphi \in L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N)$. If $|\delta|$ and $\|\varphi - \psi\|_{L^{\frac{\alpha+2}{\alpha+1}}}$ are sufficiently small, then $\|\varphi\|_{W_\infty} \le \varepsilon$, where $\varepsilon > 0$ is as in part (iii) of Theorem 6.4.1 (see Remark 6.4.6(iv)). If we denote by u the corresponding solution of (4.1.2), then u is an H^1 solution by Lemma 6.4.3. Moreover, it follows from Theorem 6.4.1(iii) and Remark 6.4.6(iv) that $t^\mu \|u(t)\|_{L^{\alpha+2}} \to c > 0$ as $t \to \infty$. In particular, we know the exact rate of decay of $\|u(t)\|_{L^{\alpha+2}}$ as $t \to \infty$. Note also that by Theorem 6.4.1(iii) and Remark 6.4.6(iv), $t^\mu \|u(t) - \mathcal{T}(t)\psi\|_{L^{\alpha+2}} \to 0$ as $t \to \infty$. This means that $u(t)$ is asymptotic to $\mathcal{T}(t)\psi$ as $t \to \infty$. Note that $\mathcal{T}(t)\psi$ is a self-similar solution of the linear Schrödinger equation; see Remark 2.6.6(iii).

REMARK 6.4.9. Here are some comments on the decay rates of $\|u(t)\|_{L^{\alpha+2}}$ that are achieved by H^1 solutions of (4.1.1). (See also Remark 7.3 in [75].)

(i) Suppose $\alpha \le 4/N$. It follows from Remark 6.4.8(ii) above that if $\beta < \mu < N\alpha/2(\alpha+2)$, then there exist H^1 solutions of (4.1.1) for which $\|u(t)\|_{L^{\alpha+2}} \approx t^{-\mu}$ as $t \to \infty$ (in the sense that $t^\mu \|u(t)\|_{L^{\alpha+2}} \to c > 0$). By Remark 6.4.8(i), $\mu = \beta$ is also achieved if $\alpha < 4/N$, and it follows from the results of Chapter 7 below that $\mu = N\alpha/2(\alpha+2)$ can also be achieved. Moreover, $\mu = N\alpha/2(\alpha+2)$ is the fastest possible decay in general (see Bégout [20] and Hayashi and Ozawa [187]). On the other hand, it is not known whether some solutions can have a slower decay than $t^{-\beta}$.

(ii) Suppose $\alpha > 4/N$. It follows from Remark 6.4.8(ii) above that if $N\alpha/4(\alpha+2) < \mu < N\alpha/2(\alpha+2)$, there there exist H^1 solutions of (4.1.1) for which $\|u(t)\|_{L^{\alpha+2}} \approx t^{-\mu}$ as $t \to \infty$. A decay like $t^{-\frac{N\alpha}{2(\alpha+2)}}$ is also possible and is the fastest possible (see (i) above). Note that the lower bound $\mu > N\alpha/4(\alpha+2)$ is also optimal. Indeed, if $u \in X_\infty$ is a solution of (4.1.1), then $u \in L^{\frac{N\alpha}{4(\alpha+2)}}((0,\infty), L^{\alpha+2}(\mathbb{R}^N))$ (see Remark 3.12 in [75]). If λ in (6.4.1) is a negative real number, the same property holds for any H^1 solution with initial value in $H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2\,dx)$; see Chapter 7 below. In both cases, it follows that $\liminf_{t\to\infty} t^{\frac{N\alpha}{4(\alpha+2)}} \|u(t)\|_{L^{\alpha+2}} = 0$.

6.5. Finite-Time Blowup

We show that, under suitable assumptions on the nonlinearity, some solutions of the nonlinear Schrödinger equation blow up in finite time. We follow the method of Glassey [148]. This is essentially a convexity method, but not purely energetic. It is based on the calculation of the variance

$$\int_{\mathbb{R}^N} |x|^2 |u(t,x)|^2\,dx .$$

That calculation is technically complicated. Therefore, for the sake of simplicity, we consider a specific type of nonlinearity. More precisely, we consider the case where g is as in Example 3.2.11. Therefore, we assume

$$g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u ,$$

where V, f, and W are as follows. The potential V is real-valued, $V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $p \ge 1$, $p > N/2$. The function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}$ and satisfies (3.2.7), (3.2.8), and (3.2.17). Extend f to $\mathbb{R}^N \times \mathbb{C}$ by (3.2.10). The potential W is even and real valued; $W \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $q \ge 1$, $q > N/4$. In particular, g is the gradient of the potential G defined by

$$G(u) = \int_{\mathbb{R}^N} \left\{ \frac{1}{2} V(x)|u(x)|^2 + F(x, u(x)) + \frac{1}{4}(W \star |u|^2)(x)|u(x)|^2 \right\} dx ,$$

and we set

$$E(u) = \frac{1}{2}\int_{\Omega} |\nabla u(x)|^2\,dx - G(u) \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

We recall (see Corollary 4.3.3 and Remark 5.3.3) that the initial-value problem for (4.1.1) is locally well posed in $H^1(\mathbb{R}^N)$, that there is conservation of charge and energy, and that there is the $H^2(\mathbb{R}^N)$ regularity if the initial value is in $H^2(\mathbb{R}^N)$.

Our blowup result is based on the following identities, which will also be essential in the next chapter to establish the pseudoconformal conservation law.

PROPOSITION 6.5.1. *Let g*

$$g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u$$

be as in Example 3.2.11. *Assume, in addition, that*

$$x \cdot \nabla V(x) \in L^\sigma(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \quad \textit{for some } \sigma \geq 1, \sigma > \frac{N}{2}, \tag{6.5.1}$$

$$f(x,u) \textit{ is independent of } x, \tag{6.5.2}$$

$$x \cdot \nabla W(x) \in L^\delta(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \quad \textit{for some } \delta \geq 1, \delta > \frac{N}{4}. \tag{6.5.3}$$

Consider $\varphi \in H^1(\mathbb{R}^N)$ *such that* $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$, *and let u be the corresponding maximal solution of* (4.1.1). *It follows that the function* $t \mapsto |\cdot|u(t,\cdot)$ *belongs to* $C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N))$. *Moreover, the function*

$$t \mapsto f(t) \equiv \int_{\mathbb{R}^N} |x|^2 |u(t,x)|^2\,dx \tag{6.5.4}$$

is in $C^2(-T_{\min}, T_{\max})$,

$$f'(t) = 4\,\mathrm{Im}\int_{\mathbb{R}^N} \overline{u}\, x \cdot \nabla u\,dx, \tag{6.5.5}$$

and

$$\begin{aligned} f''(t) = 16E(\varphi) &+ \int_{\mathbb{R}^N} (8(N+2)F(u) - 4N\,\mathrm{Re}(f(u)\overline{u}))dx \\ &+ 8\int_{\mathbb{R}^N} \Big(V + \frac{1}{2}x\cdot\nabla V\Big)|u|^2\,dx \\ &+ 4\int_{\mathbb{R}^N} \Big((W + \frac{1}{2}x\cdot\nabla W)\star|u|^2\Big)|u|^2\,dx \end{aligned} \tag{6.5.6}$$

for all $t \in (-T_{\min}, T_{\max})$.

Before proceeding to the proof, we establish the following lemma.

LEMMA 6.5.2. *Let* $g \in C(H^1(\mathbb{R}^N), H^{-1}(\mathbb{R}^N))$. *Assume that* $g(w) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *and that* $\mathrm{Im}\, g(w)\overline{w} = 0$ *a.e. in* $\mathbb{R}^N$ *for all* $w \in H^1(\mathbb{R}^N)$. *Let* $I \ni 0$ *be an interval of* $\mathbb{R}$, *let* $\varphi \in H^1(\mathbb{R}^N)$, *and let u be a weak* H^1*-solution of* (4.1.1) *on I. If* $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$, *then the function* $t \mapsto |\cdot|u(t,\cdot)$ *belongs to* $C(I, L^2(\mathbb{R}^N))$. *Furthermore, the function* f *defined by* (6.5.4) *belongs to* $W^{1,\infty}(I)$ *and the identity* (6.5.5) *holds for a.a.* $t \in I$. *If* u *is a strong* H^1*-solution of* (4.1.1) *on* I, *then* $f \in C^1(I)$ *and the identity* (6.5.5) *holds for all* $t \in I$.

PROOF. Without loss of generality, we may assume that $I = [0,T]$ with $0 < T < \infty$. Formally, the result follows by multiplying the equation by $i|x|^2\overline{u}$, taking the real part, and integrating on $\mathbb{R}^N$. However, the equation makes sense only in $H^{-1}(\mathbb{R}^N)$ and $i|x|^2\overline{u} \notin H^1(\mathbb{R}^N)$, so we need a regularization argument. Let $\varepsilon > 0$, and take the H^{-1}–H^1 duality product of equation (4.1.1) with $ie^{-2\varepsilon|x|^2}|x|^2u(t,x) \in H^1(\mathbb{R}^N)$. Setting

$$f_\varepsilon(t) = \|e^{-\varepsilon|x|^2}|x|u(t)\|_{L^2}^2 ,$$

we deduce easily that

$$f_\varepsilon'(t) = 2\,\mathrm{Im} \int_{\mathbb{R}^N} \{\nabla u \cdot \nabla(e^{-2\varepsilon|x|^2}|x|^2\overline{u}) - e^{-2\varepsilon|x|^2}|x|^2 g(u)\overline{u}\}dx .$$

Since $\mathrm{Im}(\nabla u \cdot \nabla\overline{u}) = \mathrm{Im}(g(u)\overline{u}) = 0$ a.e., we obtain that

$$\begin{aligned} f_\varepsilon'(t) &= 2\,\mathrm{Im} \int_{\mathbb{R}^N} \overline{u}\nabla u \cdot \nabla(e^{-2\varepsilon|x|^2}|x|^2)dx \\ &= 4\,\mathrm{Im} \int_{\mathbb{R}^N} \{e^{-\varepsilon|x|^2}(1-2\varepsilon|x|^2)\}e^{-\varepsilon|x|^2}\overline{u}\,x\cdot\nabla u\,dx , \end{aligned}$$

and so

$$f_\varepsilon(t) = f_\varepsilon(0) + 4\int_0^t \mathrm{Im} \int_{\mathbb{R}^N} \{e^{-\varepsilon|x|^2}(1-2\varepsilon|x|^2)\}e^{-\varepsilon|x|^2}\overline{u}\,x\cdot\nabla u\,dx\,dt . \tag{6.5.7}$$

Note that $e^{-\varepsilon|x|^2}(1-2\varepsilon|x|^2)$ is bounded in x and ε and that $\|e^{-\varepsilon|x|^2}|x|\varphi\|_{L^2} \le \|x\varphi\|_{L^2}$. Therefore, we deduce easily from (6.5.7) that

$$f_\varepsilon(t) \le \||x|\varphi\|_{L^2}^2 + C\int_0^t \|\nabla u(s)\|_{L^2}\sqrt{f_\varepsilon(s)}\,ds .$$

It follows easily that

$$\sqrt{f_\varepsilon(t)} \le \||x|\varphi\|_{L^2} + \frac{C}{2}\int_0^t \|\nabla u(s)\|_{L^2}\,ds \quad \text{for all } t \in I. \tag{6.5.8}$$

Letting $\varepsilon \downarrow 0$ and using Fatou's lemma, we see that $xu(t) \in L^2(\mathbb{R}^N)$ for all $t \in I$ and that $\||x|u(t)|\|_{L^2}$ is bounded in $t \in I$. Therefore, the function $t \mapsto |\cdot|u(t,\cdot)$ is weakly continuous $I \to L^2(\mathbb{R}^N)$ (see Section 1.1). Moreover, we may let $\varepsilon \downarrow 0$ in (6.5.7) and we obtain

$$\|xu(t)\|_{L^2}^2 = \|x\varphi\|_{L^2}^2 + 4\int_0^t \mathrm{Im} \int_{\mathbb{R}^N} \overline{u}\,x\cdot\nabla u\,dx\,dt . \tag{6.5.9}$$

Note that the right-hand side is a continuous function of t and so the function $t \mapsto |\cdot|u(t,\cdot)$ is continuous $I \to L^2(\mathbb{R}^N)$. It follows that the right-hand side of (6.5.9) is a $W^{1,\infty}$ function and the identity (6.5.5) holds a.e. If u is a strong H^1-solution, then the right-hand side of (6.5.9) is a C^1 function, so the identity (6.5.5) holds for all $t \in I$. □

COROLLARY 6.5.3. *Let g be as in Lemma 6.5.2. Assume that $g : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ is bounded on bounded sets and let $I \ni 0$ be a closed, bounded interval of $\mathbb{R}$. Let $\varphi \in H^1(\mathbb{R}^N)$ be such that $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$ and let u be a weak H^1-solution of (4.1.1) on I. Let $(\varphi_m)_{m\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ and for all $m \geq 0$, let u^m be a weak H^1-solution of (4.1.1) corresponding to the initial value φ_m. Assume further that $|\cdot|\varphi_m(\cdot) \in L^2(\mathbb{R}^N)$ for all $m \geq 0$ and that $x\varphi_m \to x\varphi$ in $L^2(\mathbb{R}^N)$ as $m \to \infty$. If $u^m \to u$ in $L^\infty(I, H^1(\mathbb{R}^N))$ as $m \to \infty$, then $xu^m \to xu$ in $C(I, L^2(\mathbb{R}^N))$ as $m \to \infty$.*

PROOF. Assume by contradiction that there exist $(t_m)_{m\geq 0} \subset I$ and $\varepsilon > 0$ such that $\|xu^m(t_m) - xu(t_m)\|_{L^2} \geq \varepsilon$. Without loss of generality, we may assume that there exists $\tau \in I$ such that $t_m \to \tau$ as $m \to \infty$. We deduce from (6.5.9) that for every $t \in I$,

$$\|xu^m(t)\|_{L^2}^2 = \|x\varphi_m\|_{L^2}^2 + 4\int_0^t \operatorname{Im} \int_{\mathbb{R}^N} \overline{u^m}\, x \cdot \nabla u^m \, dx\, dt\,. \tag{6.5.10}$$

It follows from (6.5.10) that $\||\cdot|u^m(t)\|_{L^2}$ is bounded uniformly in $t \in I$ and $m \geq 0$. Since $(u^m)_{m\geq 0}$ is bounded in $L^\infty(I, H^1(\mathbb{R}^N))$, $(g(u^m))_{m\geq 0}$ is bounded in $L^\infty(I, H^{-1}(\mathbb{R}^N))$ so that $(u^m_t)_{m\geq 0}$ is bounded in $L^\infty(I, H^{-1}(\mathbb{R}^N))$, by (4.1.1). Therefore, $(u^m)_{m\geq 0}$ is bounded in $C^{0,\frac{1}{2}}(I, L^2(\mathbb{R}^N))$. We deduce that $u^m(t_m) \to u(\tau)$ in $L^2(\mathbb{R}^N)$ as $m \to \infty$ and, since $|\cdot|u^m(t_m)$ is bounded in $L^2(\mathbb{R}^N)$, $|\cdot|u^m(t_m) \rightharpoonup u(\tau)$ in $L^2(\mathbb{R}^N)$ as $m \to \infty$. Since $u^m \to u$ in $L^\infty(I, H^1(\mathbb{R}^N))$, it then follows from (6.5.10) and (6.5.9) that $\|xu^m(t_m)\|_{L^2} \to \|xu(\tau)\|_{L^2}$ as $m \to \infty$, and so $xu^m(t_m) \to xu(\tau)$ in $L^2(\mathbb{R}^N)$, which yields a contradiction. This completes the proof. □

PROOF OF PROPOSITION 6.5.1. The first part of the statement follows from Lemma 6.5.2. It remains to show that the function f defined by (6.5.4) belongs to $C^2(-T_{\min}, T_{\max})$ and that the identity (6.5.6) holds. Formally, the result would follow by calculating the time derivative of the right-hand side of (6.5.5). This corresponds to multiplying equation (4.1.1) by $i(2r\partial_r\overline{u} + N\overline{u})$, which is not allowed since the equation only make sense in $H^{-1}(\mathbb{R}^N)$. The proof we give below is based on two regularizations. Therefore, we proceed in two steps.

STEP 1. The case $\varphi \in H^2(\mathbb{R}^N)$. Note first that by H^2 regularity, $u \in C((-T_{\min}, T_{\max}), H^2(\mathbb{R}^N)) \cap C^1((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N))$. Given $\varepsilon > 0$, consider $\theta_\varepsilon(x) = e^{-\varepsilon|x|^2}$ and let

$$h_\varepsilon(t) = \operatorname{Im} \int_{\mathbb{R}^N} \theta_\varepsilon \overline{u}\, x \cdot \nabla u \, dx \quad \text{for every } t \in (-T_{\min}, T_{\max}). \tag{6.5.11}$$

We claim that h_ε is C^1 and that

$$h'_\varepsilon(t) = -\operatorname{Im} \int_{\mathbb{R}^N} u_t\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}dx\,. \tag{6.5.12}$$

Indeed, (6.5.12) is equivalent to

$$h_\varepsilon(t) = h_\varepsilon(0) - \operatorname{Im} \int_0^t \int_{\mathbb{R}^N} u_t\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}dx\,ds. \tag{6.5.13}$$

The identity (6.5.13) holds in fact for every function u which is continuous in $H^1(\mathbb{R}^N)$ and C^1 in $L^2(\mathbb{R}^N)$. Indeed, by density we need only establish (6.5.13) for u which is C^1 in $H^1(\mathbb{R}^N)$. In this case, we deduce from (6.5.11) that

$$h'_\varepsilon(t) = -\operatorname{Im} \int_{\mathbb{R}^N} \theta_\varepsilon u_t x \cdot \nabla\overline{u}\,dx - \operatorname{Im} \int_{\mathbb{R}^N} \theta_\varepsilon u x \cdot \nabla\overline{u}_t\,dx,$$

and (6.5.13) follows by integration by parts, since

$$\theta_\varepsilon u x \cdot \nabla\overline{u_t} = \nabla \cdot (x\theta_\varepsilon u\overline{u_t}) - N\theta_\varepsilon u\overline{u_t} - \theta_\varepsilon\overline{u_t}x \cdot \nabla u - r\partial_r\theta_\varepsilon u\overline{u_t}.$$

This proves the claim. Using now equation (4.1.1), we see that

$$h'_\varepsilon(t) = -\operatorname{Re} \int_{\mathbb{R}^N} (\Delta u + g(u))\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}dx. \tag{6.5.14}$$

Next, an elementary calculation based on the identity

$$\operatorname{Re}(2\theta_\varepsilon\nabla u \cdot \nabla(r\partial_r\overline{u})) = -((N-2)\theta_\varepsilon + r\partial_r\theta_\varepsilon)|\nabla u|^2 + \nabla \cdot (x\theta_\varepsilon|\nabla u|^2)$$

shows that for every $u \in H^2(\mathbb{R}^N)$,

$$\begin{aligned}
&\operatorname{Re} \int_{\mathbb{R}^N} \Delta u\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}dx \\
&= -2\int_{\mathbb{R}^N} \theta_\varepsilon|\nabla u|^2\,dx \\
&\quad - \int_{\mathbb{R}^N} \{2r\partial_r\theta_\varepsilon|\partial_r u|^2 + ((N+1)\partial_r\theta_\varepsilon + r\partial_r^2\theta_\varepsilon)\operatorname{Re}(\overline{u}\partial_r u)\}dx.
\end{aligned} \tag{6.5.15}$$

We now calculate the various terms corresponding to $g(u)$. Since

$$\operatorname{Re}[Vu\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}] = \nabla \cdot (xV\theta_\varepsilon|u|^2) - \theta_\varepsilon(x \cdot \nabla V)|u|^2,$$

we obtain

$$\operatorname{Re} \int_{\mathbb{R}^N} Vu\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}dx = -\int_{\mathbb{R}^N} \theta_\varepsilon(x \cdot \nabla V)|u|^2\,dx. \tag{6.5.16}$$

Next,

$$\operatorname{Re}[f(u)(2\theta_\varepsilon r\partial_r\overline{u})] = \nabla \cdot (2x\theta_\varepsilon F(u)) - 2(N\theta_\varepsilon + r\partial_r\theta_\varepsilon)F(u),$$

so that

$$\begin{aligned}
&\operatorname{Re} \int_{\mathbb{R}^N} f(u)\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}dx = \\
&\qquad \int_{\mathbb{R}^N} (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)(f(u)\overline{u} - 2F(u))dx.
\end{aligned} \tag{6.5.17}$$

Finally, using the identity

$$\operatorname{Re}[u\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}] = \nabla\cdot(x\theta_\varepsilon|u|^2)\,,$$

we obtain

$$\operatorname{Re}\int_{\mathbb{R}^N}(W\star|u|^2)u\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}dx = -\int_{\mathbb{R}^N}\theta_\varepsilon|u|^2 x\cdot(\nabla W\star|u|^2)dx\,.$$

On the other hand, W is even, so that ∇W is odd. Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^N}\theta_\varepsilon|u|^2 x\cdot(\nabla W\star|u|^2)dx\\ &\quad=\frac{1}{2}\int_{\mathbb{R}^N}\theta_\varepsilon|u|^2[(x\cdot\nabla W)\star|u|^2]dx\\ &\qquad+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\theta_\varepsilon(x)-\theta_\varepsilon(y))|u(x)|^2|u(y)|^2 x\cdot\nabla W(x-y)dy\,dx\,,\end{aligned}$$

and so

$$\begin{aligned}&\operatorname{Re}\int_{\mathbb{R}^N}(W\star|u|^2)u\{2\theta_\varepsilon r\partial_r\overline{u} + (N\theta_\varepsilon + r\partial_r\theta_\varepsilon)\overline{u}\}dx\\ &\quad=-\frac{1}{2}\int_{\mathbb{R}^N}\theta_\varepsilon|u|^2[(x\cdot\nabla W)\star|u|^2]dx\\ &\qquad-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\theta_\varepsilon(x)-\theta_\varepsilon(y))|u(x)|^2|u(y)|^2 x\cdot\nabla W(x-y)dy\,dx\,.\end{aligned}\tag{6.5.18}$$

Applying (6.5.15), (6.5.16), (6.5.17), and (6.5.18), we deduce from (6.5.14) that

$$\begin{aligned}h_\varepsilon'(t) &= 2\int_{\mathbb{R}^N}\theta_\varepsilon|\nabla u|^2\,dx + \int_{\mathbb{R}^N}\theta_\varepsilon(x\cdot\nabla V)|u|^2\,dx\\ &\quad+\int_{\mathbb{R}^N}N\theta_\varepsilon(2F(u)-f(u)\overline{u})\,dx + \frac{1}{2}\int_{\mathbb{R}^N}\theta_\varepsilon|u|^2[(x\cdot\nabla W)\star|u|^2]dx\\ &\quad+\int_{\mathbb{R}^N}\Big\{r\partial_r\theta_\varepsilon(2|\partial_r u|^2 + 2F(u) - f(u)\overline{u})\\ &\qquad\qquad+((N+1)\partial_r\theta_\varepsilon + r\partial_r^2\theta_\varepsilon)\operatorname{Re}(\overline{u}\partial_r u)\Big\}dx\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\theta_\varepsilon(x)-\theta_\varepsilon(y))|u(x)|^2|u(y)|^2 x\cdot\nabla W(x-y)dy\,dx\,.\end{aligned}\tag{6.5.19}$$

Note that $\theta_\varepsilon, r\partial_r\theta_\varepsilon$, and $r^2\partial_r^2\theta_\varepsilon$ are bounded with respect to both x and ε. Furthermore, $\theta_\varepsilon\uparrow 1$, $\partial_r\theta_\varepsilon\to 0$, $r\partial_r\theta_\varepsilon\to 0$, and $r\partial_r^2\theta_\varepsilon\to 0$ as $\varepsilon\downarrow 0$. On the other hand, for every $t\in(-T_{\min},T_{\max})$, we have $u(t)\in H^2(\mathbb{R}^N)$ and $|x|u(t)\in L^2(\mathbb{R}^N)$ by Lemma 6.5.2, and so we may use the dominated convergence theorem to pass to the limit in the right-hand side of (6.5.19) as $\varepsilon \downarrow 0$, except in the last one. We obtain

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} h_\varepsilon'(t) &= 2\int_{\mathbb{R}^N} |\nabla u|^2\,dx + N\int_{\mathbb{R}^N}(2F(u) - \operatorname{Re}(f(u)\overline{u}))dx \\ &+ \int_{\mathbb{R}^N} |u|^2\,x\cdot\nabla V\,dx + \frac{1}{2}\int_{\mathbb{R}^N}|u|^2((x\cdot\nabla W)\star|u|^2)dx + L,\end{aligned} \tag{6.5.20}$$

where L is the limit as $\varepsilon \downarrow 0$ of the last term in (6.5.19). We claim that

$$L = 0. \tag{6.5.21}$$

Since also

$$\lim_{\varepsilon\downarrow 0} h_\varepsilon(t) = \operatorname{Im}\int_{\mathbb{R}^N} \overline{u}\,x\cdot\nabla u\,dx \equiv h(t),$$

we see that h is of class C^1 and that

$$\begin{aligned}h'(t) &= 2\int_{\mathbb{R}^N} |\nabla u|^2\,dx + N\int_{\mathbb{R}^N}(2F(u) - \operatorname{Re}(f(u)\overline{u}))dx \\ &+ \int_{\mathbb{R}^N} |u|^2\,x\cdot\nabla V\,dx + \frac{1}{2}\int_{\mathbb{R}^N}|u|^2((x\cdot\nabla W)\star|u|^2)dx.\end{aligned}$$

Equation (6.5.6) now follows from the above identity, (6.5.5), and conservation of energy. We finally prove the claim (6.5.21). Note that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\theta_\varepsilon(x) - \theta_\varepsilon(y))|u(x)|^2|u(y)|^2 x\cdot\nabla W(x-y)dy\,dx\right| \le \\ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|x|\frac{|\theta_\varepsilon(x)-\theta_\varepsilon(y)|}{|x-y|}|u(x)|^2|u(y)|^2|(x-y)\cdot\nabla W(x-y)|dy\,dx.\end{aligned} \tag{6.5.22}$$

Also, it follows from assumption (6.5.3) and from Young's and Sobolev's inequalities that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)|^2|u(y)|^2|(x-y)\cdot\nabla W(x-y)|dy\,dx &\le C\|u\|_{L^2}^4 + C\|u\|^4_{L^{\frac{4\delta}{2\delta-1}}} \\ &\le C\|u\|_{H^1}^4.\end{aligned}$$

Since

$$\sup_{x\neq y}|x|\frac{|\theta_\varepsilon(x)-\theta_\varepsilon(y)|}{|x-y|} < \infty \quad \text{and} \quad |x|\frac{|\theta_\varepsilon(x)-\theta_\varepsilon(y)|}{|x-y|} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0 \quad \text{a.e. in } \mathbb{R}^N\times\mathbb{R}^N,$$

we may use the dominated convergence theorem to pass to the limit in (6.5.22) as $\varepsilon \downarrow 0$, and we obtain (6.5.21).

STEP 2. Conclusion. Let $(\varphi_m)_{m\in\mathbb{N}} \subset H^2(\mathbb{R}^N)$ be such that $\varphi_m \to \varphi$ in $H^1(\mathbb{R}^N)$ and $x\varphi_m \to x\varphi$ in $L^2(\mathbb{R}^N)$ as $m \to \infty$, and let u^m be the corresponding maximal solutions of (4.1.1). Let $\Phi(t)$ denote the right-hand side of (6.5.6) and let

$\Phi_m(t)$ denote the right-hand side of (6.5.6) corresponding to the solution u^m. It follows from Step 1 that

$$\|xu^m(t)\|_{L^2}^2 = \|x\varphi_m\|_{L^2}^2 + 4t\,\mathrm{Im} \int_{\mathbb{R}^N} \overline{\varphi_m}\, x\cdot\nabla\varphi_m\,dx + \int_0^t\int_0^s \Phi_m(s)ds\,dt\,. \tag{6.5.23}$$

By continuous dependence and Corollary 6.5.3, we may let $m\to\infty$ in (6.5.23) and we obtain

$$\|xu(t)\|_{L^2}^2 = \|x\varphi\|_{L^2}^2 + 4t\,\mathrm{Im} \int_{\mathbb{R}^N} \overline{\varphi}\, x\cdot\nabla\varphi\,dx + \int_0^t\int_0^s \Phi(s)ds\,dt\,,$$

from which (6.5.6) follows. □

THEOREM 6.5.4. *Let*

$$g(u) = Vu + f(\cdot,u(\cdot)) + (W\star|u|^2)u$$

be as in Example 3.2.11. *Assume further that*

$$2(N+2)F(s) - Nsf(s) \le 0 \quad \textit{for all } s\ge 0, \tag{6.5.24}$$

$$V + \frac{1}{2}x\cdot\nabla V \le 0 \quad \textit{a.e.}, \tag{6.5.25}$$

$$W + \frac{1}{2}x\cdot\nabla W \le 0 \quad \textit{a.e.} \tag{6.5.26}$$

Let $\varphi\in H^1(\mathbb{R}^N)$ be such that $|\cdot|\,|\varphi(\cdot)\in L^2(\mathbb{R}^N)$. If $E(\varphi)<0$, then $T_{\min}<\infty$ and $T_{\max}<\infty$. In other words, the solution u of (4.1.1) *blows up in finite time for both $t>0$ and $t<0$.*

PROOF. It follows from (6.5.24), (6.5.25), (6.5.26), and Proposition 6.5.1 that for every $t\in(-T_{\min},T_{\max})$,

$$\|xu(t)\|_{L^2}^2 \le \theta(t)\,, \tag{6.5.27}$$

where

$$\theta(t) = \|x\varphi\|_{L^2}^2 + 4t\,\mathrm{Im}\int_{\mathbb{R}^N} \overline{\varphi}\, x\cdot\nabla\varphi\,dx + 8t^2E(\varphi)\,.$$

Observe that $\theta(t)$ is a second-degree polynomial and that the coefficient of t^2 is negative; therefore $\theta(t)<0$ for $|t|$ large enough. Since $\|xu(t)\|_{L^2}^2\ge 0$, we deduce from (6.5.27) that both $T_{\min}$ and $T_{\max}$ are finite. □

REMARK 6.5.5. Note that the proof of Theorem 6.5.4 does not show that $\|xu(t)\|_{L^2}\to 0$ as $t\uparrow T_{\max}$ or $t\downarrow -T_{\max}$. (See Ball [16, 15] for an interesting discussion of related phenomena.) This is sometimes the case (see Remark 6.7.3), but not always. Consider the model case $g(u)=\lambda|u|^\alpha u$ with $\lambda>0$ and $\alpha=4/N$. First, observe that by the invariance of the equation under space translation, one constructs easily a solution such that $\|xu(t)\|_{L^2}^2 \not\to 0$ as $t\uparrow T_{\max}$. Indeed, it follows from the conservation of momentum (3.1.5) that, given $x_0\in\mathbb{R}^N$,

$$\int|x-x_0|^2|u|^2 = \int|x|^2|u|^2 + |x_0|^2\int|\varphi|^2 + 2\int x\cdot x_0|\varphi|^2 + 4t\,\mathrm{Im}\int\overline{\varphi}x_0\cdot\nabla\varphi\,,$$

so that $\|(x - x_0)u\|_{L^2}$ will not converge to 0 if $|x_0|$ is large enough. Moreover, in general,

$$\inf\left\{\|(x - x_0)u(t)\|_{L^2} : t \in [0, T_{\max}), x_0 \in \mathbb{R}^N\right\} > 0\,.$$

To see this, we follow an argument of Merle [248]. Consider a real-valued, spherically symmetric function $\varphi \in H^1(\mathbb{R}^N)$ such that $x\varphi(x) \notin L^2(\mathbb{R}^N)$ and $E(\varphi) < 0$. Let $(\varphi_n)_{n\geq 0}$ be a sequence of real-valued, spherically symmetric functions such that $x\varphi_n(x) \in L^2(\mathbb{R}^N)$ for all $n \geq 0$ and $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$, and let u_n be the corresponding solutions of (4.1.1). In particular, $E(\varphi_n) \underset{n\to\infty}{\longrightarrow} E(\varphi)$ and $\|\varphi_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} \|\varphi\|_{L^2}$. Therefore, it follows from the proof of Theorem 6.5.10 below (see in particular formula (6.5.42)) that there exists a function $\Psi \in W^{4,\infty}(\mathbb{R}^N)$, $\Psi \geq 0$, such that

$$\frac{d^2}{dt^2}\int \Psi|u_n|^2 \leq 2E(\varphi) < 0 \quad \text{for } 0 \leq t < T_{\max}(\varphi_n)\,.$$

On the other hand, since φ_n is real valued, one easily verifies that

$$\frac{d}{dt}\int \Psi|u_n|^2\bigg|_{t=0} = 0\,,$$

so that

$$\int \Psi|u_n|^2 \leq 2\int \Psi|\varphi|^2 + t^2E(\varphi)$$

for $0 \leq t < T_{\max}(\varphi_n)$ and for n large enough. This implies that there exists $T^0 < \infty$ such that

$$T_{\max}(\varphi_n) \leq T^0 \quad \text{for } n \text{ large enough.} \tag{6.5.28}$$

On the other hand, for every $x_0 \in \mathbb{R}^N$, we have (see the proof of Theorem 6.5.4)

$$\|(x - x_0)u_n(t)\|_{L^2}^2 = \|(x - x_0)\varphi_n\|_{L^2}^2 + 8E(\varphi_n)t^2 \quad \text{for } 0 \leq t < T_{\max}(\varphi_n)\,.$$

In particular, for n large enough,

$$\|(x - x_0)u_n(t)\|_{L^2}^2 \geq \|(x - x_0)\varphi_n\|_{L^2}^2 + 16E(\varphi)t^2 \quad \text{for } 0 \leq t < T_{\max}(\varphi_n)\,.$$

Since $\inf_{x_0\in\mathbb{R}^N} \|(x - x_0)\varphi_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} \infty$, it now follows from (6.5.28) that

$$\inf\left\{\|(x - x_0)u_n(t)\|_{L^2} : 0 \leq t < T_{\max}(\varphi_n), x_0 \in \mathbb{R}^N\right\} \underset{n\to\infty}{\longrightarrow} \infty\,,$$

which proves the claim.

REMARK 6.5.6. The proof of Theorem 6.5.4 is based on the fact that the nonnegative quantity $\|xu(t)\|_{L^2}^2$ is dominated by the polynomial $\theta(t)$ and that the assumption $E(\varphi) < 0$ implies that $\theta(t)$ takes negative values. A necessary and sufficient condition so that $\theta(t)$ takes negative values is that

$$\left(\operatorname{Im}\int_{\mathbb{R}^N} \overline{\varphi}x\cdot\nabla\varphi\,dx\right)^2 > 2E(\varphi)\|x\varphi\|_{L^2}^2\,. \tag{6.5.29}$$

Therefore, we have

$$\text{if } E(\varphi)=0 \text{ and } \operatorname{Im}\int \overline{\varphi}\, x\cdot\nabla\varphi\,dx<0 \qquad \text{then } T_{\max}<\infty\,,$$

$$\text{if } E(\varphi)=0 \text{ and } \operatorname{Im}\int \overline{\varphi}\, x\cdot\nabla\varphi\,dx>0 \qquad \text{then } T_{\min}<\infty\,,$$

$$\text{if } E(\varphi)>0 \text{ and } \operatorname{Im}\int_{\mathbb{R}^N} \overline{\varphi}\, x\cdot\nabla\varphi\,dx<-\sqrt{2E(\varphi)}\|x\varphi\|_{L^2} \quad \text{then } T_{\max}<\infty\,,$$

$$\text{if } E(\varphi)>0 \text{ and } \operatorname{Im}\int_{\mathbb{R}^N} \overline{\varphi}\, x\cdot\nabla\varphi\,dx>\sqrt{2E(\varphi)}\|x\varphi\|_{L^2} \quad \text{then } T_{\min}<\infty.$$

REMARK 6.5.7. Note that

$$2(N+2)F(s)-Nsf(s)=-Ns^{3+\frac{4}{N}}\frac{d}{ds}(s^{-(2+\frac{4}{N})}F(s))\,,$$

and so assumption (6.5.24) is equivalent to the property that $s^{-(2+\frac{4}{N})}F(s)$ is a nondecreasing function of s. Similarly, $V+\frac{1}{2}x\cdot\nabla V=V+\frac{1}{2}r\partial_r V=\frac{1}{2r}\partial_r(r^2V)$. Therefore, assumption (6.5.25) (respectively, (6.5.26)) is equivalent to the property that $|x|^2V(x)$ (respectively, $|x|^2W(x)$) is a nonincreasing function of $|x|$.

REMARK 6.5.8. Note that the assumption $E(\varphi)<0$ is a sufficient condition for finite-time blowup, but it is not necessary. To see this, consider the model case $g(u)=\lambda|u|^\alpha u$ with $\lambda>0$ and $4/N\le\alpha<4/(N-2)$. We claim that for any $E_0>0$, there exists φ such that $E(\varphi)=E_0$ and $T_{\max}(\varphi)<\infty$. Indeed, fix a real-valued function $\theta\in C_c^\infty(\mathbb{R}^N)$ and set $\psi(x)=e^{-i|x|^2}\theta(x)$. It follows that $\psi\in C_c^\infty(\mathbb{R}^N)$ and that

$$\operatorname{Im}\int_{\mathbb{R}^N}\overline{\psi}x\cdot\nabla\psi=-\int_{\mathbb{R}^N}|x|^2\theta(x)^2<0\,. \tag{6.5.30}$$

Set now

$$A=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\psi|^2\,,\qquad B=\frac{\lambda}{\alpha+2}\int_{\mathbb{R}^N}|\psi|^{\alpha+2}\,,$$

$$C=\int_{\mathbb{R}^N}|x|^2|\psi|^2\,,\qquad D=-\operatorname{Im}\int_{\mathbb{R}^N}\overline{\psi}x\cdot\nabla\psi\,.$$

Given $\sigma,\mu>0$, to be chosen later, set $\varphi(x)=\sigma\psi(\mu x)$. It follows from (6.5.30) that

$$\operatorname{Im}\int_{\mathbb{R}^N}\overline{\varphi}x\cdot\nabla\varphi<0\,,$$

so that, in view of (6.5.29), we need only show that we can choose σ and μ so that

$$E(\varphi)=E_0\,, \tag{6.5.31}$$

$$\Big(\operatorname{Im}\int_{\mathbb{R}^N}\overline{\varphi}\,x\cdot\nabla\varphi\Big)^2>2E(\varphi)\|x\varphi\|_{L^2}^2\,. \tag{6.5.32}$$

Conditions (6.5.31)–(6.5.32) reduce to

$$\frac{\sigma^2}{\mu^N}\mu^2\left(A-\frac{\sigma^\alpha}{\mu^2}B\right)=E_0\,, \tag{6.5.33}$$

$$\frac{D^2}{C}>A-\frac{\sigma^\alpha}{\mu^2}B\,. \tag{6.5.34}$$

Fix now

$$0<\varepsilon<\min\left\{A,\frac{D^2}{C}\right\},$$

and let μ be given by

$$\mu^2=\frac{B}{A-\varepsilon}\sigma^\alpha\,.$$

In particular, (6.5.34) is satisfied and (6.5.33) reduces to

$$\varepsilon\left(\frac{B}{A-\varepsilon}\right)^{\frac{2-N}{2}}\sigma^{\frac{4-(N-2)\alpha}{2}}=E_0\,,$$

which is achieved for σ suitably chosen.

REMARK 6.5.9. Theorem 6.5.4 shows the existence of solutions for which both $T_{\max}<\infty$ and $T_{\min}<\infty$. As a matter of fact, there exist solutions for which $T_{\max}=\infty$ and $T_{\min}<\infty$ and solutions for which $T_{\max}<\infty$ and $T_{\min}=\infty$. Indeed, let $g(u)=\lambda|u|^\alpha u$ with $\lambda>0$ and $\alpha\geq 4/N$. Let $\varphi\in H^1(\mathbb{R}^N)$ with $|\cdot|\varphi(\cdot)\in L^2(\mathbb{R}^N)$ be such that $E(\varphi)<0$. It follows that the maximal solution u of (4.1.1) blows up in finite time for both $t>0$ and $t<0$ (see Theorem 6.5.4). Theorem 6.3.4 implies that if b is large enough, then the maximal solution $\widetilde{u}_b$ of (4.1.1) with initial value φ_b given by (6.3.12) is positively global and decays as $t\to\infty$. Of course, $E(\varphi_b)\geq 0$ for such b's, and one may wonder if $\widetilde{u}_b$ still blows up at a finite negative time. The answer is yes, as the following argument shows. Changing φ_b to $\overline{\varphi_b}$ (which changes $\widetilde{u}_b(t)$ to $\overline{\widetilde{u}_b(-t)}$), it suffices to show that if $E(\varphi)<0$, then for all $b>0$ the solution v of (4.1.1) with initial value

$$\psi(x)=\varphi(x)e^{-i\frac{b|x|^2}{4}}$$

blows up at a positive finite time. Let $T_{\max}(\psi)$ be the maximal existence time of v, and let $f(t)=\||\cdot|v(t,\cdot)\|_{L^2}^2$. It follows from formulae (6.5.5) and (6.5.6) that

$$f(t)=f(0)+tf'(0)+8E(\psi)t^2-\lambda\frac{4(N\alpha-4)}{\alpha+2}\int_0^t\int_0^s\int_{\mathbb{R}^N}|v|^{\alpha+2}\,dx\,d\sigma\,ds$$

for all $0\leq t<T_{\max}(\psi)$, and so

$$f(t)\leq f(0)+tf'(0)+8E(\varphi)t^2\quad\text{for all } 0\leq t<T_{\max}(\psi).$$

Setting $P(t)=f(0)+tf'(0)+8E(u(0))t^2$ for all $t\geq 0$, a straightforward calculation shows that

$$P(t)=\|x\varphi\|_{L^2}^2+4t\left(F(\varphi)-\frac{b}{2}\|x\varphi\|_{L^2}^2\right)+8t^2\left(E(\varphi)+\frac{b^2}{8}\|x\varphi\|_{L^2}^2-\frac{b}{2}F(\varphi)\right)$$

with

$$F(\varphi) = \operatorname{Im} \int_{\mathbb{R}^N} x\overline{\varphi} \nabla \varphi \, dx .$$

In particular,

$$P\left(\frac{1}{b}\right) = \frac{8}{b^2} E(\varphi) < 0 ,$$

and we deduce easily from (6.3.13) that $T_{\max}(\psi) < 1/b$ (see the proof of Theorem 6.5.4). Hence the result follows.

The condition for finite-time blowup in Theorem 6.5.4 is $E(\varphi) < 0$. However, the argument is based on the study of $\|xu(t,x)\|_{L^2}^2$, and this quantity is defined only if $x\varphi(x) \in L^2(\mathbb{R}^N)$, but not for a general $\varphi \in H^1(\mathbb{R}^N)$. The question as to whether negative energy implies finite-time blowup for general H^1 solutions is open (see Gonçalves Ribeiro [152] and Merle and Raphael [249] for partial results). However, Ogawa and Tsutsumi [275] have shown the following result in this direction.

THEOREM 6.5.10. *Let $g(u) = \lambda |u|^\alpha u$ with $\lambda > 0$. Assume $N \geq 2$ and*

$$\frac{4}{N} \leq \alpha < \frac{4}{N-2} \quad (2 \leq \alpha \leq 4 \text{ if } N = 2) .$$

If $\varphi \in H^1(\mathbb{R}^N)$ is such that $E(\varphi) < 0$ and if φ is spherically symmetric, then $T_{\min} < \infty$ and $T_{\max} < \infty$; i.e., the solution u of (4.1.1) *blows up in finite time for both $t > 0$ and $t < 0$.*

The proof is in some way an adaptation of the proof of Theorem 6.5.4. Instead of calculating $\|xu(t,x)\|_{L^2}^2$, we calculate $\|M(x)u(t,x)\|_{L^2}^2$, where $M : \mathbb{R}^N \to \mathbb{R}$ is a function such that $M(x) = |x|$ for $|x| \leq R$ and M is constant for $|x|$ large. Next, we use the decay properties of the spherically symmetric functions of $H^1(\mathbb{R}^N)$ to estimate certain integrals for $|x|$ large that appear in the calculation of $\|M(x)u(t,x)\|_{L^2}^2$. Note that, as opposed to the case $x\varphi \in L^2(\mathbb{R}^N)$, the appropriate function $M(x)$ depends on the initial value φ.

The proof makes use of the following lemma.

LEMMA 6.5.11. *Let $N \geq 1$ and let $k \in C^1([0,\infty))$ be a nonnegative function such that $r^{-(N-1)}k(r) \in L^\infty(0,\infty)$ and $r^{-(N-1)}(k'(r))^- \in L^\infty(0,\infty)$. There exists a constant C such that*

$$\|k^{\frac{1}{2}}u\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|_{L^2(\mathbb{R}^N)}^{\frac{1}{2}} \big(\|r^{-(N-1)}ku_r\|_{L^2(\mathbb{R}^N)}^{\frac{1}{2}} + \|r^{-(N-1)}(k')^-\|_{L^\infty}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^N)}^{\frac{1}{2}}\big)$$

for all spherically symmetric functions $u \in H^1(\mathbb{R}^N)$.

PROOF. By density, we may assume that $u \in \mathcal{D}(\mathbb{R}^N)$. For $s > 0$,

$$\begin{aligned}
k(s)|u(s)|^2 &= -\int_s^\infty \frac{d}{d\sigma}(k(\sigma)|u(\sigma)|^2) d\sigma \\
&= -\int_s^\infty k'(\sigma)|u(\sigma)|^2 \, d\sigma - 2\int_s^\infty k(s) \operatorname{Re}\big(u(\sigma)\overline{u}_r(\sigma)\big) d\sigma \\
&\leq \int_s^\infty (k'(\sigma))^- |u(\sigma)|^2 \, d\sigma + 2\int_s^\infty k(\sigma)|u(\sigma)||u_r(\sigma)| d\sigma .
\end{aligned}$$

Therefore,

$$k(s)|u(s)|^2 \le C\|r^{-(N-1)}(k')^-\|_{L^\infty}\|u\|^2_{L^2(\mathbb{R}^N)} + C\|u\|_{L^2(\mathbb{R}^N)}\|r^{-(N-1)}ku_r\|_{L^2(\mathbb{R}^N)},$$

and the result follows. □

PROOF OF THEOREM 6.5.10. By scaling, we may assume $\lambda = 1$. Let u be as in the statement of the theorem. Consider a function $\Psi \in W^{4,\infty}(\mathbb{R}^N)$, and set

$$V(t) = \frac{1}{2}\int_{\mathbb{R}^N} \Psi(x)|u(t,x)|^2\,dx \quad \text{for all } t \in (-T_{\min}, T_{\max}).$$

We claim that

$$\begin{aligned}\frac{d^2}{dt^2}V(t) = 2\int_{\mathbb{R}^N}(H(\Psi)\nabla u, \nabla u)dx - \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}\Delta\Psi|u|^{\alpha+2}\,dx \\ -\frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\Psi|u|^2\,dx\end{aligned} \tag{6.5.35}$$

for all $t \in (-T_{\min}, T_{\max})$, where the Hessian matrix $H(\Psi)$ is given by $H(\Psi) = (\partial_j\partial_k\Psi)_{1\le j,k\le N}$. Assuming $\varphi \in H^2(\mathbb{R}^N)$, it follows that u is an H^2 solution (see Remark 5.3.3). In this case, (6.5.35) follows from elementary calculations (see Kavian [208]). The general case follows by approximating φ in $H^1(\mathbb{R}^N)$ by a sequence $(\varphi_n)_{n\ge0} \subset H^2(\mathbb{R}^N)$ and using the continuous dependence. Next, we rewrite (6.5.35):

$$\begin{aligned}\frac{d^2}{dt^2}V(t) = 2N\alpha E(u(t)) - 2\int_{\mathbb{R}^N}\Big\{\frac{N\alpha}{2}|\nabla u|^2 - (H(\Psi)\nabla u, \nabla u)\Big\}dx \\ + \frac{\alpha}{\alpha+2}\int_{\mathbb{R}^N}(2N - \Delta\Psi)|u|^{\alpha+2}\,dx - \frac{1}{2}\int_{\mathbb{R}^N}\Delta^2\Psi|u|^2\,dx.\end{aligned} \tag{6.5.36}$$

Let now $\rho \in \mathcal{D}(\mathbb{R})$ be such that $\rho(x) \equiv \rho(4-x)$, $\rho \ge 0$, $\int_{\mathbb{R}}\rho = 1$, $\operatorname{supp}(\rho) \subset (1,3)$, and $\rho' \ge 0$ on $(-\infty, 2)$. We define the function θ by

$$\theta(r) = r - \int_0^r (r-\sigma)\rho(\sigma)d\sigma \quad \text{for } r \ge 0.$$

We consider $\varepsilon \in (0,1)$, to be specified later, and we set

$$\Psi(x) \equiv \Psi(r) = \frac{1}{\varepsilon}\theta(\varepsilon r^2)$$

and

$$\gamma(x) \equiv \gamma(r) = 1 - \theta'(\varepsilon r^2) - 2\varepsilon r^2\theta''(\varepsilon r^2) = \int_0^{\varepsilon r^2}\rho(s)ds + 2\varepsilon r^2\rho(\varepsilon r^2)$$

for $x \in \mathbb{R}^N$ and $r = |x|$. Elementary calculations show that

$$\begin{cases}(H(\Psi)\nabla u, \nabla u) = 2(1-\gamma(r))|u_r|^2, \\ \Delta\Psi = 2N(1-\gamma(r)) + 4(1-N)\varepsilon r^2\theta''(\varepsilon r^2),\end{cases} \tag{6.5.37}$$

and that

$$\Delta^2\Psi = \varepsilon\big(4N(N+2)\theta''(\varepsilon r^2) + 16(N+2)\varepsilon r^2\theta'''(\varepsilon r^2) + 16(\varepsilon r^2)^2\theta''''(\varepsilon r^2)\big).$$

In particular, there exists a constant a such that

$$\|\Delta^2\Psi\|_{L^\infty} \le 2a\varepsilon. \tag{6.5.38}$$

It now follows from (6.5.36), (6.5.37), and (6.5.38) that

$$\frac{d^2}{dt^2}V(t) \le 2N\alpha E(\varphi) - 4\int\gamma(r)|u_r|^2 + \frac{2N\alpha}{\alpha+2}\int\gamma(r)|u|^{\alpha+2} + a\varepsilon\|u\|_{L^2}^2, \tag{6.5.39}$$

where we have used the above relations and the properties $\frac{N\alpha}{4} \ge 1$ and $\theta'' \le 0$. We claim that there exist b and c such that

$$\frac{2N\alpha}{\alpha+2}\int\gamma(r)|u|^{\alpha+2} \le b\varepsilon^{\frac{(N-1)\alpha}{4}}\|u\|_{L^2}^{\frac{\alpha+4}{2}}\Big(\int\gamma|u_r|^2\Big)^{\frac{\alpha}{4}} + c\varepsilon^{\frac{N\alpha}{4}}\|u\|_{L^2}^{\alpha+2}. \tag{6.5.40}$$

Indeed, we first observe that $\gamma(r) \le 1 + 2\sup_{s\ge0} s\rho(s)$, so that

$$\begin{aligned}\int\gamma(r)|u|^{\alpha+2} &\le C\|\gamma^{\frac14}u\|_{L^\infty}^{\alpha}\|u\|_{L^2}^2\\ &\le C\|u\|_{L^2}^{\frac{\alpha+4}{2}}\|r^{-(N-1)}\gamma^{\frac12}u_r\|_{L^2}^{\frac{\alpha}{2}} + C\|u\|_{L^2}^{\alpha+2}\|r^{-(N-1)}\gamma^{-\frac12}(\gamma')^-\|_{L^\infty}^{\frac{\alpha}{2}}.\end{aligned}$$

The first inequality follows from the property $\alpha \le 4$ and the second from Lemma 6.5.11 (one easily verifies that $\gamma^{\frac12} \in C^1([0,\infty))$). Observe that $\gamma(r) \equiv 0$ for $r \le \varepsilon^{-\frac12}$, so that $\|r^{-(N-1)}\gamma^{\frac12}u_r\|_{L^2} \le \varepsilon^{\frac{N-1}{2}}\|\gamma^{\frac12}u_r\|_{L^2}$. Next, note that $\gamma \ge 1/2$ for $\varepsilon r^2 \ge 2$. Furthermore,

$$\gamma'(r) = 6\varepsilon r\rho(\varepsilon r^2) + 4\varepsilon^2 r^3\rho'(\varepsilon r^2),$$

so that $\gamma'(r) \ge 0$ for $\varepsilon r^2 \le 2$ and

$$\gamma'(r) \ge -4\varepsilon^2 r^3|\rho'(\varepsilon r^2)| \ge -4\varepsilon^{\frac12}\|s^{\frac32}\rho'(s)\|_{L^\infty(0,\infty)}.$$

Thus $\|r^{-(N-1)}\gamma^{-\frac12}(\gamma')^-\|_{L^\infty} \le C\varepsilon^{\frac{N}{2}}$ and (6.5.40) follows. Using now (6.5.39), (6.5.40), and conservation of charge, we see that

$$\begin{aligned}\frac{d^2}{dt^2}V(t) &\le 2N\alpha E(\varphi) - 4\int\gamma(r)|u_r|^2\\ &\quad + b\varepsilon^{\frac{(N-1)\alpha}{4}}\|\varphi\|_{L^2}^{\frac{\alpha+4}{2}}\Big(\int\gamma|u_r|^2\Big)^{\frac{\alpha}{4}} + c\varepsilon^{\frac{N\alpha}{4}}\|\varphi\|_{L^2}^{\alpha+2} + a\varepsilon\|\varphi\|_{L^2}^2.\end{aligned}$$

Finally, since $\alpha \le 4$, we may apply the inequality $x^{\frac{\alpha}{4}} \le x+1$ to obtain

$$\begin{aligned}\frac{d^2}{dt^2}V(t) &\le 2N\alpha E(\varphi) - \big(4 - b\varepsilon^{\frac{(N-1)\alpha}{4}}\|\varphi\|_{L^2}^{\frac{\alpha+4}{2}}\big)\int\gamma(r)|u_r|^2\\ &\quad + b\varepsilon^{\frac{(N-1)\alpha}{4}}\|\varphi\|_{L^2}^{\frac{\alpha+4}{2}} + c\varepsilon^{\frac{N\alpha}{2}}\|\varphi\|_{L^2}^{\alpha+2} + a\varepsilon\|\varphi\|_{L^2}^2.\end{aligned} \tag{6.5.41}$$

We note that the constants a, b, and c do not depend on φ and ε. Since $E(\varphi) \le 0$ and $\alpha \le 4$, it follows immediately from (6.5.41) that one can choose $\varepsilon > 0$ depending only on φ through $\|\varphi\|_{L^2}$ and $E(\varphi)$ such that

$$\frac{d^2}{dt^2}V(t) \le N\alpha E(\varphi) \quad \text{for all } t \in (-T_{\min}, T_{\max}). \tag{6.5.42}$$

Since $V(t) \geq 0$, (6.5.42) implies that $T_{\min} < \infty$ and $T_{\max} < \infty$. $\square$

REMARK 6.5.12. There are two limitations in the above proof. The first one is $\alpha \leq 4$. If $\alpha > 4$, powers of $\|\gamma^{\frac{1}{2}} u_r\|_{L^2}$ larger than 2 appear with positive coefficients in (6.5.41). This is due to the homogeneity in Lemma 6.5.11. The other limitation is $N \geq 2$, since if $N = 1$ the power of ε in the second and third terms of the right-hand side of (6.5.41) vanishes. This is due to the fact that the radially symmetric functions in dimension 1 do not have any decay property. However, in the critical case $N = 1$, $\alpha = 4$, Ogawa and Tsutsumi [276] have proved that all negative energy H^1 solutions blow up in finite time without any symmetry assumption. Their method is a more sophisticated version of the above argument. (See also Martel [240] for certain extensions.)

We now give a lower estimate for blowing up solutions (see [70]).

THEOREM 6.5.13. *Suppose $g(u) = \lambda |u|^\alpha u$ with $\lambda > 0$ and*

$$\frac{4}{N} \leq \alpha < \frac{4}{N-2} \quad \Bigl(\frac{4}{N} \leq \alpha < \infty \text{ if } N = 1\Bigr).$$

If $\varphi \in H^1(\mathbb{R}^N)$ is such that $T_{\max} < \infty$, then there exists $\delta > 0$ such that

$$\|\nabla u(t)\|_{L^2} \geq \frac{\delta}{(T_{\max} - t)^{\frac{1}{\alpha} - \frac{N-2}{4}}} \quad \text{for } 0 \leq t < T_{\max}. \tag{6.5.43}$$

A similar estimate holds near $-T_{\min}$ if $T_{\min} < \infty$.

PROOF. Generally speaking, every time one proves local existence by a fixed point argument, the proof also gives a lower estimate of the blowup. Here, we do not go through the entire local existence argument, but instead we give a direct proof. Set $r = \alpha + 2$ and let q be such that (q, r) is an admissible pair. Let φ be as above, and let u be the corresponding solution of (4.1.1). It follows from Remark 1.3.1(v) that

$$\|\nabla(|u|^\alpha u)\|_{L^{r'}} \leq C \|u\|_{L^r}^\alpha \|\nabla u\|_{L^r}. \tag{6.5.44}$$

By conservation of energy,

$$\lambda \|u\|_{L^r}^r = -rE(\varphi) + \frac{r}{2} \|\nabla u\|_{L^2}^2 .$$

Therefore

$$\lambda \|u\|_{L^r}^\alpha \leq C(1 + \|\nabla u\|_{L^2}^2)^{\frac{\alpha}{r}} \leq C(1 + \|\nabla u\|_{L^2})^{\frac{2\alpha}{r}} .$$

From (6.5.44) and the above inequality, we deduce that for any $0 < t < \tau < T_{\max}$,

$$\begin{aligned} \|\nabla(|u|^\alpha u)\|_{L^{q'}((t,\tau),L^{r'})} &\leq C(1 + \|\nabla u\|_{L^\infty((t,\tau),L^2)})^{\frac{2\alpha}{r}} \|\nabla u\|_{L^{q'}((t,\tau),L^r)} \\ &\leq C(\tau - t)^{\frac{q-q'}{qq'}} (1 + \|\nabla u\|_{L^\infty((t,\tau),L^2)})^{\frac{2\alpha}{r}} \|\nabla u\|_{L^q((t,\tau),L^r)} . \end{aligned}$$

Set now

$$f_t(\tau) = 1 + \|\nabla u\|_{L^\infty((t,\tau),L^2)} + \|\nabla u\|_{L^q((t,\tau),L^r)} ,$$

so that, by the above inequality,

$$\|\nabla(|u|^\alpha u)\|_{L^{q'}((t,\tau),L^{r'})} \leq C(\tau - t)^{\frac{q-q'}{qq'}} f_t(\tau)^{1+\frac{2\alpha}{r}} . \tag{6.5.45}$$

On the other hand, it follows from Strichartz's estimates that

$$\|\nabla u\|_{L^\infty((t,\tau),L^2)} + \|\nabla u\|_{L^q((t,\tau),L^r)} \le C\|\nabla u(t)\|_{L^2} + C\|\nabla(|u|^\alpha u)\|_{L^{q'}((t,\tau),L^{r'})}$$

for $0 < t < \tau < T_{\max}$. By (6.5.45), this implies that

$$f_t(\tau) \le C(1+\|\nabla u(t)\|_{L^2})+C(\tau-t)^{\frac{q-q'}{qq'}} f_t(\tau)^{1+\frac{2\alpha}{r}} \quad \text{for } 0<t<\tau<T_{\max}. \tag{6.5.46}$$

Consider now $t \in (0, T_{\max})$. Note that if $T_{\max} < \infty$, it follows from the blowup alternative that $f_t(\tau) \to \infty$ as $\tau \uparrow T_{\max}$. Note also that f_t is continuous and nondecreasing on $(t, T_{\max})$ and that

$$f_t(\tau) \underset{\tau\downarrow t}{\longrightarrow} 1 + \|\nabla u(t)\|_{L^2}.$$

Therefore, there exists $\tau_0 \in (t, T_{\max})$ such that $f_t(\tau_0) = (C+1)(1+\|\nabla u(t)\|_{L^2})$, where C is the constant in (6.5.46). Choosing $\tau = \tau_0$ in (6.5.46) yields

$$\begin{aligned} 1+\|\nabla u(t)\|_{L^2} &\le C(1+C)^{1+\frac{2\alpha}{r}}(\tau_0-t)^{\frac{q-q'}{qq'}}(1+\|\nabla u(t)\|_{L^2})^{1+\frac{2\alpha}{r}} \\ &\le (1+C)^{2+\frac{2\alpha}{r}}(T_{\max}-t)^{\frac{q-q'}{qq'}}(1+\|\nabla u(t)\|_{L^2})^{1+\frac{2\alpha}{r}}, \end{aligned}$$

and so

$$1+\|\nabla u(t)\|_{L^2} \ge \frac{1}{(1+C)^{1+\frac{r}{\alpha}}(T_{\max}-t)^{\frac{r(q-q')}{2\alpha qq'}}}.$$

Hence the result follows, since $t \in [0, T_{\max})$ is arbitrary and $\frac{r(q-q')}{2\alpha qq'} = \frac{1}{\alpha} - \frac{N-2}{4}$. □

Before proceeding further, we establish an immediate consequence of the above result concerning the blowing up of certain L^p norms of the solution.

COROLLARY 6.5.14. *Suppose $g(u) = \lambda|u|^\alpha u$ with $\lambda > 0$ and*

$$\frac{4}{N} \le \alpha < \frac{4}{N-2} \quad \left(\frac{4}{N} \le \alpha < \infty \text{ if } N = 1\right).$$

If $\varphi \in H^1(\mathbb{R}^N)$ is such that $T_{\max} < \infty$, then $\|u(t)\|_{L^p} \underset{t\uparrow T_{\max}}{\longrightarrow} \infty$ for all $p > \frac{N\alpha}{2}$. Moreover,

$$\|u(t)\|_{L^p} \ge \frac{\delta}{(T_{\max}-t)^{\frac{1}{\alpha}-\frac{N}{2p}}} \quad \text{for } 0<t<T_{\max} \text{ if } \frac{N\alpha}{2} < p \le \alpha+2 \tag{6.5.47}$$

and

$$\|u(t)\|_{L^p} \ge \frac{\delta}{(T_{\max}-t)^{\frac{4-(N-2)\alpha}{\alpha^2}(\frac{1}{2}-\frac{1}{p})}} \quad \text{for } 0<t<T_{\max} \text{ if } p \ge \alpha+2. \tag{6.5.48}$$

A similar estimate holds near $-T_{\min}$ if $T_{\min} < \infty$.

REMARK 6.5.15. Note that if $N \ge 3$ and $p > \frac{2N}{N-2}$, or if $N = 2$ and $p = \infty$, then it may happen that $\|u(t)\|_{L^p} = \infty$ for some (or all) $t \in (-T_{\min}, T_{\max})$. Clearly, this does not contradict the above estimates. Note, however, that $u \in L^q_{\text{loc}}((-T_{\min}, T_{\max}), W^{1,r}(\mathbb{R}^N))$ for every admissible pair (q, r), so that by Sobolev's embedding theorem, $\|u(t)\|_{L^p} < \infty$ for a.a. $t \in (-T_{\min}, T_{\max})$ provided $N \le 3$ or $N \ge 4$ and $p < \frac{2N}{N-4}$.

PROOF OF COROLLARY 6.5.14. Suppose first $\frac{N\alpha}{2} < p \le \alpha + 2$. By Gagliardo-Nirenberg's inequality,

$$\|u\|_{L^{\alpha+2}}^{\alpha+2} \le C\|\nabla u\|_{L^2}^{2-\mu}\|u\|_{L^p}^{\alpha+\mu} \quad \text{with } \mu = \frac{4p - 2N\alpha}{2N - (N-2)p}.$$

By conservation of energy and the above inequality, we obtain

$$\|\nabla u(t)\|_{L^2}^2 \le 2E(\varphi) + \frac{2\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \le C + C\|\nabla u(t)\|_{L^2}^{2-\mu}\|u(t)\|_{L^p}^{\alpha+\mu}$$

for all $0 < t < T_{\max}$. Since $\|\nabla u(t)\|_{L^2} \underset{t\uparrow T_{\max}}{\longrightarrow} \infty$, it follows that

$$\|\nabla u(t)\|_{L^2}^{\mu} \le C\|u(t)\|_{L^p}^{\alpha+\mu}.$$

By Theorem 6.5.13, this implies

$$\|u(t)\|_{L^p} \ge \frac{\varepsilon}{(T_{\max} - t)^{\frac{\mu}{\alpha+\mu}(\frac{1}{\alpha} - \frac{N-2}{4})}}.$$

Inequality (6.5.47) follows, since $\frac{\mu}{\alpha+\mu}(\frac{1}{\alpha} - \frac{N-2}{4}) = \frac{1}{\alpha} - \frac{N}{2p}$.

Suppose now $p \ge \alpha + 2$. It follows from Hölder's inequality that

$$\|u\|_{L^{\alpha+2}}^{\alpha+2} \le \|u\|_{L^p}^{\frac{\alpha p}{p-2}}\|u\|_{L^2}^{\frac{2(p-(\alpha+2))}{p-2}}.$$

Therefore, by conservation of charge and energy,

$$\|\nabla u(t)\|_{L^2}^2 \le 2E(\varphi) + \frac{2\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \le C + C\|u(t)\|_{L^p}^{\frac{\alpha p}{p-2}}\|\varphi\|_{L^2}^{\frac{2(p-(\alpha+2))}{p-2}}$$

for all $0 < t < T_{\max}$. The lower bound (6.5.48) now follows from Theorem 6.5.13 and the above inequality. □

REMARK 6.5.16. Theorem 6.5.13 and Corollary 6.5.14 give lower estimates of $\|\nabla u\|_{L^2}$ and $\|u\|_{L^p}$ near the blowup. They do not give any upper estimate. It is interesting to compare these results with the corresponding ones for the heat equation. If one considers the equation $u_t - \Delta u = |u|^{p-1}u$ with a Dirichlet boundary condition, then a simple argument (even simpler than the proof of Theorem 6.5.13) gives the lower estimate $\|u\|_{L^\infty} \ge (T_{\max} - t)^{-\frac{1}{\alpha}}$. If $\alpha < \frac{4}{N-2}$, then it is known that this is the actual blowup rate of the solutions (see [361, 126, 195, 352]). However for larger α's, some solutions blow up faster (see [196]). A lower estimate is obtained as well for $\|u\|_{L^p}$, $p > N\alpha/2$. In some cases it is known that $\|u\|_{L^p}$ also blows up for $p = N\alpha/2$ (see [362]) and that $\|u\|_{L^p}$ remains bounded for $p < N\alpha/2$ (see [113]).

REMARK 6.5.17. In the case $\alpha > 4/N$, one does not know the exact blowup rate of any blowing up solution. In addition, one does not know whether $\|u\|_{L^p}$ blows up for $2 < p \le N\alpha/2$. On the other hand, there is an upper estimate of integral form (see Merle [243]). More precisely, if $\varphi \in H^1(\mathbb{R}^N)$ and $x\varphi(x) \in L^2(\mathbb{R}^N)$, then

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2|u(t,x)|^2\,dx = 4N\alpha E(\varphi) - 2(N\alpha - 4)\int_{\mathbb{R}^N} |\nabla u(t,x)|^2\,dx$$
$$\le a - b\|\nabla u(t)\|_{L^2}^2 \quad \text{for some constants } a, b > 0.$$

Since $\|xu(t)\|_{L^2}^2 \geq 0$, this implies that

$$\int_0^{T_{\max}} \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \, dt < \infty .$$

Since

$$\int_0^{T_{\max}} \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \, dt = \int_0^{T_{\max}} (T_{\max} - t) \|\nabla u(t)\|_{L^2}^2 \, dt ,$$

it follows immediately from Hölder's inequality that

$$\int_0^{T_{\max}} \|\nabla u(t)\|_{L^2}^{\mu} \, dt < \infty \quad \text{for } 0 \leq \mu < 1.$$

If $\varphi \in H^1(\mathbb{R}^N)$ and φ is spherically symmetric, then one obtains the same estimate. Indeed, by using the fact that $\alpha > 4/N$, one can improve (6.5.42) to

$$\frac{d^2}{dt^2} V(t) \leq N\alpha E(\varphi) - (N\alpha - 4) \|\nabla u\|_{L^2}^2 ,$$

and the conclusion is the same.

REMARK 6.5.18. In the case $\alpha = 4/N$, then (6.5.43) becomes

$$\|\nabla u(t)\|_{L^2} \geq \frac{\delta}{(T_{\max} - t)^{\frac{1}{2}}}, \tag{6.5.49}$$

and (6.5.47) and (6.5.48) become

$$\|u(t)\|_{L^p} \geq \frac{\delta}{(T_{\max} - t)^{\frac{N}{2}(\frac{1}{2} - \frac{1}{p})}} \quad \text{for } p > 2. \tag{6.5.50}$$

In particular, $\|u\|_{L^p}$ blows up for $p > 2$. Since $\|u\|_{L^2}$ is constant, estimate (6.5.50) is optimal with respect to p. On the other hand, it is known that the blowup rate given by (6.5.49) and (6.5.50) is not always optimal, since some solutions blow up twice as fast (see Remark 6.7.3 and Bourgain and Wang [41]). Moreover, in space dimension $N = 1$, Perelman [297] has constructed a family of blowing up solutions for which

$$\left(\frac{\log |\log(T_{\max} - t)|}{T_{\max} - t} \right)^{-\frac{1}{4}} \|u(t)\|_{L^\infty} \underset{t \uparrow T_{\max}}{\longrightarrow} c > 0 ,$$

which is very close to but different from the lower estimate (6.5.50). Merle and Raphael [250] recently obtained the upper estimate

$$\|\nabla u(t)\|_{L^2}^2 \leq C \left(\frac{\log |\log(T_{\max} - t)|}{T_{\max} - t} \right)^{\frac{1}{2}}$$

for a certain large class of blowing up solutions, which is very close to the lower estimate (6.5.49). These results show in particular that at least two different blow up rates are actually achieved. This was established in space dimension $N = 1$, but there are strong indications that it might also hold in higher dimensions.

REMARK 6.5.19. There is an abundant literature devoted to the determination of the blowup rate by means of numerical computations. See Frisch, Sulem, and Sulem [114], Le Mesurier, Papanicolaou, Sulem, and Sulem [226, 228, 227], McLaughlin, Papanicolaou, Sulem, and Sulem [242], and Patera, Sulem, and Sulem

[293]. See also [336] for a survey of some of this literature. For a given equation, the computed blowup rates vary widely according to the authors. For example, in the case $\alpha = N = 2$, the rates (as of 1993) range from $(T_m - t)^{-\frac{1}{2}}$ to $(T_m - t)^{-\frac{3}{2}}$ (see in particular Table 1 of [336], p. 409). Furthermore, for a given author, the rate may vary from one paper to another. On the other hand, it seems that the "fast" blowup rate of the pseudoconformally self-similar solutions in the critical case (see Remark 6.7.3) was never reported numerically. This last point is usually interpreted as the exceptional character of this blowup rate. This does not seem to be correct, however, since it has been shown to have some "stability"; see Bourgain and Wang [41].

6.6. The Critical Case: Sharp Existence and Blowup Results

In this section we consider the model nonlinearity

$$g(u) = \lambda |u|^\alpha u\,, \tag{6.6.1}$$

where

$$\lambda > 0\,, \qquad \alpha = \frac{4}{N}\,. \tag{6.6.2}$$

We recall that the H^1 solutions of (4.1.1) are global and bounded in $H^1(\mathbb{R}^N)$ provided $\varphi \in H^1(\mathbb{R}^N)$ satisfies $\|\varphi\|_{L^2} < \delta$ for some $\delta > 0$ (see Remark 6.1.3). In fact, one can determine the optimal δ. Let R be the (unique) spherically symmetric, positive ground state of the elliptic equation

$$-\Delta R + R = |R|^{\frac{4}{N}} R \quad \text{in } \mathbb{R}^N \tag{6.6.3}$$

(see, for example, Definition 8.1.13 and Theorems 8.1.4, 8.1.5, and 8.1.6). Note that any ground state of (6.6.3) is of the form $e^{i\theta} R(x-y)$ for some $\theta \in \mathbb{R}$ and $y \in \mathbb{R}^N$. We have the following result of M. Weinstein [356].

THEOREM 6.6.1. *Assume* (6.6.1)–(6.6.2) *and let R be the spherically symmetric, positive ground state of* (6.6.3). *If $\varphi \in H^1(\mathbb{R}^N)$ is such that*

$$\lambda^{\frac{1}{\alpha}} \|\varphi\|_{L^2} < \|R\|_{L^2}\,,$$

then the maximal H^1 solution u of (4.1.1) *is global and* $\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} < \infty$.

REMARK 6.6.2. The condition $\|\varphi\|_{L^2} < \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$ is sharp, in the sense that for any $\rho > \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$ (in fact, even for $\rho = \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$, see Remark 6.7.3 below) there exists $\varphi \in H^1(\mathbb{R}^N)$ such that $\|\varphi\|_{L^2} = \rho$ and such that u blows up in finite time for both $t < 0$ and $t > 0$. Indeed, let $\psi(x) = R(\sqrt{\lambda} x)$, so that $\|\psi\|_{L^2} = \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$ and ψ is a solution of

$$-\Delta\psi + \lambda\psi = \lambda |\psi|^\alpha \psi\,.$$

It follows that $E(\psi) = 0$ (see formula (8.1.21)). Let $\rho > \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$, set $\gamma = \lambda^{\frac{1}{\alpha}} \rho / \|R\|_{L^2} > 1$, and consider $\varphi_\rho = \gamma\psi$. It follows that $\|\varphi_\rho\|_{L^2} = \rho$ and

$$E(\varphi_\rho) = \gamma^{\alpha+2} E(\psi) - \frac{\gamma^{\alpha+2} - \gamma^2}{2} \|\nabla\psi\|_{L^2}^2 < 0\,,$$

and so the corresponding solution u_ρ of (4.1.1) blows up in finite time (see Theorem 6.5.4). Note that there are certain extensions of Theorem 6.6.1 to nonlinearities of the form $g(u) = \lambda|u|^\alpha u$ with $\alpha > 4/N$; see Bégout [19].

REMARK 6.6.3. In space dimension $N = 1$, an elementary calculation shows that

$$R(x) = \frac{3^{\frac{1}{4}}}{\sqrt{\cosh(2x)}}\,, \tag{6.6.4}$$

in particular,

$$\|R\|_{L^2}^2 = \pi\sqrt{3}\,.$$

Therefore, it follows from Theorem 6.6.1 that if $\varphi \in H^1(\mathbb{R}^N)$ is such that $\lambda^{\frac{1}{\alpha}}\|\varphi\|_{L^2} < 3^{\frac{1}{4}}\sqrt{\pi}$, then the solution u of (4.1.1) is global and $\sup_{t\in\mathbb{R}}\|u(t)\|_{H^1} < \infty$.

PROOF OF THEOREM 6.6.1. By conservation of charge and energy and by Lemma 8.4.2 below, we have for all $t \in (-T_{\min}, T_{\max})$,

$$\begin{aligned}
\frac{1}{2}\|\nabla u(t)\|_{L^2}^2 &\le E(\varphi) + \frac{\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \\
&\le E(\varphi) + \frac{\lambda}{2\|R\|_{L^2}^{\alpha}}\|\nabla u(t)\|_{L^2}^2\|u(t)\|_{L^2}^{\alpha} \\
&\le E(\varphi) + \frac{\lambda\|\varphi\|_{L^2}^{\alpha}}{2\|R\|_{L^2}^{\alpha}}\|\nabla u(t)\|_{L^2}^2\,,
\end{aligned}$$

and so

$$\frac{1}{2}\Big(1 - \frac{\lambda\|\varphi\|_{L^2}^{\alpha}}{\|R\|_{L^2}^{\alpha}}\Big)\|\nabla u(t)\|_{L^2}^2 \le E(\varphi)\,.$$

Hence the result follows, by using the blowup alternative. □

REMARK 6.6.4. Assume $\lambda > 0$ and let d be the supremum of the μ's such that $\|\varphi\|_{L^2} < \mu$ implies global existence of the corresponding L^2 solution (see Remark 4.7.5). Then it is clear that $d \le \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$, where R is as in Theorem 6.6.1. This follows from Remark 6.6.2 and from the regularity property (iii) of Theorem 4.7.1. Whether or not $d = \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$ is an open question. However, one can show that if $\varphi \in L^2(\mathbb{R}^N)$ satisfies $\|\varphi\|_{L^2} < \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$, and if, in addition, $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$, then $T_{\min} = T_{\max} = \infty$ and $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ for every admissible pair (q,r). (Here, $T_{\max}, T_{\min}$ are the existence times given in Theorem 4.7.1.) Indeed, consider a sequence $(\varphi_n)_{n\ge 0} \subset H^1(\mathbb{R}^N)$ with $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N)$ and $x\varphi_n(x)$ bounded in $L^2(\mathbb{R}^N)$. The corresponding solutions u_n satisfy $|\cdot|u_n(\cdot) \in C(\mathbb{R}, L^2(\mathbb{R}^N))$ (see Lemma 6.5.2), and from the pseudoconformal conservation law we see that (see formula (7.2.8))

$$8t^2E(v_n(t)) = \|x\varphi_n\|_{L^2}^2 \quad \text{for all } t \in \mathbb{R},$$

where $v_n(t) = e^{-\frac{i|x|^2}{4t}}u_n(t)$. In particular, $\|v_n(t)\|_{L^2} = \|u_n(t)\|_{L^2} = \|\varphi_n\|_{L^2}$, so that there exists $\varepsilon > 0$ such that $\|v_n(t)\|_{L^2} \le \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2} - \varepsilon$ for n large enough. It follows that there exists C such that $\|\nabla v_n(t)\|_{L^2}^2 \le CE(v_n(t))$ for all $t \in \mathbb{R}$ (see the proof of Theorem 6.6.1). By Lemma 8.4.2, this implies that

$$\|v_n(t)\|_{L^{\alpha+2}}^{\alpha+2} \le CE(v_n(t))\|\varphi_n\|_{L^2}^{\frac{4}{N}} \le \frac{C}{t^2} \quad \text{for all } t \in \mathbb{R}.$$

We conclude as in Remark 4.7.4 above.

Theorem 4.7.1 has an immediate application to the study of blowup solutions.

THEOREM 6.6.5. *Assume* (6.6.1)–(6.6.2). *Let* $\varphi \in L^2(\mathbb{R}^N)$ *and let* u *be the corresponding maximal* L^2 *solution of* (4.1.1) *given by Theorem* 4.7.1. *If* $T_{\max} < \infty$ *and if* $(t_n)_{n\geq 0}$ *is any sequence such that* $t_n \uparrow T_{\max}$, *then* $u(t_n)$ *does not have any strong limit in* $L^2(\mathbb{R}^N)$. *A similar statement holds for* $T_{\min}$.

PROOF. Assume that $u(t_n) \to w$ in $L^2(\mathbb{R}^N)$. By continuous dependence (see Theorem 4.7.1),

$$T_{\max}(u(t_n)) \geq \frac{1}{2} T_{\max}(w) > 0 \quad \text{for } n \text{ large enough.}$$

This implies that

$$T_{\max}(\varphi) \geq t_n + \frac{1}{2} T_{\max}(w) \quad \text{for } n \text{ large.}$$

This is absurd, since $t_n \to T_{\max}$. □

In fact, one can prove a stronger result which implies the above theorem (see [71]).

THEOREM 6.6.6. *Assume* (6.6.1)–(6.6.2). *There exists* $\rho > 0$ *with the following property. Let* $\varphi \in L^2(\mathbb{R}^N)$ *and let* u *be the corresponding maximal* L^2 *solution of* (4.1.1) *given by Theorem* 4.7.1. *If* $T_{\max} < \infty$ *and if* $\mathcal{L}$ *is the set of weak* L^2 *limit points of* $u(t)$ *as* $t \uparrow T_{\max}$, *then* $\|w\|_{L^2}^2 \leq \|\varphi\|_{L^2}^2 - \rho^2$ *for all* $w \in \mathcal{L}$. *A similar statement holds for* $T_{\min}$.

PROOF. It follows from Step 1 of the proof of Theorem 4.7.1 (see in particular (4.7.4)) that there exists $\delta > 0$ such that if

$$\|\mathcal{T}(\cdot)\phi\|_{L^{\alpha+2}((0,\tau),L^{\alpha+2})} < \delta\,,$$

then $T_{\max}(\phi) > \tau$. Letting $\phi = u(t)$, we deduce that

$$\|\mathcal{T}(\cdot)u(t)\|_{L^{\alpha+2}((0,T_{\max}-t),L^{\alpha+2})} \geq \delta \quad \text{for all } t \in [0, T_{\max}).$$

Therefore, given any $\psi \in L^2(\mathbb{R}^N)$ and any $t \in [0, T_{\max})$,

$$\begin{aligned}\delta &\leq \|\mathcal{T}(\cdot)(u(t) - \psi)\|_{L^{\alpha+2}((0,T_{\max}-t),L^{\alpha+2})} + \|\mathcal{T}(\cdot)\psi\|_{L^{\alpha+2}((0,T_{\max}-t),L^{\alpha+2})} \\ &\leq c\|u(t) - \psi\|_{L^2} + \|\mathcal{T}(\cdot)\psi\|_{L^{\alpha+2}((0,T_{\max}-t),L^{\alpha+2})}\,,\end{aligned}$$

where c is the constant in the corresponding Strichartz estimate. Since

$$\|\mathcal{T}(\cdot)\psi\|_{L^{\alpha+2}((0,T_{\max}-t),L^{\alpha+2})} \underset{t\uparrow T_{\max}}{\longrightarrow} 0\,,$$

it follows that

$$\liminf_{t\uparrow T_{\max}} \|u(t) - \psi\|_{L^2} \geq \frac{\delta}{c}\,.$$

Therefore, if $u(t_n) \rightharpoonup \psi$ for some sequence $t_n \uparrow T_{\max}$, then

$$\begin{aligned}\frac{\delta^2}{c^2} &\le \liminf_{n\to\infty} \|u(t_n) - \psi\|_{L^2}^2 \\ &= \liminf_{n\to\infty} (u(t_n) - \psi, u(t_n) - \psi)_{L^2} \\ &= \liminf_{n\to\infty} \left(\|u(t_n)\|_{L^2}^2 + \|\psi\|_{L^2}^2 - 2(u(t_n), \psi)_{L^2}\right) \\ &= \|\varphi\|_{L^2}^2 - \|\psi\|_{L^2}^2 ,\end{aligned}$$

and the result follows. □

For H^1 spherically symmetric blowup solutions in dimension $N \ge 2$, there is a minimal amount of concentration of the L^2 norm at the origin, as the following result shows (see Merle and Tsutsumi [251], Y. Tsutsumi [344], and M. Weinstein [360]).

THEOREM 6.6.7. *Assume* (6.6.1)–(6.6.2). *Suppose further that* $N \ge 2$ *and let* R *be the spherically symmetric, positive ground state of equation* (6.6.3). *Let* $\gamma : (0,\infty) \to (0,\infty)$ *be any function such that* $\gamma(s) \underset{s\downarrow 0}{\longrightarrow} \infty$ *and* $s^{\frac{1}{2}}\gamma(s) \underset{s\downarrow 0}{\longrightarrow} 0$. *Finally, let* $\varphi \in H^1(\mathbb{R}^N)$ *and let* u *be the maximal* H^1 *solution of* (4.1.1). *If* φ *is spherically symmetric and such that* $T_{\max} < \infty$, *then*

$$\liminf_{t\uparrow T_{\max}} \|u(t)\|_{L^2(\Omega_t)} \ge \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2} ,$$

where $\Omega_t = \{x \in \mathbb{R}^N : |x| < |T_{\max} - t|^{\frac{1}{2}}\gamma(T_{\max} - t)\}$. *A similar statement holds for* $T_{\min}$.

As a consequence of Theorem 6.6.7, we have the following result.

COROLLARY 6.6.8. *Under the assumptions of Theorem* 6.6.7, *if* $T_{\max} < \infty$ *and if* $\mathcal{L}$ *is the set of weak* L^2 *limit points of* $u(t)$ *as* $t \uparrow T_{\max}$, *then*

$$\|w\|_{L^2}^2 \le \|\varphi\|_{L^2}^2 - \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 \quad \textit{for all } w \in \mathcal{L}.$$

A similar statement holds for $T_{\min}$.

REMARK 6.6.9. Note that the minimal loss of L^2 norm given by Corollary 6.6.8 is optimal. Indeed, there exist solutions that blow up in finite time, and for which $\|\varphi\|_{L^2}^2 = \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2$ (see Remark 6.7.3). Corollary 6.6.8 improves the conclusion of Theorem 6.6.6 for H^1 spherically symmetric solutions, in the sense that it gives the optimal value of ρ.

PROOF OF COROLLARY 6.6.8. Assume $t_n \uparrow T_{\max}$ and $u(t_n) \rightharpoonup w$ in $L^2(\mathbb{R}^N)$. Given $\varepsilon > 0$, $u(t_n) \rightharpoonup w$ in $L^2(\{|x| > \varepsilon\})$, and so

$$\|w\|_{L^2(\{|x|>\varepsilon\})}^2 \le \liminf_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|>\varepsilon\})}^2 .$$

On the other hand,

$$\begin{aligned}\|u(t_n)\|_{L^2(\{|x|>\varepsilon\})}^2 &= \|u(t_n)\|_{L^2}^2 - \|u(t_n)\|_{L^2(\{|x|<\varepsilon\})}^2 \\ &= \|\varphi\|_{L^2}^2 - \|u(t_n)\|_{L^2(\{|x|<\varepsilon\})}^2 \\ &\le \|\varphi\|_{L^2}^2 - \|u(t_n)\|_{L^2(\Omega_t)}^2 ,\end{aligned}$$

and the result follows from Theorem 6.6.7. □

PROOF OF THEOREM 6.6.7. Set $\omega(t) = \|\nabla u(t)\|_{L^2}^{-1}$ so that $\omega(t) \underset{t\uparrow T_{\max}}{\longrightarrow} 0$. We claim that

$$\liminf_{t\uparrow T_{\max}} \|u(t)\|_{L^2(\{|x|<\omega(t)\gamma(T_{\max}-t)\})} \geq \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2} . \tag{6.6.5}$$

The result follows from (6.6.5) and (6.5.49), since γ is arbitrary. We prove claim (6.6.5) by contradiction, so we assume that there exists $t_n \uparrow T_{\max}$ such that

$$\lim_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|<\omega(t_n)\gamma(T_{\max}-t_n)\})} < \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2} . \tag{6.6.6}$$

We set

$$v_n(x) = \omega(t_n)^{\frac{N}{2}} u(t_n, \omega(t_n)x) ,$$

so that

$$\begin{cases} \|v_n\|_{L^2} = \|u(t_n)\|_{L^2} = \|\varphi\|_{L^2}, \\ \|\nabla v_n\|_{L^2} = 1, \\ E(v_n) = \omega(t_n)^2 E(u(t_n)) = \omega(t_n)^2 E(\varphi) \underset{n\to\infty}{\longrightarrow} 0. \end{cases} \tag{6.6.7}$$

It follows in particular that

$$E(v_n) = \frac{1}{2} - \frac{\lambda}{\alpha+2} \|v_n\|_{L^{\alpha+2}}^{\alpha+2} ,$$

so that

$$\|v_n\|_{L^{\alpha+2}}^{\alpha+2} \underset{n\to\infty}{\longrightarrow} \frac{\alpha+2}{2\lambda} \neq 0 . \tag{6.6.8}$$

By (6.6.7), $(v_n)_{n\geq 0}$ is a bounded sequence in $H^1(\mathbb{R}^N)$, so that there exist a subsequence, which we still denote by $(v_n)_{n\geq 0}$, and $w \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Since the v_n's are spherically symmetric, we deduce that $v_n \underset{n\to\infty}{\longrightarrow} w$ in $L^{\alpha+2}(\mathbb{R}^N)$ (see Proposition 1.7.1). In particular, $E(w) \leq 0$, and by (6.6.8), $w \neq 0$. By applying Lemma 8.4.2 below, we obtain

$$\lambda^{\frac{1}{\alpha}} \|w\|_{L^2} \geq \|R\|_{L^2} . \tag{6.6.9}$$

Given $M > 0$,

$$\begin{aligned} \|w\|_{L^2(\{|x|<M\})} &= \lim_{n\to\infty} \|v_n\|_{L^2(\{|x|<M\})} \\ &= \lim_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|<M\omega(t_n)\})} \\ &\leq \liminf_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|<\omega(t_n)\gamma(T_{\max}-t_n)\})} , \end{aligned}$$

since $\gamma(s) \to \infty$ as $s \downarrow 0$. Since M is arbitrary, by applying (6.6.9) we obtain

$$\liminf_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|<\omega(t_n)\gamma(T_{\max}-t_n)\})} \geq \|w\|_{L^2} \geq \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2} ,$$

which contradicts (6.6.6). This completes the proof. □

In fact, Corollary 6.6.8 can be generalized to nonradial solutions (and also to the space dimension $N = 1$). More precisely, we have the following result.

THEOREM 6.6.10. *Assume (6.6.1)–(6.6.2). Let R be the spherically symmetric, positive ground state of equation (6.6.3). Finally, let $\varphi \in H^1(\mathbb{R}^N)$ and let u be the maximal H^1 solution of (4.1.1). If $T_{\max} < \infty$ and if $\mathcal{L}$ is the set of weak L^2 limit points of $u(t)$ as $t \uparrow T_{\max}$, then*

$$\|w\|_{L^2}^2 \le \|\varphi\|_{L^2}^2 - \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 \quad \text{for all } w \in \mathcal{L}.$$

A similar statement holds for $T_{\min}$.

By conservation of energy and the blowup of $\|\nabla u(t)\|_{L^2}$, Theorem 6.6.10 is an immediate consequence of the following proposition.

PROPOSITION 6.6.11. *Let $(u_n)_{n\ge 0} \subset H^1(\mathbb{R}^N) \setminus \{0\}$ and $u \in L^2(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. If furthermore $\|\nabla u_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} \infty$ and*

$$\limsup_{n\to\infty} \frac{E(u_n)}{\|\nabla u_n\|_{L^2}^2} \le 0\,,$$

then $\|u\|_{L^2}^2 \le \liminf_{n\to\infty} \|u_n\|_{L^2}^2 - \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2$.

We will use the following lemma.

LEMMA 6.6.12. *If $(u_n)_{n\ge 0} \subset H^1(\mathbb{R}^N)$ is such that*

(i) $\|u_n\|_{L^2}^2 = a > 0$,

(ii) $0 < \inf_{n\ge 0} \|\nabla u_n\|_{L^2} \le \sup_{n\ge 0} \|\nabla u_n\|_{L^2} < \infty$,

(iii) $\limsup_{n\to\infty} E(u_n) \le 0$,

then $\mu \ge \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2$, where $\mu = \mu((u_n)_{n\ge 0})$ is defined by (1.7.6).

PROOF OF PROPOSITION 6.6.11. (Assuming Lemma 6.6.12.) Let $(u_n)_{n\ge 0}$ be as in the statement of Proposition 6.6.11. Set $a = \liminf_{n\to\infty} \|u_n\|_{L^2}^2$. By considering a subsequence, we may assume that

$$\|u_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} a\,.$$

Set $\omega_n = \|\nabla u_n\|_{L^2}^{-1}$ and define $v_n(x) = \omega_n^{\frac{2}{N}} u_n(\omega_n x)$. It follows that $\|v_n\|_{L^2}^2 = \|u_n\|_{L^2}^2$, $\|\nabla v_n\|_{L^2}^2 = 1$, and

$$\limsup_{n\to\infty} E(v_n) = \limsup_{n\to\infty} \frac{E(u_n)}{\|\nabla u_n\|_{L^2}^2} \le 0\,.$$

We first show that $a > 0$. Indeed,

$$E(v_n) \ge \frac{1}{2}\left(1 - \lambda \frac{\|v_n\|_{L^2}^\alpha}{\|R\|_{L^2}^\alpha}\right) \|\nabla v_n\|_{L^2}^2\,,$$

by Lemma 8.4.2, which implies that $\lambda \|v_n\|_{L^2}^\alpha \ge \|R\|_{L^2}^\alpha$, and so $a \ge \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2$. We now set

$$w_n = \frac{\sqrt{a}}{\|v_n\|_{L^2}} v_n$$

so that $(w_n)_{n\geq 0}$ satisfies the assumptions of Lemma 6.6.12. Therefore,

$$\mu((w_n)_{n\geq 0}) \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 .$$

We apply Lemma 1.7.5 to the sequence $(w_n)_{n\geq 0}$. Given $\varepsilon > 0$, it follows from the above inequality that there exists T such that

$$\rho(w_n, T) \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 - \varepsilon \quad \text{for } n \text{ large enough.}$$

Therefore, setting $y_n = y(w_n, T)$ with $y(\cdot,\cdot)$ defined by Lemma 1.7.4(ii),

$$\int_{\{|x-y_n|<T\}} |w_n(x)|^2 \, dx \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 - \varepsilon \quad \text{for } n \text{ large,}$$

which means that

$$\frac{a}{\|u_n\|_{L^2}^2} \int_{\{|x-z_n|\leq t_n\}} |u_n(x)|^2 \, dx \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 - \varepsilon$$

with $z_n = \omega_n y_n$ and $t_n = \omega_n T$. Note that $t_n \underset{n\to\infty}{\longrightarrow} 0$. By possibly extracting a subsequence, we may assume that either $|z_n| \underset{n\to\infty}{\longrightarrow} \infty$ or $z_n \underset{n\to\infty}{\longrightarrow} z$ for some $z \in \mathbb{R}^N$. In the first case, consider $M > 0$. Since $u_n \rightharpoonup u$ in $L^2(\{|x| \leq M\})$, we deduce that

$$\begin{aligned}
\|u\|_{L^2(\{|x|\leq M\})}^2 &\leq \liminf_{n\to\infty} \|u_n\|_{L^2(\{|x|\leq M\})}^2 \\
&= \liminf_{n\to\infty} \left\{ \|u_n\|_{L^2}^2 - \|u_n\|_{L^2(\{|x|\geq M\})}^2 \right\} \\
&= a - \limsup_{n\to\infty} \|u_n\|_{L^2(\{|x|\geq M\})}^2 \\
&\leq a - \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 + \varepsilon .
\end{aligned}$$

The result follows by letting $M \uparrow \infty$, then $\varepsilon \downarrow 0$. In the second case, consider $\delta > 0$. Since $u_n \rightharpoonup u$ in $L^2(\{|x| \geq \delta\})$, we see that

$$\begin{aligned}
\|u\|_{L^2(\{|x-z|\geq \delta\})}^2 &\leq \liminf_{n\to\infty} \|u_n\|_{L^2(\{|x-z|\geq \delta\})}^2 \\
&= \liminf_{n\to\infty} \left\{ \|u_n\|_{L^2}^2 - \|u_n\|_{L^2(\{|x-z|\leq \delta\})}^2 \right\} \\
&= a - \limsup_{n\to\infty} \|u_n\|_{L^2(\{|x-z|\leq \delta\})}^2 \\
&\leq a - \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 + \varepsilon .
\end{aligned}$$

The result follows by letting $\delta \downarrow 0$, then $\varepsilon \downarrow 0$. This completes the proof. □

PROOF OF LEMMA 6.6.12. We claim that there exists $\delta > 0$, depending only on N and λ with the following property. If $(u_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$ is such that

$$\|u_n\|_{L^2}^2 = a > 0 , \tag{6.6.10}$$

$$0 < \inf_{n\geq 0} \|\nabla u_n\|_{L^2} \leq \sup_{n\geq 0} \|\nabla u_n\|_{L^2}^2 < \infty , \tag{6.6.11}$$

$$\limsup_{n\to\infty} E(u_n) \leq 0 , \tag{6.6.12}$$

$$\mu((u_n)_{n\geq 0}) < \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 , \tag{6.6.13}$$

then $a > \mu((u_n)_{n\geq 0})$, and there exists a sequence $(\widetilde{u}_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$ satisfying (6.6.11), (6.6.12), and (6.6.13), and such that $\|\widetilde{u}_n\|_{L^2}^2 = a - \beta$ for some $\beta \geq \delta$. The result follows, since if $(u_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$ is as in the statement of the lemma, and if $\mu((u_n)_{n\geq 0}) < \lambda^{-\frac{2}{\alpha}}\|R\|_{L^2}^2$, then we may apply the claim k times to obtain $a - k\delta > \mu((u_n)_{n\geq 0})$, which is absurd for k large. Therefore, we need only prove the claim, and we consider $(u_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$ satisfying (6.6.10)–(6.6.13). The property $a > \mu((u_n)_{n\geq 0})$ follows from (6.6.11), (6.6.12), (6.6.13), and Lemma 8.4.2. Next, we apply Lemma 1.7.5, then Proposition 1.7.6(iii) to the sequence $(u_n)_{n\geq 0}$, and we consider the corresponding sequences $(v_k)_{k\geq 0}$ and $(w_k)_{k\geq 0}$. We set

$$\delta = \left(\frac{\alpha+2}{2K}\right)^{\frac{2}{\alpha}} > 0, \tag{6.6.14}$$

where the constant K is given by (1.7.17). We first show that

$$\mu := \mu((u_n)_{n\geq 0}) \geq \delta, \tag{6.6.15}$$

where δ is defined by (6.6.14). Indeed, it follows from (1.7.17) and (6.6.14) that

$$E(u_{n_k}) \geq \frac{1}{2}\left(1 - \left(\frac{\rho(u_{n_k}, t_k)}{\delta}\right)^{\frac{\alpha}{2}}\right)\int |\nabla u_{n_k}|^2 - \frac{K}{(\alpha+2)t_k^2}\rho(u_{n_k}, t_k)^{\frac{\alpha}{2}}.$$

Assuming by contradiction $\mu < \delta$, we obtain by letting $k \to \infty$ and applying Lemma 1.7.5(ii) and (6.6.11),

$$\limsup_{n\to\infty} E(u_n) \geq \frac{1}{2}\left(1 - \left(\frac{\mu}{\delta}\right)^{\frac{\alpha}{2}}\right)\inf_{n\geq 0}\int |\nabla u_n|^2 > 0,$$

which is absurd. Next, since $|w_k| \leq |u_{n_k}|$ by (1.7.12), it is not difficult to deduce from Lemma 1.7.5(ii) and (6.6.13) that

$$\mu((w_k)_{k\geq 0}) \leq \mu < \lambda^{-\frac{2}{\alpha}}\|R\|_{L^2}^2. \tag{6.6.16}$$

Also, it follows from Lemma 8.4.2, (6.6.13), and (1.7.14), that there exists $\sigma > 0$ such that

$$E(v_k) \geq \sigma\|\nabla v_k\|_{L^2}^2 \quad \text{for } k \text{ large}. \tag{6.6.17}$$

On the other hand, it follows from (1.7.15) and (1.7.16) that

$$\liminf_{k\to\infty}\{E(u_{n_k}) - E(v_k) - E(w_k)\} \geq 0. \tag{6.6.18}$$

Inequalities (6.6.12), (6.6.17), and (6.6.18) imply that

$$\limsup_{k\to\infty} E(w_k) \leq 0. \tag{6.6.19}$$

Next, we deduce from (1.7.13) and (6.6.11) that

$$\|\nabla w_k\|_{L^2} \leq C\|u_{n_k}\|_{H^1} \leq C. \tag{6.6.20}$$

Finally, we show that

$$\liminf_{k\to\infty}\|\nabla w_k\|_{L^2} > 0. \tag{6.6.21}$$

To prove this, we argue by contradiction and we assume that there exists a subsequence, which we still denote by $(w_k)_{k\geq 0}$, such that $\|\nabla w_k\|_{L^2} \to 0$ as $k \to \infty$. It follows that $E(w_k) \to 0$ as $k \to \infty$, so that (6.6.12), (6.6.18), and (6.6.17) now imply that $\|\nabla v_k\|_{L^2} \to 0$ as $k \to \infty$. Using (1.7.16), we deduce that $\|u_{n_k}\|_{L^{\alpha+2}} \to 0$, so that, by (6.6.11),

$$\limsup_{n\to\infty} E(u_n) \geq \frac{1}{2} \liminf_{n\to\infty} \|\nabla u_n\|_{L^2}^2 > 0\,,$$

which contradicts (6.6.12). Hence we have proved (6.6.21). Setting

$$\widetilde{u}_k = \frac{\sqrt{a-\mu}}{\|w_k\|_{L^2}} w_k\,,$$

we see that $0 < \|\widetilde{u}_k\|_{L^2}^2 = a - \mu \leq a - \delta$, and we deduce from (1.7.14), (6.6.20), (6.6.21), (6.6.19), and (6.6.16) that the sequence $(\widetilde{u}_k)_{k\geq 0}$ satisfies estimates (6.6.11), (6.6.12), and (6.6.13). This completes the proof. □

REMARK 6.6.13. For more information on the blowup in the critical case, see the series of papers of Nawa [264, 265, 266, 267, 268, 269, 270].

6.7. The Pseudoconformal Transformation and Applications

In this section we consider the model nonlinearity

$$g(u) = \lambda |u|^\alpha u, \tag{6.7.1}$$

where

$$\lambda \in \mathbb{R}, \qquad \alpha = \frac{4}{N}. \tag{6.7.2}$$

In this case, the pseudoconformal conservation law, introduced by Ginibre and Velo [133], becomes an exact conservation law. This conservation law is associated to a group of transformations which leaves invariant the set of solutions of (4.1.1) (see Ginibre and Velo [139]). We describe below this group of transformations (the pseudoconformal transformation).

It will be convenient to use the Hilbert space

$$\Sigma = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx) = \{u \in H^1(\mathbb{R}^N) : |\cdot|u(\cdot) \in L^2(\mathbb{R}^N)\} \tag{6.7.3}$$

equipped with the norm

$$\|u\|_\Sigma^2 = \|u\|_{H^1}^2 + \|xu\|_{L^2}^2\,. \tag{6.7.4}$$

Let now $b \in \mathbb{R}$. Given $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, we define the conjugate variables $(s,y) \in \mathbb{R} \times \mathbb{R}^N$ by

$$s = \frac{t}{1+bt}\,, \quad y = \frac{x}{1+bt}\,, \quad \text{or equivalently} \quad t = \frac{s}{1-bs}\,, \quad x = \frac{y}{1-bs}\,.$$

Given u defined on $(-S_1, S_2) \times \mathbb{R}^N$ with $0 \leq S_1, S_2 \leq \infty$, we set

$$T_1 = \begin{cases} \infty & \text{if } bS_1 \leq -1 \\ \frac{S_1}{1+bS_1} & \text{if } bS_1 > -1, \end{cases} \qquad T_2 = \begin{cases} \infty & \text{if } bS_2 \geq 1 \\ \frac{S_2}{1-bS_2} & \text{if } bS_2 < 1. \end{cases} \tag{6.7.5}$$

We define u_b on $(-T_1, T_2)$ by

$$u_b(t,x) = (1+bt)^{-\frac{N}{2}} e^{i\frac{b|x|^2}{4(1+bt)}} u\left(\frac{t}{1+bt}, \frac{x}{1+bt}\right),$$

or equivalently

$$u_b(t,x) = (1-bs)^{\frac{N}{2}} e^{i\frac{b|y|^2}{4(1-bs)}} u(s,y)\,. \tag{6.7.6}$$

Note that

$$\|u_b(t)\|_{L^2} = \|u(s)\|_{L^2}\,, \tag{6.7.7}$$

and, more generally,

$$\|u_b(t)\|_{L^{\beta+2}} = (1-bs)^{\frac{N\beta}{2(\beta+2)}} \|u(s)\|_{L^{\beta+2}} \quad \text{if } \beta \ge 0. \tag{6.7.8}$$

In particular,

$$\|u_b(t)\|_{L^{\alpha+2}} = (1-bs)^{\frac{2}{\alpha+2}} \|u(s)\|_{L^{\alpha+2}} \tag{6.7.9}$$

so that if $bs_1 > -1$ and $bs_2 < 1$, then

$$\|u_b\|_{L^{\alpha+2}((-t_1,t_2)L^{\alpha+2})} = \|u\|_{L^{\alpha+2}((-s_1,s_2)L^{\alpha+2})} \tag{6.7.10}$$

with $t_1 = \frac{s_1}{1+bs_1}$ and $t_2 = \frac{s_2}{1-bs_2}$. Next, if $u \in C((-S_1,S_2),\Sigma)$, then it is clear that $u_b \in C((-T_1,T_2),\Sigma)$. In addition,

$$\|xu_b(t)\|_{L^2} = (1-bs)^{-1}\|yu(s)\|_{L^2}\,, \tag{6.7.11}$$

$$\|\nabla u_b(t)\|_{L^2} = \frac{1}{2}\|(-by + 2i(1-bs)\nabla)u(s)\|_{L^2}\,, \tag{6.7.12}$$

$$\|\nabla u(s)\|_{L^2} = \frac{1}{2}\|(bx + 2i(1+bt)\nabla)u_b(t)\|_{L^2}\,. \tag{6.7.13}$$

The interest of the above transformation lies in the following result.

THEOREM 6.7.1. *Suppose $u \in C((-S_1,S_2),L^2(\mathbb{R}^N)) \cap L^{\alpha+2}_{\text{loc}}((-S_1,S_2),L^{\alpha+2}(\mathbb{R}^N))$ is a solution of* (4.1.1) *(see Theorem* 4.7.1*). Let $b \in \mathbb{R}$, let T_1, T_2 be defined by* (6.7.5)*, and let u_b be defined by* (6.7.6)*. It follows that*

$$u_b \in C\big((-T_1,T_2),L^2(\mathbb{R}^N)\big) \cap L^{\alpha+2}_{\text{loc}}\big((-T_1,T_2),L^{\alpha+2}(\mathbb{R}^N)\big)$$

is also a solution of (4.1.1)*. If, in addition, $u \in C((-S_1,S_2),\Sigma)$, then $u_b \in C((-T_1,T_2),\Sigma)$.*

PROOF. It is clear that $u_b \in C((-T_1,T_2),L^2(\mathbb{R}^N))$. In addition, it follows from (6.7.10) that $u_b \in L^{\alpha+2}_{\text{loc}}((-T_1,T_2),L^{\alpha+2}(\mathbb{R}^N))$. Furthermore, one shows that if $0 \le S_1, S_2 < \infty$ and if $bS_1 > -1$ and $bS_2 < 1$, then the mapping $u \mapsto u_b$ is continuous $C([-S_1,S_2],L^2(\mathbb{R}^N)) \cap L^{\alpha+2}((-S_1,S_2),L^{\alpha+2}(\mathbb{R}^N)) \to C([-T_1,T_2],L^2(\mathbb{R}^N)) \cap L^{\alpha+2}((-T_1,T_2),L^{\alpha+2}(\mathbb{R}^N))$.

Let now $u \in C([-S_1,S_2],L^2(\mathbb{R}^N)) \cap L^{\alpha+2}((-S_1,S_2),L^{\alpha+2}(\mathbb{R}^N))$ be a solution of (4.1.1), with S_1 and S_2 as above. Let $\varphi = u(0)$. We have in particular $T_{\min}(\varphi) > S_1$ and $T_{\max}(\varphi) > S_2$. Consider $(\varphi_n)_{n\ge0} \subset H^2(\mathbb{R}^N)$ such that $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N)$. By continuous dependence (Theorem 4.7.1(v)), $T_{\min}(\varphi_n) > S_1$ and $T_{\max}(\varphi_n) > S_2$ for n large enough. We denote by u^n the corresponding solutions of (4.1.1). We first observe that $u^n \in C((-S_1,S_2),H^1(\mathbb{R}^N))$ by Theorem 4.7.1(iii). Applying then Remark 5.3.3, we deduce that $u^n \in C((-S_1,S_2),H^2(\mathbb{R}^N))$; i.e., u is an H^2 solution. It follows that u satisfies equation (4.1.1) a.e. on $(-S_1,S_2)\times\mathbb{R}^N$. A tedious, but straightforward, calculation shows that $(u^n)_b$ satisfies (4.1.1) a.e. on $(-T_1,T_2)\times\mathbb{R}^N$.

The conclusion follows from continuous dependence (Theorem 4.7.1(v)) and from the continuity property mentioned above. □

REMARK 6.7.2. Note that the pseudoconformal transformation preserves both the space $L^2(\mathbb{R}^N)$ and the space Σ. On the other hand, it *does not* preserve the space $H^1(\mathbb{R}^N)$.

REMARK 6.7.3. The pseudoconformal transformation has a simple application (see M. Weinstein [359]), which yields interesting information on the blowup. Assume for simplicity that $\lambda = 1$ and let ψ be a nontrivial solution of (6.6.3). (Note that $\psi(x)$ has exponential decay as $|x| \to \infty$; see, for example, Theorem 8.1.1.) It follows that $u(t,x) = e^{it}\psi(x)$ is the solution of (4.1.1) with $\varphi = \psi$, and that $T_{\max}(\varphi) = T_{\min}(\varphi) = +\infty$. We set $v(t,x) = u_{-1}(t,x)$; i.e.,

$$v(t,x) = (1-t)^{-\frac{N}{2}} e^{-i\frac{|x|^2}{4(1-t)}} e^{i\frac{t}{1-t}} \psi\Big(\frac{x}{1-t}\Big) \quad \text{for } x \in \mathbb{R}^N,\ t < 1. \tag{6.7.14}$$

Therefore,

$$\|v(t)\|_{L^p} = (1-t)^{-\frac{N(p-2)}{2p}} \|\psi\|_{L^p} \quad \text{for all } t < 1,\ 1 \le p \le \infty. \tag{6.7.15}$$

Thus, $\|v\|_{L^{\alpha+2}((0,1),L^{\alpha+2})} = +\infty$ so that $T_{\max} = 1$. Furthermore, it follows from (6.7.12) that

$$\|\nabla v(t)\|_{L^2}^2 = \frac{1}{4} \int_{\mathbb{R}^N} \Big| \Big(x + \frac{2i}{1-t}\nabla\Big)\psi(x)\Big|^2\, dx$$

so that

$$(1-t)\|\nabla v(t)\|_{L^2} \underset{t\uparrow 1}{\longrightarrow} \|\nabla \psi\|_{L^2}. \tag{6.7.16}$$

We deduce in particular from (6.7.15) and (6.7.16) that v blows up twice as fast as the lower estimates (6.5.49) and (6.5.50). This implies that, at least in the case $\alpha = 4/N$, the lower estimates (6.5.49) and (6.5.50) are not optimal for all the blowup solutions.

It also follows from (6.7.15) that $\|v(t)\|_{L^p} \underset{t\uparrow 1}{\longrightarrow} 0$ if $1 \le p < 2$, so that

$$v(t) \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^N) \text{ as } t \uparrow 1. \tag{6.7.17}$$

In particular, the loss of L^2 norm at the blowup is equal to $\|R\|_{L^2}$ if ψ is a ground state of (6.6.3), but it is larger if ψ is an excited state. (Note that excited states exist if $N \ge 2$; see [25].) Therefore, the loss of L^2 norm given by Theorem 6.6.10 is not always optimal.

Note also that by (6.7.11),

$$\|xv(t)\|_{L^2} = (1-t)\|x\psi\|_{L^2} \underset{t\uparrow 1}{\longrightarrow} 0 \tag{6.7.18}$$

(cf. Remark 6.5.5.). In particular, $v(t) \underset{t\uparrow 1}{\longrightarrow} 0$ in $L^2(\{|x| \ge \varepsilon\})$ for any $\varepsilon > 0$. Furthermore, one easily verifies that $v(t) \underset{t\uparrow 1}{\longrightarrow} 0$ in $H^1(\{|x| \ge \varepsilon\})$ and in $L^\infty(\{|x| \ge \varepsilon\})$ (this last point because ψ has exponential decay). Therefore, $v(t)$ blows up only at $x = 0$. Furthermore, it follows from an easy calculation that $|v(t)|^2 \underset{t\uparrow 1}{\longrightarrow} \|\psi\|_{L^2}^2 \delta$ in $\mathcal{D}'(\mathbb{R}^N)$, where δ is the Dirac measure at $x = 0$.

Finally, we observe that formula (6.7.14) also makes sense for $t > 1$, and that v given by (6.7.14) is also a solution of (4.1.1) for $t > 1$. As a matter of fact, the properties of v as $t \uparrow 1$ and as $t \downarrow 1$ are similar. Formula (6.7.14) gives (formally) an extension of the solution v beyond the blowup time $T_{\max} = 1$. We know that v satisfies (4.1.1) on $(-\infty, 1)$ and on $(1, \infty)$, and we investigate in what sense v may be a solution near $t = 1$. Note first that $v \in C((-\infty, 1) \cup (1, \infty), L^2(\mathbb{R}^N))$ and that $\|v(t)\|_{L^2} = \|\psi\|_{L^2}$ for $t \neq 1$, so that by property (6.7.17) v is discontinuous in $L^2(\mathbb{R}^N)$ at $t = 1$. On the other hand, $v \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^N))$, so that $\Delta v \in L^\infty(\mathbb{R}, H^{-2}(\mathbb{R}^N))$. Furthermore, it follows from (6.7.15) that $\||v(t)|^\alpha v(t)\|_{L^1} = c|1-t|^{\frac{N-4}{2}}$. Therefore, if we assume $N \geq 3$, then $|v|^\alpha v \in L^1_{\text{loc}}(\mathbb{R}, L^1(\mathbb{R}^N))$. If m is an integer such that $L^1(\mathbb{R}^N) \hookrightarrow H^{-m}(\mathbb{R}^N)$ (so that in particular $m \geq 2$), then we deduce that $|v|^\alpha v \in L^1_{\text{loc}}(\mathbb{R}, H^{-m}(\mathbb{R}^N))$. Therefore, $u_t \in L^1_{\text{loc}}(\mathbb{R}, H^{-m}(\mathbb{R}^N))$, so that $u \in C(\mathbb{R}, H^{-m}(\mathbb{R}^N))$. Thus $v(t) \to 0$ in $H^{-m}(\mathbb{R}^N)$ as $t \to 1$. This implies easily that v satisfies (4.1.1) in $\mathcal{D}'(\mathbb{R}, H^{-m}(\mathbb{R}^N))$. Therefore, we see that v can be extended in a reasonable sense beyond the blowup time $T_{\max} = 1$. However, the meaning of this extension is not quite clear. Indeed, if we define

$$\widetilde{v}(t) = \begin{cases} v(t) & \text{if } t < 1 \\ 0 & \text{if } t \geq 1, \end{cases}$$

then the above argument shows that $\widetilde{v}$ is also an extension of v beyond $T_{\max} = 1$, which satisfies (4.1.1) in $\mathcal{D}'(\mathbb{R}, H^{-m}(\mathbb{R}^N))$. As a matter of fact, one can define many such extensions. For example, since equation (4.1.1) is invariant by space translation and by multiplication by a constant of modulus 1, we see easily that for any $y \in \mathbb{R}^N$ and $\omega \in \mathbb{R}$,

$$\widetilde{v}(t) = \begin{cases} v(t) & \text{if } t < 1 \\ e^{i\omega} v(t, \cdot - y) & \text{if } t \geq 1 \end{cases}$$

satisfies (4.1.1) in $\mathcal{D}'(\mathbb{R}, H^{-m}(\mathbb{R}^N))$ and is also an extension of v beyond $T_{\max} = 1$. About this problem, see Merle [245].

REMARK 6.7.4. In space dimension $N = 1$, the solutions considered in the above remark are completely explicit. Indeed, it follows from formula (6.6.4) that

$$v(t,x) = \frac{1}{\sqrt{1-t}} e^{-i\frac{x^2}{4(1-t)}} e^{i\frac{t}{1-t}} \frac{3^{\frac{1}{4}}}{\sqrt{\cosh\left(\frac{2x}{1-t}\right)}}$$

is a solution of the Schrödinger equation $iu_t + u_{xx} + |u|^4 u = 0$ that blows up at $t = 1$.

REMARK 6.7.5. Let $v(t)$ be as in Remark 6.7.3. Given $y \in \mathbb{R}^N$, set $v_y(t) = v(t, \cdot - y)$, so that v_y is a solution of (4.1.1) for which $T_{\max} = 1$, and that blows up at the point $y \in \mathbb{R}^N$. Given $(y_\ell)_{1 \leq \ell \leq k}$ with $y_j \neq y_\ell$ for $j \neq \ell$,

$$w(t) = \sum_{\ell=1}^{k} v_{y_\ell}(t)$$

is a function that blows up at $t = 1$, and only at the points y_ℓ. On the other hand, since (4.1.1) is nonlinear, w is not a solution of (4.1.1). However, Merle [244] shows that there exists a solution u of (4.1.1) on $[0, 1)$ for which $T_{\max} = 1$ and which is asymptotic as $t \uparrow 1$ to w. This shows in a way the stability of the type of blowup displayed by v.

REMARK 6.7.6. Assume for simplicity $\lambda = 1$ and let R be the positive, spherically symmetric ground state of (6.6.3). It follows that for any $\gamma \in \mathbb{R}$, $\mu > 0$, and $y \in \mathbb{R}^N$, $v(t,x) = e^{i\gamma} e^{i\mu^2 t} R(\mu(x-y))$ is a solution of (4.1.1). For any $b < 0$ and $x_1 \in \mathbb{R}^N$, $u(t,x) = v_b(t, x - x_1)$ is therefore also a solution. An easy calculation shows that

$$\begin{aligned} u(t,x) &= \Bigl(\frac{\omega}{T_{\max}-t}\Bigr)^{\frac{N}{2}} e^{i\theta - i\frac{|x-x_1|^2}{4(T_{\max}-t)} + i\frac{\omega^2}{T_{\max}-t}} \\ &\quad \times R\Bigl(\frac{\omega}{T_{\max}-t}((x-x_1)-(T_{\max}-t)x_0)\Bigr) \end{aligned} \tag{6.7.19}$$

with $T_{\max} = -1/b$, $\omega = \mu T_{\max}$, $x_0 = y/T_{\max}$, and $\theta = \gamma - \mu^2 T_{\max}$. Let now $\varphi \in H^1(\mathbb{R}^N)$ be such that $\|\varphi\|_{L^2} = \|R\|_{L^2}$ and such that $T_{\max} < \infty$. It follows from Merle [246] that there exist $\theta \in \mathbb{R}$, $\omega > 0$, $x_0, x_1 \in \mathbb{R}^N$ such that u is given by (6.7.19). Similarly, if $T_{\min} < \infty$, then there exist $\theta \in \mathbb{R}$, $\omega > 0$, $x_0, x_1 \in \mathbb{R}^N$ such that u is given by

$$u(t,x) = \Bigl(\frac{\omega}{T_{\min}+t}\Bigr)^{\frac{N}{2}} e^{i\theta + i\frac{|x-x_1|^2}{4(T_{\min}+t)} - i\frac{\omega^2}{T_{\min}+t}} R\Bigl(\frac{\omega}{T_{\min}+t}((x-x_1)-(T_{\min}+t)x_0)\Bigr).$$

In other words, the only solutions that blow up on the critical sphere are those obtained from the ground state by the pseudoconformal transformation. Note in particular that if u is a solution on the critical L^2 sphere, then $T_{\max}$ and $T_{\min}$ cannot both be finite.

6.8. Comments

We begin with some examples of applications of the results of the preceding sections.

REMARK 6.8.1. Let $g(u) = \lambda |u|^\alpha u$ with $\lambda \in \mathbb{R}$ and $0 < \alpha < 4/(N-2)$ ($0 < \alpha < \infty$ if $N = 1$).

(i) If $\lambda < 0$, then all solutions of (4.1.1) are global.
(ii) If $\lambda > 0$ and $\alpha < 4/N$, then all solutions of (4.1.1) are global.
(iii) If $\lambda > 0$ and $\alpha \geq 4/N$, then the solution of (4.1.1) is global if $\|\varphi\|_{H^1}$ is small enough. On the other hand, given $\psi \in H^1(\mathbb{R}^N)$, $\psi \neq 0$, the solution of (4.1.1) with $\varphi = k\psi$ blows up in finite time, provided $|k|$ is large enough.

Statements (i) and (ii) follow from Corollary 6.1.2. The first part of (iii) follows from Corollary 6.1.5. Finally, the last part of (iii) follows from Theorem 6.5.4. Indeed, it is clear that (6.5.24) is satisfied, and that $E(k\psi) < 0$ for $|k|$ large enough.

REMARK 6.8.2. Let $g(u) = \lambda(|x|^{-\nu} \star |u|^2)u$, where $\lambda \in \mathbb{R}$, and $0 < \nu < \min\{N, 4\}$.

(i) If $\lambda < 0$, then all solutions of (4.1.1) are global.
(ii) If $\lambda > 0$ and $0 < \nu < 2$, then all solutions of (4.1.1) are global.
(iii) If $\lambda > 0$ and $\nu \geq 2$, then the solution of (4.1.1) is global if $\|\varphi\|_{H^1}$ is small enough. On the other hand, given $\psi \in H^1(\mathbb{R}^N)$, $\psi \neq 0$, the solution of (4.1.1) with $\varphi = k\psi$ blows up in finite time, provided $|k|$ is large enough.

Statements (i) and (ii) follow from Corollary 6.1.2. The first part of (iii) follows from Corollary 6.1.5. Finally, the last part of (iii) follows from Theorem 6.5.4. Indeed, it is clear that (6.5.26) is satisfied, and that $E(k\psi) < 0$ for $|k|$ large enough.

REMARK 6.8.3. Let $g(u) = \lambda|u|^\alpha u + \beta(|x|^{-\nu} \star |u|^2)u$, $\lambda, \beta \in \mathbb{R}$, $0 < \alpha < 4/(N-2)$, and $0 < \nu < \min\{N, 4\}$.

(i) The solution of (4.1.1) is global if $\|\varphi\|_{H^1}$ is small enough.
(ii) If $\lambda, \beta < 0$, then all solutions of (4.1.1) are global.
(iii) If $\lambda < 0$, $\beta > 0$, and $0 < \nu < 2$, then all solutions of (4.1.1) are global.
(iv) If $\lambda > 0$, $\beta < 0$, and $\alpha < 4/N$, then all solutions of (4.1.1) are global.
(v) If $\lambda, \beta > 0$, $\alpha < 4/N$, and $0 < \nu < 2$, then all solutions of (4.1.1) are global.
(vi) If $\lambda, \beta > 0$, $\alpha \ge 4/N$, and $\nu \ge 2$, then given $\psi \in H^1(\mathbb{R}^N)$, $\psi \ne 0$, the solution of (4.1.1) with $\varphi = k\psi$ blows up in finite time, provided $|k|$ is large enough.
(vii) If $\lambda > 0$, $\beta < 0$, $\alpha \ge 4/N$, $\alpha > 2$, and $\nu \le 2$, then given $\psi \in H^1(\mathbb{R}^N)$, $\psi \ne 0$, the solution of (4.1.1) with $\varphi = k\psi$ blows up in finite time, provided $|k|$ is large enough.
(viii) If $\lambda < 0$, $\beta > 0$, $\alpha \le 4/N$, $\alpha < 2$, and $\nu \ge 2$, then given $\psi \in H^1(\mathbb{R}^N)$, $\psi \ne 0$, the solution of (4.1.1) with $\varphi = k\psi$ blows up in finite time, provided $|k|$ is large enough.

Statement (i) follows from Corollary 6.1.5. Claims (ii), (iii), (iv), and (v) follow from Corollary 6.1.2. Finally, (vi), (vii), and (viii) follow from Theorem 6.5.4.

REMARK 6.8.4. There are some finite-time blowup results in strict subdomains $\Omega \subset \mathbb{R}^N$. For example, assume $\Omega \subset \mathbb{R}^N$ is smooth, bounded, and star-shaped about the origin, and $g(u) = |u|^\alpha u$ for some $4/N \le \alpha < 4/(N-2)$. It follows that a solution $u \in C^1([0,T], L^2(\Omega)) \cap C([0,T], H^2(\Omega) \cap H_0^1(\Omega))$ of (3.1.1) blows up in finite time, provided $E(\varphi) < 0$ (see Kavian [208], proposition 1.2). In addition, if Ω is a smooth, bounded domain of $\mathbb{R}^2$ and $g(u) = |u|^2 u$, then for every $x_0 \in \Omega$ there exists a solution that blows up like the singular solution in $\mathbb{R}^N$ (the rescaled ground state of Remark 6.7.3) at x_0. See Burq, Gérard, and Tzvetkov [50].

REMARK 6.8.5. Consider the problem (4.1.1) with $g(u) = \lambda|u|^\alpha u$, $\lambda \in \mathbb{R}$, and $\alpha > 0$. Suppose first that $\alpha < 4/N$. It follows from Theorem 4.6.1 that for every $\varphi \in L^2(\mathbb{R}^N)$, the corresponding L^2 solution of (4.1.1) is global. By H^s regularity (Theorem 5.1.1), we deduce that if $0 < s < \min\{N/2, 1\}$, then for every $\varphi \in H^s(\mathbb{R}^N)$, the corresponding H^s solution of (4.1.1) given by Theorem 4.9.1 is global. Fix now $0 < s < \min\{N/2, 1\}$ and assume $4/N < \alpha < 4/(N-2s)$ and $\lambda < 0$. It follows (see Remark 6.8.1(i) above) that for every $\varphi \in H^1(\mathbb{R}^N)$, the corresponding H^1 solution of (4.1.1) is global. On the other hand, it follows from Theorem 4.9.1 that for every $\varphi \in H^s(\mathbb{R}^N)$, there exists a local H^s solution u of (4.1.1). One may expect that the H^s solution is global, but there is no equivalent of the conservation of energy at the H^s level. It is possible to show, however, global existence for all $\varphi \in H^s(\mathbb{R}^N)$ in some cases; see Bourgain [38] and Colliander et al. [88, 89, 90, 91]. See also Vargas and Vega [350] for a related result of global existence for all initial values in a space strictly larger than $L^2(\mathbb{R}^N)$ for the cubic one-dimensional Schrödinger equation.

CHAPTER 7

Asymptotic Behavior in the Repulsive Case

In this chapter we continue the study of the global properties of the solutions of (4.1.1). We have seen in the preceding chapter that for certain nonlinearities and initial values, the solution of (4.1.1) satisfies $u \in L^q(\mathbb{R}, W^{1,r}(\mathbb{R}^N))$ for every admissible pair (q, r). See, e.g., Theorem 6.2.1. This implies that $u(t)$ has a certain decay as $t \to \infty$. If g is "sufficiently" superlinear near 0, for example if $g(u) = \lambda|u|^\alpha u$ with α "sufficiently" large, then $g(u)$ will have a stronger decay. One may then expect that the term $g(u)$ becomes negligible in equation (4.1.1) and that the solution $u(t)$ behaves as $t \to \infty$ like a solution of the linear Schrödinger equation. This turns out to be the case, under appropriate assumptions on g, and the scattering theory formalizes this kind of property. Of course, we have been vague in saying that the solution $u(t)$ behaves like a solution of the linear Schrödinger equation, since this can be measured in various different topologies. This gives rise to different scattering theories.

In Section 7.1, we describe the basic notions of the scattering theory.

In Sections 7.2–7.4, we develop a scattering theory in the weighted Sobolev space Σ defined by (6.7.3)–(6.7.4). We first establish the pseudoconformal conservation law (Section 7.2), then deduce the decay properties of solutions (Section 7.3), and develop the scattering theory (Section 7.4).

In Section 7.5, we apply the pseudoconformal transformation in order to obtain some further results (both positive results and counterexamples).

In Sections 7.6–7.8, we develop a scattering theory in the energy space $H^1(\mathbb{R}^N)$. We first derive the Morawetz estimate (Section 7.6). This is the essential tool to obtain the decay properties of the solutions (Section 7.7) on which the scattering theory is based (Section 7.8).

7.1. Basic Notions of Scattering Theory

In this section we introduce basic notions of scattering theory. Consider a Banach space X in which the equation (4.1.1) can be solved locally. For example, X can be $H^1(\mathbb{R}^N)$, $L^2(\mathbb{R}^N)$, $H^s(\mathbb{R}^N)$, or the space Σ defined by (6.7.3)–(6.7.4), depending on the nonlinearity g. See Chapter 4.

Let $\varphi \in X$ be such that the corresponding solution u of (4.1.1) is defined for all $t \geq 0$, i.e., $T_{\max} = \infty$. If the limit

$$u_+ = \lim_{t \to \infty} \mathcal{T}(-t)u(t) \tag{7.1.1}$$

exists in X, we say that u_+ is the scattering state of φ (at $+\infty$). Also, if $\varphi \in X$ is such that the solution of (4.1.1) is defined for all $t \leq 0$; i.e., $T_{\min} = \infty$, and if the limit

$$u_- = \lim_{t \to -\infty} \mathcal{T}(-t)u(t) \tag{7.1.2}$$

exists in X, we say that u_- is the scattering state of φ at $-\infty$.

We observe that saying that φ has a scattering state at $\pm\infty$ is a way of saying that $u(t)$ behaves as $t \to \pm\infty$ like the solution $\mathcal{T}(t)u_\pm$ of the *linear* Schrödinger equation.

We set

$$\mathcal{R}_+ = \{\varphi \in X : T_{\max} = \infty \text{ and the limit (7.1.1) exists}\} \tag{7.1.3}$$

and

$$\mathcal{R}_- = \{\varphi \in X : T_{\min} = \infty \text{ and the limit (7.1.2) exists}\}\,. \tag{7.1.4}$$

In other words, $\mathcal{R}_\pm$ is the set of initial values φ which have a scattering state at $\pm\infty$. We define the operators

$$U_\pm : \mathcal{R}_\pm \to X \quad \text{mapping } \varphi \mapsto u_\pm \tag{7.1.5}$$

and we set

$$\mathcal{U}_\pm = U_\pm(\mathcal{R}_\pm)\,. \tag{7.1.6}$$

If the mappings $U_\pm$ are injective, we set

$$\Omega_\pm = U_\pm^{-1} : \mathcal{U}_\pm \to \mathcal{R}_\pm\,. \tag{7.1.7}$$

The mappings $\Omega_\pm$ are called the wave operators. Next, we set

$$\mathcal{O}_\pm = U_\pm(\mathcal{R}_+ \cap \mathcal{R}_-)\,. \tag{7.1.8}$$

Finally, the scattering operator $\mathbf{S}$ is the mapping

$$\mathbf{S} = U_+\Omega_- : \mathcal{O}_- \to \mathcal{O}_+\,. \tag{7.1.9}$$

In other words, $u_+ = \mathbf{S}u_-$ if and only if there exists $\varphi \in \Sigma$ such that $T_{\max} = T_{\min} = \infty$ and such that $\mathcal{T}(-t)u(t) \to u_\pm$ as $t \to \pm\infty$.

REMARK 7.1.1. Note that the operators and the sets that we defined above depend on the space X in which the convergence (7.1.1) or (7.1.2) takes place.

REMARK 7.1.2. We observe that for the linear Schrödinger equation; i.e., when $g(u) \equiv 0$, all the operators $U_\pm, \Omega_\pm, \mathbf{S}$ defined above coincide with the identity on X. Note, however, that, in the general case $g \neq 0$, these operators are nonlinear.

REMARK 7.1.3. Assume that

$$g(\overline{u}) = \overline{g(u)} \quad \text{for all } u \in X.$$

It follows that changing t to $-t$ in the equation (4.1.1) corresponds to changing u to $\overline{u}$, which means changing φ to $\overline{\varphi}$. So we see that

$$\begin{aligned}
\mathcal{R}_- &= \overline{\mathcal{R}_+} = \{\varphi \in X : \overline{\varphi} \in \mathcal{R}_+\}\,,\\
\mathcal{U}_- &= \overline{\mathcal{U}_+} = \{v \in X : \overline{v} \in \mathcal{U}_+\}\,,\\
\mathcal{O}_- &= \overline{\mathcal{O}_+} = \{v \in X : \overline{v} \in \mathcal{O}_+\}\,,
\end{aligned}$$

and that $U_-\varphi = \overline{U_+\overline{\varphi}}$ and $\Omega_-\varphi = \overline{\Omega_+\overline{\varphi}}$.

7.2. The Pseudoconformal Conservation Law

Throughout this section we consider a nonlinearity g is as in Example 3.2.11. Therefore, we assume

$$g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u\,,$$

where V, f, and W are as follows. The potential V is real valued, $V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $p \geq 1$, $p > N/2$. $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}$ and satisfies (3.2.7), (3.2.8), and (3.2.17). Extend f to $\mathbb{R}^N \times \mathbb{C}$ by (3.2.10). The potential W is even and real valued, $W \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $q \geq 1$, $q > N/4$. In particular, g is the gradient of the potential G defined by

$$G(u) = \int_{\mathbb{R}^N} \left\{ \frac{1}{2}V(x)|u(x)|^2 + F(x, u(x)) + \frac{1}{4}(W \star |u|^2)(x)|u(x)|^2 \right\} dx\,, \tag{7.2.1}$$

and we set

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx - G(u) \quad \text{for all } u \in H^1(\mathbb{R}^N). \tag{7.2.2}$$

We recall (see Corollary 4.3.3) that the initial-value problem for (4.1.1) is locally well posed in $H^1(\mathbb{R}^N)$ and that there is conservation of charge and energy. Moreover, if $\varphi \in \Sigma$ with Σ defined by (6.7.3)–(6.7.4), then $u \in C((-T_{\min}, T_{\max}), \Sigma)$ (see Remark 6.5.2). Also, if $\varphi \in H^2(\mathbb{R}^N)$, then $u \in C((-T_{\min}, T_{\max}), H^2(\mathbb{R}^N))$ (see Remark 5.3.3).

The following "pseudoconformal conservation law," discovered by Ginibre and Velo [133, 134], is essential for the study of the asymptotic behavior of solutions.

THEOREM 7.2.1. *Let Σ be defined by* (6.7.3)–(6.7.4) *and let*

$$g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u$$

be as in Example 3.2.11. *If $\varphi \in \Sigma$ and if $u \in C((-T_{\min}, T_{\max}), \Sigma)$ is the corresponding solution of* (4.1.1), *then*

$$\|(x + 2it\nabla)u(t)\|_{L^2}^2 - 8t^2 G(u(t)) = \|x\varphi\|_{L^2}^2 - \int_0^t s\theta(s)ds\,, \tag{7.2.3}$$

where G is defined by (7.2.1) *and*

$$\begin{aligned}\theta(t) = \bigg\{ &\int_{\mathbb{R}^N} (8(N+2)F(u) - 4N\,\mathrm{Re}(f(u)\overline{u}))dx \\ &+ 8\int_{\mathbb{R}^N} (V + \frac{1}{2}x \cdot \nabla V)|u|^2\,dx \\ &+ 4\int_{\mathbb{R}^N} ((W + \frac{1}{2}x \cdot \nabla W) \star |u|^2)|u|^2\,dx \bigg\}\end{aligned} \tag{7.2.4}$$

for all $t \in (-T_{\min}, T_{\max})$.

REMARK 7.2.2. Note that by Proposition 6.5.1, the left-hand side of (7.2.3) makes sense.

PROOF OF THEOREM 7.2.1. Let

$$h(t) = \|(x + 2it\nabla)u(t)\|_{L^2}^2 - 8t^2 G(u(t)).$$

We have

$$\begin{aligned} h(t) &= \|xu(t)\|_{L^2}^2 + 4t^2\|\nabla u(t)\|_{L^2}^2 - 4t \operatorname{Im} \int_{\mathbb{R}^N} \overline{u}\, x \cdot \nabla u \, dx - 8t^2 G(u(t)) \\ &= \|xu(t)\|_{L^2}^2 - 4t \operatorname{Im} \int_{\mathbb{R}^N} \overline{u}\, x \cdot \nabla u \, dx + 8t^2 E(\varphi), \end{aligned}$$

where the energy E is defined by (7.2.2). Applying Proposition 6.5.1, we deduce that $h \in C^1(-T_{\min}, T_{\max})$ and that

$$\begin{aligned} h'(t) = \frac{d}{dt}\|xu(t)\|_{L^2}^2 &- 4 \operatorname{Im} \int_{\mathbb{R}^N} \overline{u}\, x \cdot \nabla u \, dx \\ &- 4t \frac{d}{dt} \operatorname{Im} \int_{\mathbb{R}^N} \overline{u}\, x \cdot \nabla u \, dx + 16tE(\varphi) = -t\theta(t), \end{aligned}$$

and (7.2.3) follows after integration on $(0, t)$. □

REMARK 7.2.3. Note that when $V \equiv 0$, $f(s) = \lambda |s|^{\frac{4}{N}} s$ for some $\lambda \in \mathbb{R}$ and $W(x) = \mu |x|^{-2}$ for some $\mu \in \mathbb{R}$ ($\mu = 0$, if $N = 1, 2$), (7.2.3) is an exact conservation law, which is

$$\|(x + 2it\nabla)u(t)\|_{L^2}^2 - 4t^2 \int_{\mathbb{R}^N} \Big\{ \frac{\lambda N}{N+2} |u|^{2+\frac{4}{N}} + \frac{\mu}{2} (|x|^{-2} \star |u|^2)|u|^2 \Big\} dx = \|x\varphi\|_{L^2}^2. \tag{7.2.5}$$

It corresponds to the invariance of the equation under a group of transformations. See Section 6.7 and see Ginibre and Velo [139] and Olver [285].

REMARK 7.2.4. Note that if $t \neq 0$, then

$$(x + 2it\nabla)w = 2ite^{i\frac{|x|^2}{4t}} \nabla (e^{-i\frac{|x|^2}{4t}} w), \tag{7.2.6}$$

and so

$$\|(x + 2it\nabla)w\|_{L^2}^2 = 4t^2 \|\nabla(e^{-i\frac{|x|^2}{4t}} w)\|_{L^2}^2.$$

Therefore, if we set

$$v(t, x) = e^{-i\frac{|x|^2}{4t}} u(t, x), \tag{7.2.7}$$

then (7.2.3) is equivalent to

$$8t^2 E(v(t)) = \|x\varphi\|_{L^2}^2 - \int_0^t s\theta(s) ds. \tag{7.2.8}$$

That equivalent formulation of (7.2.3) will be helpful later.

REMARK 7.2.5. Let $f \in C(\mathbb{C},\mathbb{C})$ satisfy $f(0)=0$ and $|f(v)-f(u)| \le C(|u|^\alpha + |v|^\alpha)|v-u|$ for all $u,v \in \mathbb{C}$, where $0 \le \alpha < 4/(N-2)$ ($0 \le \alpha < \infty$ if $N=1$). Assume further that $f(e^{i\theta}u) = e^{i\theta} f(u)$ for all $u \in \mathbb{C}$ and $\theta \in \mathbb{R}$. It follows from (7.2.6) that

$$|(x+2it\nabla)f(u)| = 2|t|\big|\nabla(e^{-i\frac{|x|^2}{4t}} f(u))\big| = 2|t||\nabla(f(v))|,$$

where v is defined by (7.2.7). From the above identity and Remark 1.3.1(vii), it follows that

$$\|(x+2it\nabla)f(u)\|_{L^{\frac{\alpha+2}{\alpha+1}}} \le C|t|\|v\|^\alpha_{L^{\alpha+2}}\|\nabla v\|_{L^{\alpha+2}}.$$

Since $|v|=|u|$ and $2|t||\nabla v| = |(x+2it\nabla)u|$, we obtain

$$\|(x+2it\nabla)f(u)\|_{L^{\frac{\alpha+2}{\alpha+1}}} \le C\|u\|^\alpha_{L^{\alpha+2}}\|(x+2it\nabla)u\|_{L^{\alpha+2}}. \tag{7.2.9}$$

Note that when $\alpha=0$; i.e., when f is globally Lipschitz continuous,

$$\|(x+2it\nabla)f(u)\|_{L^2} \le C\|(x+2it\nabla)u\|_{L^2}. \tag{7.2.10}$$

The constants C in the above inequalities are independent of u and t.

7.3. Decay of Solutions in the Weighted L^2 Space

In this section we apply the pseudoconformal conservation law to the study of the asymptotic behavior of solutions. For simplicity, we restrict our attention to the model case

$$g(u) = -\eta|u|^\alpha u, \tag{7.3.1}$$

where

$$\eta > 0, \quad 0 < \alpha < \frac{4}{N-2} \quad (0<\alpha<\infty \text{ if } N=1). \tag{7.3.2}$$

and we refer to Section 7.9 and Ginibre and Velo [133, 132, 134, 139] for more general results. Note that in this case,

$$T_{\min}(\varphi) = T_{\max}(\varphi) = \infty \quad \text{for all } \varphi \in H^1(\mathbb{R}^N) \tag{7.3.3}$$

(see Remark 6.8.1(i)).

Furthermore, it follows from (7.2.8) that (7.2.3) is equivalent to

$$\begin{aligned} &8t^2\Big\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v(t)|^2\,dx + \frac{\eta}{\alpha+2}\int_{\mathbb{R}^N} |v(t)|^{\alpha+2}\,dx\Big\} = \\ &\|x\varphi\|^2_{L^2} + 4\eta\frac{4-N\alpha}{\alpha+2}\int_0^t s\int_{\mathbb{R}^N} |v(s)|^{\alpha+2}\,dx\,ds \quad \text{for all } t\in\mathbb{R}, \end{aligned} \tag{7.3.4}$$

where v is defined by (7.2.7). We have the following result.

THEOREM 7.3.1. *Assume (7.3.1)—(7.3.2). If $\varphi \in H^1(\mathbb{R}^N)$ satisfies $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$ and if $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ is the corresponding solution of (4.1.1), then the following properties hold:*

(i) *If $\alpha \ge 4/N$, then for every $2 \le r \le \frac{2N}{N-2}$ ($2 \le r \le \infty$ if $N=1$; $2 \le r < \infty$ if $N=2$), there exists C such that*

$$\|u(t)\|_{L^r} \le C|t|^{-N(\frac{1}{2}-\frac{1}{r})} \quad \text{for all } t\in\mathbb{R}. \tag{7.3.5}$$

(ii) *If $\alpha < 4/N$, then for every $2 \le r \le \frac{2N}{N-2}$ ($2 \le r \le \infty$ if $N = 1$; $2 \le r < \infty$ if $N = 2$), there exists C such that*

$$\|u(t)\|_{L^r} \le C|t|^{-N(\frac{1}{2}-\frac{1}{r})(1-\theta(r))} \quad \text{for all } t \in \mathbb{R}, \tag{7.3.6}$$

where

$$\theta(r) = \begin{cases} 0 & \text{if } 2 \le r \le \alpha + 2 \\ \frac{(r-(\alpha+2))(4-N\alpha)}{(r-2)(2\alpha+4-N\alpha)} & \text{if } r > \alpha + 2. \end{cases}$$

PROOF. If $\alpha \ge 4/N$, we deduce from (7.3.4) that

$$8t^2 \int_{\mathbb{R}^N} |\nabla v(t)|^2 \, dx \le \|x\varphi\|_{L^2}^2 \quad \text{for all } t \in \mathbb{R}.$$

Applying conservation of charge and Gagliardo-Nirenberg's inequality, we obtain

$$\begin{aligned} \|u(t)\|_{L^r} = \|v(t)\|_{L^r} &\le C\|\nabla v(t)\|_{L^2}^{N(\frac{1}{2}-\frac{1}{r})} \|v(t)\|_{L^2}^{1-N(\frac{1}{2}-\frac{1}{r})} \\ &\le C|t|^{-N(\frac{1}{2}-\frac{1}{r})} \|\varphi\|_{L^2}^{1-N(\frac{1}{2}-\frac{1}{r})} \\ &\le C|t|^{-N(\frac{1}{2}-\frac{1}{r})}. \end{aligned}$$

Hence (i) is established. Assume now $\alpha < 4/N$. We consider the case $t \ge 1$, the argument being the same for $t < -1$. It follows from (7.3.4) that

$$8t^2 E(v(t)) = 8E(v(1)) + 4\eta \frac{4 - N\alpha}{\alpha + 2} \int_1^t s \int_{\mathbb{R}^N} |v(s)|^{\alpha+2} \, dx \, ds.$$

This implies that

$$h(t) \le C + \frac{4 - N\alpha}{2} \int_1^t \frac{1}{s} h(s) \quad \text{where} \quad h(t) = t^2 \int_{\mathbb{R}^N} |v(t)|^{\alpha+2} \, dx.$$

Applying Gronwall's Lemma, we deduce that

$$h(t) \le Ct^{\frac{4-N\alpha}{2}},$$

from which we deduce

$$\|v(t)\|_{L^{\alpha+2}} \le Ct^{-N(\frac{1}{2}-\frac{1}{\alpha+2})}. \tag{7.3.7}$$

Applying (7.3.4) and (7.3.7), we obtain

$$8t^2 \int_{\mathbb{R}^N} |\nabla v(t)|^2 \, dx \le C + C\int_0^t s^{1-\frac{N\alpha}{2}} \, ds \le C + Ct^2 t^{-\frac{N\alpha}{2}},$$

and so

$$\|\nabla v(t)\|_{L^2} \le Ct^{-\frac{N\alpha}{4}}. \tag{7.3.8}$$

Applying (7.3.7), Hölder's inequality, and conservation of charge, we deduce that for every $2 \le r \le \alpha + 2$,

$$\begin{aligned}\|u(t)\|_{L^r} = \|v(t)\|_{L^r} &\le C\|v(t)\|_{L^{\alpha+2}}^{\frac{2(\alpha+2)}{\alpha}(\frac{1}{2}-\frac{1}{r})}\|v(t)\|_{L^2}^{1-\frac{2(\alpha+2)}{\alpha}(\frac{1}{2}-\frac{1}{r})} \\ &\le C|t|^{-N(\frac{1}{2}-\frac{1}{r})}\|\varphi\|_{L^2}^{1-\frac{2(\alpha+2)}{\alpha}(\frac{1}{2}-\frac{1}{r})} \\ &\le C|t|^{-N(\frac{1}{2}-\frac{1}{r})}.\end{aligned}$$

This implies (7.3.6) for $2 \le r \le \alpha+2$. For $\alpha+2 < r \le \frac{2N}{N-2}$, it follows from (7.3.7), (7.3.8), and Gagliardo-Nirenberg's inequality that

$$\begin{aligned}\|u(t)\|_{L^r} = \|v(t)\|_{L^r} &\le C\|\nabla v(t)\|_{L^2}^{\frac{2N(r-\alpha-2)}{r(2\alpha+4-N\alpha)}}\|v(t)\|_{L^{\alpha+2}}^{1-\frac{2N(r-\alpha-2)}{r(2\alpha+4-N\alpha)}} \\ &\le Ct^{-N(\frac{1}{2}-\frac{1}{r})(1-\theta(r))}.\end{aligned}$$

Hence (7.3.6) follows for $r > \alpha + 2$. This completes the proof. □

REMARK 7.3.2. Theorem 7.3.1 implies in particular that if $\varphi \in H^1(\mathbb{R}^N)$ and $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$, then the corresponding solution u of (4.1.1) converges weakly to 0 in $L^2(\mathbb{R}^N)$ as $|t| \to \infty$. Indeed, u is bounded in L^2 and converges to 0 strongly in $L^{\alpha+2}$.

REMARK 7.3.3. Note that for $\alpha \ge 4/N$, u has the same decay properties in $L^r(\mathbb{R}^N)$ as the solutions of the linear Schrödinger equation (see Proposition 2.2.3) for every $2 \le r < 2N/(N-2)$. When $\alpha < 4/N$, the decay properties are the same for $r \le \alpha + 2$.

COROLLARY 7.3.4. *Assume* (7.3.1)–(7.3.2). *Assume further that* $\alpha > \alpha_0$, *where* α_0 *is defined by* (6.3.3). *Let* $\varphi \in H^1(\mathbb{R}^N)$ *satisfy* $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$, *let* u *be the maximal solution of* (4.1.1), *and set*

$$v(t) = (x + 2it\nabla)u(t) \quad \text{for } t \in \mathbb{R}. \tag{7.3.9}$$

It follows that $u \in L^q(\mathbb{R}, W^{1,r}(\mathbb{R}^N))$ *and* $v \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ *for every admissible pair* (q, r).

REMARK 7.3.5. Note that it follows from Proposition 6.5.1 that v is well defined, and that $v \in C(\mathbb{R}, L^2(\mathbb{R}^N))$.

For the proof of Corollary 7.3.4, we will use the following lemma.

LEMMA 7.3.6. *Under the assumptions of Corollary* 7.3.4, *it follows that* $v \in L^q_{\mathrm{loc}}(\mathbb{R}, L^r(\mathbb{R}^N))$ *for every admissible pair* (q, r).

PROOF. Given $\varepsilon > 0$, let

$$g_\varepsilon(u) = -\eta\frac{|u|^\alpha}{1+\varepsilon|u|^\alpha}u \quad \text{for } u \in \mathbb{C}.$$

Let u^ε be the maximal solution of (4.1.1) with g replaced by g_ε. Note that for every $\varepsilon > 0$, g_ε is globally Lipschitz continuous $\mathbb{C} \to \mathbb{C}$. Note also (see, e.g., Corollary 6.1.2) that u_ε is globally defined and that, by conservation of charge and energy,

$$\sup_{\substack{t\in\mathbb{R}\\ \varepsilon>0}} \|u^\varepsilon(t)\|_{H^1} < \infty . \tag{7.3.10}$$

Next, we observe that $|g(u) - g_\varepsilon(u)| \le |g(u)|$ and that $|g(u) - g_\varepsilon(u)| \to 0$ as $\varepsilon \downarrow 0$ for every $u \in \mathbb{C}$. It follows easily that $u^\varepsilon \to u$ in $L^\infty_{\text{loc}}(\mathbb{R}, L^p(\mathbb{R}^N))$ as $\varepsilon \downarrow 0$ for every $2 \le p < 2N/(N-2)$ ($2 \le p \le \infty$ if $N = 1$). (See, e.g., Step 3 of the proof of Theorem 4.4.6, and in particular the proof of (4.4.29).) Using the conservation of energy for both u and u_ε, one deduces easily that $u^\varepsilon \to u$ in $L^\infty_{\text{loc}}(\mathbb{R}, H^1(\mathbb{R}^N))$ as $\varepsilon \downarrow 0$. Therefore, we need only estimate the solutions u^ε independently of ε. It follows from Lemma 6.5.2 that $v^\varepsilon(t) \equiv (x + 2it\nabla)u^\varepsilon(t) \in C(\mathbb{R}, L^2(\mathbb{R}^N))$. Furthermore, applying formula (2.5.5), we obtain

$$v^\varepsilon(t) = \mathcal{T}(t)x\varphi + i\int_0^t \mathcal{T}(t-s)(x + 2is\nabla)g_\varepsilon(u^\varepsilon(s))ds . \tag{7.3.11}$$

Since g_ε is globally Lipschitz continuous, it follows from (7.2.10) that there exists C_ε such that

$$\|(x + 2it\nabla)g_\varepsilon(w)\|_{L^2} \le C_\varepsilon\|(x + 2it\nabla)w\|_{L^2} .$$

Therefore, $(x + 2it\nabla)g_\varepsilon(u^\varepsilon) \in L^\infty_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^N))$. Applying Strichartz's estimates, we deduce from (7.3.11) that $v^\varepsilon \in L^q_{\text{loc}}(\mathbb{R}, L^r(\mathbb{R}^N))$ for every admissible pair (q, r). Next, we observe that there exists C independent of ε such that

$$|g_\varepsilon(v) - g_\varepsilon(u)| \le C(|u|^\alpha + |v|^\alpha)|v - u| \quad \text{for all } u, v \in \mathbb{C}.$$

Therefore, we deduce from (7.2.9) and (7.3.10) that there exists C independent of t and ε such that

$$\|(x + 2it\nabla)g_\varepsilon(w)\|_{L^{\frac{\alpha+2}{\alpha+1}}} \le C\|(x + 2it\nabla)w\|_{L^{\alpha+2}} \quad \text{for all } w \in H^1(\mathbb{R}^N).$$

It then follows from Strichartz's estimates and (7.3.11) that if $\tau > 0$, if (q, r) is any admissible pair, and if (γ, ρ) is the admissible pair such that $\rho = \alpha + 2$, then there exists a constant C independent of ε and τ such that

$$\|v^\varepsilon\|_{L^q((-\tau,\tau),L^r)} \le C\|x\varphi\|_{L^2} + C\tau^{\frac{q-q'}{qq'}}\|v^\varepsilon\|_{L^\gamma((-\tau,\tau),L^\rho)} . \tag{7.3.12}$$

We first let $(q, r) = (\gamma, \rho)$. We deduce that if $\tau > 0$ is sufficiently small, then $\|v^\varepsilon\|_{L^\gamma((-\tau,\tau),L^\rho)} \le 2C\|x\varphi\|_{L^2}$. Applying again (7.3.12) with an arbitrary admissible pair (q, r), we obtain that $\|v^\varepsilon\|_{L^q((-\tau,\tau),L^r)}$ is bounded as $\varepsilon \downarrow 0$. By letting $\varepsilon \downarrow 0$, we conclude easily that $v \in L^q((-\tau, \tau), L^r(\mathbb{R}^N))$ for every admissible pair (q, r). By time-translation invariance of the equation, this implies that if $t_0 \in \mathbb{R}$, then $(x + 2i(t - t_0)\nabla)u \in L^q((t_0 - \tau, t_0 + \tau), L^r(\mathbb{R}^N))$ for every admissible pair (q, r). Since $\nabla u \in L^q((t_0 - \tau, t_0 + \tau), L^r(\mathbb{R}^N))$ by Remark 4.4.3, we conclude that $v \in L^q((t_0 - \tau, t_0 + \tau), L^r(\mathbb{R}^N))$, which completes the proof. □

Proof of Corollary 7.3.4. We proceed in two steps.

Step 1. $u \in L^q(\mathbb{R}, W^{1,r}(\mathbb{R}^N))$ for every admissible pair (q, r). Note first that $u \in L^q_{\text{loc}}(\mathbb{R}, W^{1,r}(\mathbb{R}^N))$ by Remark 4.4.3 or Theorem 4.4.6. Consider now

$r = \alpha + 2$, and let q be such that (q,r) is an admissible pair. We have

$$u(t) = \mathcal{T}(t)\varphi - i\eta \int_0^t \mathcal{T}(t-s)|u|^\alpha u(s)ds.$$

Therefore, it follows from Strichartz's estimates that for every $t \geq T \geq 0$,

$$\|u\|_{L^q((0,t),W^{1,r})} \leq C\|\varphi\|_{H^1} + C\||u|^\alpha u\|_{L^{q'}((0,T),W^{1,r'})} + C\||u|^\alpha u\|_{L^{q'}((T,t),W^{1,r'})},$$

where C is independent of t and T. Since

$$\||u|^\alpha u\|_{W^{1,r'}} \leq C\|u\|_{L^r}^\alpha \|u\|_{W^{1,r}},$$

we deduce from Hölder's inequality that for every $T \in [0,t]$,

$$\begin{aligned}\|u\|_{L^q((0,t),W^{1,r})} \leq C\|\varphi\|_{H^1} &+ C\Big(\int_0^T \|u(s)\|_{L^r}^{\frac{\alpha q}{q-2}} ds\Big)^{\frac{q-2}{q}} \|u\|_{L^q((0,T),W^{1,r})} \\ &+ C\Big(\int_T^t \|u(s)\|_{L^r}^{\frac{\alpha q}{q-2}} ds\Big)^{\frac{q-2}{q}} \|u\|_{L^q((T,t),W^{1,r})}.\end{aligned} \tag{7.3.13}$$

By Theorem 7.3.1, $\|u(s)\|_{L^r} \leq Cs^{-\frac{2}{q}}$, and so

$$\|u(s)\|_{L^r}^{\frac{\alpha q}{q-2}} \leq Cs^{-\frac{2\alpha}{q-2}}.$$

Note that since $\alpha > \alpha_0$, we have $2\alpha > q-2$. Therefore for T large enough,

$$C\Big(\int_T^t \|u(s)\|_{L^r}^{\frac{\alpha q}{q-2}} ds\Big)^{\frac{q-2}{q}} \leq \frac{1}{2}. \tag{7.3.14}$$

On the other hand, $u \in L^\infty((0,T),H^1(\mathbb{R}^N)) \cap L^q((0,T),W^{1,r}(\mathbb{R}^N))$. Therefore, it follows from (7.3.13) and (7.3.14) that

$$\|u\|_{L^q((0,t),W^{1,r})} \leq C + \frac{1}{2}\|u\|_{L^q((0,t),W^{1,r})}.$$

Letting $t \uparrow \infty$, we obtain

$$\|u\|_{L^q((0,\infty),W^{1,r})} \leq 2C.$$

One proves as well that $u \in L^q((-\infty,0),W^{1,r})$. Applying again Strichartz's estimates, one obtains the result for every admissible pair.

STEP 2. $v \in L^q(\mathbb{R},L^r(\mathbb{R}^N))$ for every admissible pair (q,r). Note that $v \in L^q_{\mathrm{loc}}(\mathbb{R},L^r(\mathbb{R}^N))$ by Lemma 7.3.6. Consider now $r = \alpha+2$, and let q be such that (q,r) is an admissible pair. It follows from formula (2.5.5) that

$$v(t) = \mathcal{T}(t)x\varphi - i\eta \int_0^t \mathcal{T}(t-s)(x+2is\nabla)|u|^\alpha u(s)ds.$$

Therefore, we deduce from Strichartz's estimates that for every $t > 0$,

$$\|v\|_{L^q((0,t),L^r)} \leq C\|x\varphi\|_{L^2} + C\|(x+2is\nabla)|u|^\alpha u\|_{L^{q'}((0,t),L^{r'})},$$

where C is independent of t. By applying (7.2.9), we obtain

$$
\begin{aligned}
\|v\|_{L^q((0,t),L^r)} \le C\|x\varphi\|_{L^2} &+ C\Big(\int_0^T \|u(s)\|_{L^r}^{\frac{\alpha q}{q-2}}\,ds\Big)^{\frac{q-2}{q}} \|v\|_{L^q((0,T),L^r)} \\
&+ C\Big(\int_T^t \|u(s)\|_{L^r}^{\frac{\alpha q}{q-2}}\,ds\Big)^{\frac{q-2}{q}} \|v\|_{L^q((T,t),L^r)}
\end{aligned}
\tag{7.3.15}
$$

for every $T \in [0,t]$. One concludes as in Step 1 above. □

7.4. Scattering Theory in the Weighted L^2 Space

In this section we still assume (7.3.1)–(7.3.2). We show that the scattering states, the wave operators, and the scattering operators are defined on the space Σ defined by (6.7.3)–(6.7.4) provided $\alpha > \alpha_0$ defined by (6.3.3). The results of this section are due to Ginibre and Velo [133] for $\alpha \ge 4/N$ and to Y. Tsutsumi [341] for $\alpha > \alpha_0$. In Section 7.5 below, we will obtain a similar result for $\alpha = \alpha_0$, but by a different method using explicitly the pseudoconformal transformation.

THEOREM 7.4.1. *Assume* (7.3.1)–(7.3.2). *Assume further that* $\alpha > \alpha_0$, *where* α_0 *is defined by* (6.3.3) *and let* Σ *be the Hilbert space defined by* (6.7.3)–(6.7.4). *If* $\varphi \in \Sigma$ *and if* u *is the maximal solution of* (4.1.1), *then there exist* $u^+, u^- \in \Sigma$ *such that*

$$\|\mathcal{T}(-t)u(t) - u^{\pm}\|_{\Sigma} \underset{t\to\pm\infty}{\longrightarrow} 0\,.$$

In addition,

$$\|u^+\|_{L^2} = \|u^-\|_{L^2} = \|\varphi\|_{L^2} \quad \textit{and} \quad \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u^+|^2 = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u^-|^2 = E(\varphi)\,.$$

PROOF. Let $v(t) = \mathcal{T}(-t)u(t)$. We have

$$v(t) = \varphi - i\eta\int_0^t \mathcal{T}(-s)|u|^\alpha u(s)ds\,.$$

Therefore for $0 < t < \tau$,

$$v(t) - v(\tau) = -i\eta\int_\tau^t \mathcal{T}(-s)|u|^\alpha u(s)ds. \tag{7.4.1}$$

It follows from Strichartz's estimates that

$$\|v(t) - v(\tau)\|_{H^1} = \|\mathcal{T}(t)(v(t) - v(\tau))\|_{H^1} \le C\||u|^\alpha u\|_{L^{q'}((t,\tau),W^{1,r'})}\,,$$

where (q,r) is the admissible pair such that $r = \alpha + 2$. Thus,

$$\|v(t) - v(\tau)\|_{H^1} \underset{t,\tau\to\infty}{\longrightarrow} 0$$

(see the proof of Corollary 7.3.4). Therefore, there exists $u^+ \in H^1(\mathbb{R}^N)$ such that $v(t) \to u^+$ in H^1 as $t \to \infty$. One shows as well that there exists $u^- \in H^1(\mathbb{R}^N)$ such that $v(t) \to u^-$ in H^1 as $t \to -\infty$. Finally, it follows from formulae (7.4.1) and (2.5.5) that

$$x(v(t) - v(\tau)) = -i\eta\int_\tau^t \mathcal{T}(-s)(x + 2is\nabla)|u|^\alpha u(s)ds\,.$$

By Strichartz's estimates,

$$\|x(v(t)-v(\tau))\|_{L^2} = \|\mathcal{T}(t)x(v(t)-v(\tau))\|_{L^2} \le C\|(x+2is\nabla)|u|^\alpha u\|_{L^{q'}((t,\tau),L^{r'})},$$

where (q,r) is the admissible pair such that $r=\alpha+2$, and so

$$\|x(v(t)-v(\tau))\|_{L^2} \underset{t,\tau\to\infty}{\longrightarrow} 0$$

(see the proof of Corollary 7.3.4.) Therefore, $x(v(t)-u^+)\to 0$ in L^2 as $t\to\infty$. One shows as well that $x(v(t)-u^-)\to 0$ in L^2 as $t\to-\infty$. The other properties follow immediately from conservation of charge and energy. □

REMARK 7.4.2. Theorem 7.4.1 means that the mappings $U_+ : \varphi \mapsto u^+$ and $U_- : \varphi\mapsto u^-$ are well defined $\Sigma\to\Sigma$. In fact, one can show with similar estimates that U_+ and U_- are continuous. Note that $U_\pm$ are nonlinear operators.

REMARK 7.4.3. By Corollary 2.3.6,

$$u^\pm = \varphi - i\eta\int_0^{\pm\infty} \mathcal{T}(-s)|u|^\alpha u(s)ds\,.$$

In particular,

$$u(t) = \mathcal{T}(t)u^\pm + i\eta\int_t^{\pm\infty}\mathcal{T}(t-s)|u|^\alpha u(s)ds \quad \text{for all } t\in\mathbb{R}. \tag{7.4.2}$$

We now construct the wave operators $\Omega_\pm$ that are the inverses of the operators $U_\pm$.

THEOREM 7.4.4. *Assume* (7.3.1)–(7.3.2). *Assume further that* $\alpha>\alpha_0$, *where* α_0 *is defined by* (6.3.3), *and let* Σ *be the Hilbert space defined by* (6.7.3)–(6.7.4).

(i) *For every* $u^+\in\Sigma$, *there exists a unique* $\varphi\in\Sigma$ *such that the maximal solution* $u\in C(\mathbb{R},H^1(\mathbb{R}^N))$ *of* (4.1.1) *satisfies* $\|\mathcal{T}(-t)u(t)-u^+\|_\Sigma\to 0$ *as* $t\to+\infty$.

(ii) *For every* $u^-\in\Sigma$, *there exists a unique* $\varphi\in\Sigma$ *such that the maximal solution* $u\in C(\mathbb{R},H^1(\mathbb{R}^N))$ *of* (4.1.1) *satisfies* $\|\mathcal{T}(-t)u(t)-u^-\|_\Sigma\to 0$ *as* $t\to-\infty$.

PROOF. We prove (i), the proof of (ii) being similar. The idea of the proof is to solve equation (7.4.2) by a fixed-point argument. To that end, we introduce the functions $\omega(t)=\mathcal{T}(t)u^+$ and $z(t)=(x+2it\nabla)\omega(t,x)$. Let (q,r) be the admissible pair such that $r=\alpha+2$. It follows from Strichartz's estimates and Corollary 2.5.4 that $\omega\in L^q(\mathbb{R},W^{1,r}(\mathbb{R}^N))$, $z\in L^q(\mathbb{R},L^r(\mathbb{R}^N))$, and that $\|\omega(t)\|_{L^r}\le C|t|^{-2/q}$. Let

$$K = \|\omega\|_{L^q(\mathbb{R},W^{1,r})} + \|z\|_{L^q(\mathbb{R},L^r)} + \sup_{t\in\mathbb{R}}|t|^{\frac{2}{q}}\|\omega(t)\|_{L^r}\,. \tag{7.4.3}$$

Given $S>0$, set

$$I=(S,\infty)\,,$$

and let

$$E=\big\{u\in L^q(I,W^{1,r}(\mathbb{R}^N)) : (x+2it\nabla)u(t,x)\in L^q(I,L^r(\mathbb{R}^N)) \text{ and}$$
$$\|u\|_{L^q(I,W^{1,r})} + \|(x+2it\nabla)u(t)\|_{L^q(I,L^r)} + \sup_{t\in I}|t|^{\frac{2}{q}}\|u(t)\|_{L^r}\le 2K\big\}$$

and

$$\mathrm{d}(u,v) = \|v-u\|_{L^q(I,L^r)} \quad \text{for } u,v \in E\,.$$

It is easily checked that (E,d) is a complete metric space. Given $u \in E$,

$$\|g(u(t))\|_{W^{1,r'}} \le C\|u(t)\|_{L^r}^{\alpha}\|u(t)\|_{W^{1,r}} \le C(2K)^{\alpha}t^{-\frac{2\alpha}{q}}\|u(t)\|_{W^{1,r}}\,,$$

and so by Hölder's inequality,

$$\begin{aligned}\|g(u)\|_{L^{q'}(I,W^{1,r'})} &\le C(2K)^{\alpha}\Big(\int_S^{\infty} s^{-\frac{2\alpha}{q-2}}ds\Big)^{\frac{q-2}{q}}\|u\|_{L^q(I,W^{1,r})}\\ &\le C(2K)^{\alpha+1}S^{1-\frac{2\alpha}{q-2}}\,,\end{aligned}$$

since $2\alpha > q-2$. It then follows from Corollary 2.3.6 that $\mathcal{J}(u)$ defined by

$$\mathcal{J}(u)(t) = -i\int_t^{\infty}\mathfrak{T}(t-s)g(u(s))ds \tag{7.4.4}$$

makes sense, that

$$\mathcal{J}(u) \in C([S,\infty),H^1(\mathbb{R}^N)) \cap L^q(I,W^{1,r}(\mathbb{R}^N))\,, \tag{7.4.5}$$

and that

$$\|\mathcal{J}(u)\|_{L^q(I,W^{1,r})} \le \frac{K}{3} \quad \text{for } S \text{ large enough.} \tag{7.4.6}$$

Furthermore,

$$(x+2it\nabla)\mathcal{J}(u)(t) = -i\int_t^{\infty}\mathfrak{T}(t-s)\big[(x+2is\nabla)g(u(s))\big]ds\,,$$

by formula (2.5.5). Since

$$\|(x+2is\nabla)g(u(s))\|_{L^{r'}} \le C\|u(s)\|_{L^r}^{\alpha}\|(x+2is\nabla)u(s)\|_{L^r}\,,$$

by (7.2.9), one concludes as above that

$$(x+2it\nabla)\mathcal{J}(u) \in C([S,\infty),L^2(\mathbb{R}^N)) \cap L^q(I,L^r(\mathbb{R}^N))\,, \tag{7.4.7}$$

and that

$$\|(x+2it\nabla)\mathcal{J}(u)\|_{L^q(I,L^r)} \le \frac{K}{3} \quad \text{for } S \text{ large enough.} \tag{7.4.8}$$

Finally, it follows from (2.2.4) that

$$\|\mathcal{J}(u)(t)\|_{L^r} \le \eta\int_t^{\infty}|t-s|^{-\frac{2}{q}}\|u(s)\|_{L^r}^{\alpha+1}ds \le C(2K)^{\alpha+1}S^{1-\frac{2(\alpha+1)}{q}}t^{-\frac{2}{q}}\,,$$

since $2(\alpha+1) > q$. Therefore for S large enough,

$$\sup\big\{t^{\frac{2}{q}}\|\mathcal{J}(u)(t)\|_{L^r} : t \in [S,\infty)\big\} \le \frac{K}{3}\,. \tag{7.4.9}$$

Applying (7.4.3), (7.4.6), (7.4.8), and (7.4.9), we deduce that $\mathcal{A}$ defined by

$$\mathcal{A}(u)(t) = \mathfrak{T}(t)u_+ + \mathcal{J}(u)(t)\,,$$

maps E to itself if S is large enough. One easily verifies with similar estimates that if S is large enough, then

$$\mathrm{d}(\mathcal{A}(u),\mathcal{A}(v)) \le \frac{1}{2}\mathrm{d}(u,v) \quad \text{for all } u,v\in E. \tag{7.4.10}$$

Applying Banach's fixed-point theorem, we deduce that $\mathcal{A}$ has a fixed point $u\in E$ that satisfies equation (7.4.2) on $[S,\infty)$. It follows from (7.4.5), (7.4.7), Strichartz's estimates, and Corollary 2.5.4 that $u\in C([S,\infty),H^1(\mathbb{R}^N))$ and that $(x+2it\nabla)u\in C([S,\infty),L^2(\mathbb{R}^N))$. In particular, $\psi=u(S)\in\Sigma$. Note also that

$$u(t+S)=\mathcal{T}(t)\psi+i\int_0^t \mathcal{T}(t-s)g(u(s+S))ds.$$

Therefore, u is the solution of the problem

$$\begin{cases} iu_t+\Delta u+g(u)=0\\ u(S)=\psi. \end{cases}$$

Note that, by Remark 6.8.1, the solution u is global. In particular, $u(0)$ is well defined, and by Proposition 6.5.1, $u(0)\in\Sigma$. It follows from equation (7.4.2) that

$$\mathcal{T}(-t)u(t)-u^+=-i\int_t^\infty \mathcal{T}(s)g(u(s))ds.$$

Since $u\in E$, it is not difficult to show with the above estimates that $\|\mathcal{T}(-t)u(t)-u^+\|_\Sigma\to 0$ as $t\to\infty$. Therefore, $\varphi=u(0)$ satisfies the conclusions of the theorem.

It remains to show uniqueness. Let $\varphi_1,\varphi_2\in\Sigma$, let u_1 and u_2 be the corresponding solutions of (4.1.1), and assume that $\|\mathcal{T}(-t)u_j(t)-u^+\|_\Sigma\to 0$ as $t\to\infty$ for $j=1,2$. It follows from Remark 7.4.3 that u_j is a solution of (7.4.2). Furthermore, it follows from Theorem 7.3.1 and Corollary 7.3.4 that $u_j\in L^q(\mathbb{R},W^{1,r}(\mathbb{R}^N))$, $(x+2it\nabla)u_j\in L^q(\mathbb{R},L^r(\mathbb{R}^N))$, and that $|t|^{2/q}\|u_j(t)\|_{L^r}$ is bounded. In a similar way to the proof of (7.4.10), one obtains that $u_1(t)=u_2(t)$ for t sufficiently large. By uniqueness for the Cauchy problem at finite time, we conclude that $\varphi_1=\varphi_2$. □

REMARK 7.4.5. It follows from Theorem 7.4.4 that the wave operators $\Omega_+: u^+\mapsto\varphi$ and $\Omega_-: u^-\mapsto\varphi$ are well-defined $\Sigma\to\Sigma$. In fact, one can show with similar estimates that Ω_+ and Ω_- are continuous. By Theorems 7.4.1 and 7.4.4, $U_\pm\Omega_\pm=\Omega_\pm U_\pm=I$ on Σ, where $U_\pm$ is defined by Remark 7.4.2. In particular, $\Omega_\pm:\Sigma\to\Sigma$ is one-to-one with continuous inverse $(\Omega_\pm)^{-1}=U_\pm$.

THEOREM 7.4.6. *Assume* (7.3.1)–(7.3.2). *Assume further that* $\alpha>\alpha_0$, *where* α_0 *is defined by* (6.3.3), *and let* Σ *be the Hilbert space defined by* (6.7.3)–(6.7.4). *For every* $u^-\in\Sigma$, *there exists a unique* $u^+\in\Sigma$ *with the following property. There exists (a unique)* $\varphi\in\Sigma$ *such that the maximal solution* $u\in C(\mathbb{R},\Sigma)$ *of* (4.1.1) *satisfies* $\mathcal{T}(-t)u(t)\to u^\pm$ *in* Σ *as* $t\to\pm\infty$. *The scattering operator*

$$\mathbf{S}:\Sigma\to\Sigma \quad \textit{mapping} \quad u^-\mapsto u^+$$

is continuous, one-to-one, and its inverse is continuous $\Sigma\to\Sigma$. *In addition,* $\|u^+\|_{L^2}=\|u^-\|_{L^2}$ *and* $\|\nabla u^+\|_{L^2}=\|\nabla u^-\|_{L^2}$ *for every* $u^-\in\Sigma$.

PROOF. The result follows from Theorems 7.4.1 and 7.4.4 and Remark 7.4.5, by setting $\mathbf{S}=U_+\Omega_-$. Note that $\mathbf{S}^{-1}=U_-\Omega_+$. □

REMARK 7.4.7. Since $\mathcal{T}(t)$ is an isometry of $H^1(\mathbb{R}^N)$, the property $\|\mathcal{T}(-t)u(t)-u^{\pm}\|_{H^1} \to 0$ is equivalent to $\|u(t)-\mathcal{T}(t)u^{\pm}\|_{H^1} \to 0$. In general, it is not known whether the property $\|\mathcal{T}(-t)u(t)-u^{\pm}\|_{\Sigma} \to 0$ is equivalent to $\|u(t)-\mathcal{T}(t)u^{\pm}\|_{\Sigma} \to 0$. On this question, see Bégout [18].

7.5. Applications of the Pseudoconformal Transformation

In this section we consider equation (4.1.1) in the model case

$$g(u) = \lambda |u|^\alpha u, \tag{7.5.1}$$

where

$$\lambda \in \mathbb{R}, \quad 0 < \alpha < \frac{4}{N-2} \quad (0 < \alpha < \infty \text{ if } N = 1). \tag{7.5.2}$$

We complete the results of the preceding section by using the pseudoconformal transformation. More precisely, we apply the pseudoconformal transformation (6.7.6) with $b < 0$, and we suppose for convenience that $b = -1$. Moreover, throughout this section we systematically consider the variables $(s, y) \in \mathbb{R} \times \mathbb{R}^N$ defined by

$$s = \frac{t}{1-t}, \quad y = \frac{x}{1-t}, \quad \text{or, equivalently,} \quad t = \frac{s}{1+s}, \quad x = \frac{y}{1+s}.$$

Given $0 \le a < b \le \infty$ and u defined on $(a, b) \times \mathbb{R}^N$, we set

$$v(t,x) = (1-t)^{-\frac{N}{2}} u\Big(\frac{t}{1-t}, \frac{x}{1-t}\Big) e^{-i\frac{|x|^2}{4(1-t)}} \tag{7.5.3}$$

for $x \in \mathbb{R}^N$ and $\frac{a}{1+a} < t < \frac{b}{1+b}$. In particular, if u is defined on $(0, \infty)$, then v is defined on $(0, 1)$. Transformation (7.5.3) can also be written, using the variables (s, y), as

$$v(t,x) = (1+s)^{\frac{N}{2}} u(s,y) e^{-i\frac{|y|^2}{4(1+s)}}. \tag{7.5.4}$$

One easily verifies that $u \in C([a,b], \Sigma)$ if and only if $v \in C([\frac{a}{1+a}, \frac{b}{1+b}], \Sigma)$ ($0 \le a < b < \infty$ are given), where the space Σ is defined by (6.7.3)–(6.7.4).

Furthermore, a straightforward calculation (see Theorem 6.7.1) shows that u satisfies (4.1.1) on (a, b) if and only if v satisfies the equation

$$iv_t + \Delta v + \lambda (1-t)^{\frac{N\alpha-4}{2}} |v|^\alpha v = 0 \tag{7.5.5}$$

on the interval $(\frac{a}{1+a}, \frac{b}{1+b})$. Note that the term $(1-t)^{\frac{N\alpha-4}{2}}$ is regular, except possibly at $t = 1$, where it is singular for $\alpha < 4/N$. Furthermore, the following identities hold (see (6.7.11)–(6.7.13)):

$$\|v(t)\|_{L^{\beta+2}}^{\beta+2} = (1+s)^{\frac{N\beta}{2}} \|u(s)\|_{L^{\beta+2}}^{\beta+2}, \quad \beta \ge 0, \tag{7.5.6}$$

$$\|\nabla v(t)\|_{L^2}^2 = \frac{1}{4} \|(y + 2i(1+s)\nabla) u(s)\|_{L^2}^2, \tag{7.5.7}$$

$$\|\nabla u(s)\|_{L^2}^2 = \frac{1}{4} \|(x - 2i(1-t)\nabla) v(t)\|_{L^2}^2. \tag{7.5.8}$$

It follows from (7.5.6) and conservation of charge for (4.1.1) that

$$\frac{d}{dt} \|v(t)\|_{L^2} = 0. \tag{7.5.9}$$

Moreover, if we set

$$\begin{aligned} E_1(t) &= \frac{1}{2}\|\nabla v(t)\|_{L^2}^2 - (1-t)^{\frac{N\alpha-4}{2}} \frac{\lambda}{\alpha+2}\|v(t)\|_{L^{\alpha+2}}^{\alpha+2}, \\ E_2(t) &= (1-t)^{\frac{4-N\alpha}{2}} E_1(t) \\ &= (1-t)^{\frac{4-N\alpha}{2}} \frac{1}{2}\|\nabla v(t)\|_{L^2}^2 - \frac{\lambda}{\alpha+2}\|v(t)\|_{L^{\alpha+2}}^{\alpha+2}, \\ E_3(t) &= \frac{1}{8}\|(x-2i(1-t)\nabla)v(t)\|_{L^2}^2 - (1-t)^{\frac{N\alpha}{2}} \frac{\lambda}{\alpha+2}\|v(t)\|_{L^{\alpha+2}}^{\alpha+2}, \end{aligned}$$

it follows that

$$\frac{d}{dt}E_1(t) = -(1-t)^{\frac{N\alpha-6}{2}} \frac{4-N\alpha}{2} \frac{\lambda}{\alpha+2}\|v(t)\|_{L^{\alpha+2}}^{\alpha+2}, \tag{7.5.10}$$

$$\frac{d}{dt}E_2(t) = (1-t)^{\frac{2-N\alpha}{2}} \frac{N\alpha-4}{4}\|\nabla v(t)\|_{L^2}^2, \tag{7.5.11}$$

$$\frac{d}{dt}E_3(t) = 0. \tag{7.5.12}$$

Indeed, (7.5.10) and (7.5.11) are equivalent, and both are equivalent to the pseudoconformal conservation law for u, by (7.5.6) and (7.5.7). Similarly, the identity (7.5.12) is equivalent to the conservation of energy for u, by using (7.5.6) and (7.5.8).

The results that we present in this section are based on the following observation.

PROPOSITION 7.5.1. *Assume* (7.5.1)–(7.5.2) *and let* Σ *be defined by equations* (6.7.3)–(6.7.4). *Let* $u \in C([0,\infty),\Sigma)$ *be a solution of equation* (4.1.1) *and let* $v \in C([0,1),\Sigma)$ *be the corresponding solution of* (7.5.5) *defined by* (7.5.3). *It follows that* $\mathcal{T}(-s)u(s)$ *has a strong limit in* Σ *(respectively, in* $L^2(\mathbb{R}^N)$*) as* $s \to \infty$ *if and only if* $v(t)$ *has a strong limit in* Σ *(respectively, in* $L^2(\mathbb{R}^N)$*) as* $t \uparrow 1$*, in which case*

$$\lim_{s\to\infty} \mathcal{T}(-s)u(s) = e^{i\frac{|x|^2}{4}}\mathcal{T}(-1)v(1) \quad \text{in } \Sigma \quad (\text{respectively, in } L^2(\mathbb{R}^N)). \tag{7.5.13}$$

PROOF. We define the dilation D_β, $\beta > 0$, by $D_\beta u(x) = \beta^{\frac{N}{2}}u(\beta x)$ and the multiplier M_σ, $\sigma \in \mathbb{R}$, by $M_\sigma u(x) = e^{i\frac{\sigma|x|^2}{4}}u(x)$. With this notation, and using the explicit kernel

$$\mathcal{T}(t)u = \frac{1}{(4\pi i t)^{\frac{N}{2}}}\int_{\mathbb{R}^N} e^{i\frac{|x-y|^2}{4t}}u(y)dy,$$

elementary calculations show that

$$\mathcal{T}(\tau)D_\beta = D_\beta\mathcal{T}(\beta^2\tau) \quad \text{for all } \tau \in \mathbb{R} \text{ and } \beta > 0, \tag{7.5.14}$$

and that

$$\mathcal{T}(\tau)M_\sigma = M_{\frac{\sigma}{1+\sigma\tau}}D_{\frac{1}{1+\sigma\tau}}\mathcal{T}\Big(\frac{\tau}{1+\sigma\tau}\Big) \quad \text{for all } \tau,\sigma \in \mathbb{R} \tag{7.5.15}$$

such that $1+\sigma\tau > 0$. We note that by (7.5.3),

$$v(t) = M_{-\frac{1}{1-t}}D_{\frac{1}{1-t}}u\Big(\frac{t}{1-t}\Big) \quad \text{for } 0 \le t < 1.$$

Therefore, we deduce from (7.5.14) and (7.5.15) that

$$\mathcal{T}(-t)v(t) = M_{-1}\mathcal{T}\Big(-\frac{t}{1-t}\Big)u\Big(\frac{t}{1-t}\Big) \quad \text{for all } t \in [0,1),$$

which we rewrite in the form

$$\mathcal{T}(-s)u(s) = e^{i\frac{|x|^2}{4}}\mathcal{T}\Big(-\frac{s}{1+s}\Big)v\Big(\frac{s}{1+s}\Big).$$

Hence the result is established. □

The following result implies that if $\alpha \le 2/N$, then no scattering theory can be developed for equation (4.1.1) (see Barab [17], Strauss [322, 325], and Tsutsumi and Yajima [348]).

THEOREM 7.5.2. *Assume* (7.5.1)–(7.5.2) *and let* Σ *be defined by* (6.7.3)–(6.7.4). *Assume further*

$$\alpha \le \frac{2}{N} \quad (\alpha \le 1 \textit{ if } N = 1).$$

Let $\varphi \in \Sigma$ *and let* $u \in C(\mathbb{R}, \Sigma)$ *be the corresponding solution of* (4.1.1). *If* $\varphi \ne 0$, *then* $\mathcal{T}(-t)u(t)$ *does not have any strong limit in* $L^2(\mathbb{R}^N)$ *as either* $t \to \infty$ *or* $t \to -\infty$. *In other words, no nontrivial solution of* (4.1.1) *has scattering states, even for the* $L^2(\mathbb{R}^N)$ *topology.*

PROOF. We consider the case $t \to \infty$, the argument for $t \to -\infty$ being the same. We argue by contradiction and we assume $\mathcal{T}(-t)u(t) \underset{t\to\infty}{\longrightarrow} u_+$ in $L^2(\mathbb{R}^N)$. In particular,

$$\|u_+\|_{L^2} = \|u(t)\|_{L^2} = \|\varphi\|_{L^2} > 0.$$

On the other hand, we deduce from (7.5.13) that $v(t) \underset{t\uparrow 1}{\longrightarrow} w$ in $L^2(\mathbb{R}^N)$ with

$$w = \mathcal{T}(1)\big(e^{-i\frac{|x|^2}{4}}u_+\big) \ne 0.$$

Since $\alpha + 1 \le 2$, we have

$$|v(t)|^\alpha v(t) \underset{t\uparrow 1}{\longrightarrow} |w|^\alpha w \ne 0 \quad \text{in } L^{\frac{2}{\alpha+1}}(\mathbb{R}^N).$$

Let $\theta \in \mathcal{D}(\mathbb{R}^N)$ be such that

(7.5.16) $$\langle i|w|^\alpha w, \theta\rangle = 1.$$

It follows from (7.5.5) that

$$\begin{aligned}\frac{d}{dt}\langle v(t),\theta\rangle &= \langle i\Delta v,\theta\rangle + \lambda(1-t)^{\frac{N\alpha-4}{2}}\langle i|v|^\alpha v,\theta\rangle \\ &= \langle iv,\Delta\theta\rangle + \lambda(1-t)^{\frac{N\alpha-4}{2}}\langle i|v|^\alpha v,\theta\rangle.\end{aligned}$$

Therefore, by (7.5.16), and since v is bounded in $L^2(\mathbb{R}^N)$,

$$\Big|\frac{d}{dt}\langle v(t),\theta\rangle\Big| \ge \frac{1}{2}|\lambda|(1-t)^{\frac{N\alpha-4}{2}} - C \quad \text{for } 1-\varepsilon \le t < 1,\ \varepsilon > 0 \text{ small enough.}$$

Since $(N\alpha - 4)/2 \le -1$, it follows that $|\langle v(t),\theta\rangle| \to \infty$ as $t \uparrow 1$, which is absurd. □

REMARK 7.5.3. In the case $N = 1$ and $1 < \alpha \le 2$, we have the following result. Let $\varphi \in \Sigma$ and let $u \in C(\mathbb{R}, \Sigma)$ be the corresponding solution of (4.1.1) with $g(u) = \lambda|u|^\alpha u$. If $\varphi \neq 0$, then $\mathcal{T}(-t)u(t)$ does not have any strong limit in Σ as either $t \to \infty$ or $t \to -\infty$. The proof is similar. One needs only observe that, since $v(t)$ is bounded in Σ (hence in $H^1(\mathbb{R})$), $v(t) \to w$ as $t \uparrow 1$ in $L^p(\mathbb{R})$ for every $2 \le p \le \infty$, and so $|v(t)|^\alpha v(t) \to |w|^\alpha w$ as $t \uparrow 1$ in $L^2(\mathbb{R})$.

If $\alpha > 2/N$ and if $\lambda \le 0$, then every solution in Σ of (4.1.1) with $g(u) = \lambda|u|^\alpha u$ has scattering states in $L^2(\mathbb{R}^N)$, as the following result shows (see Tsutsumi and Yajima [348]).

THEOREM 7.5.4. *Assume* (7.5.1)–(7.5.2) *and let* Σ *be defined by* (6.7.3)–(6.7.4). *Let* $\varphi \in \Sigma$ *and let* $u \in C(\mathbb{R}, \Sigma)$ *be the corresponding solution of* (4.1.1). *If* $\lambda \le 0$, *then there exist* $u_\pm \in L^2(\mathbb{R}^N)$ *such that*

$$\mathcal{T}(-t)u(t) \underset{t \to \pm\infty}{\longrightarrow} u_\pm \quad \text{in } L^2(\mathbb{R}^N).$$

REMARK 7.5.5. Here are some comments on Theorem 7.5.4.

(i) If $\alpha > \alpha_0$, then it follows from Theorem 7.4.1 that $u_\pm \in \Sigma$ and that the convergence holds in Σ. The same conclusion holds in some other situations: if $\alpha = \alpha_0$, see Theorem 7.5.11; if $\alpha > 4/(N+2)$ ($\alpha > 2$ if $N = 1$) and if $\|\varphi\|_\Sigma$ is small enough, see Theorem 7.5.7. On the other hand, if $\alpha \le 4/(N+2)$, or if $\alpha < \alpha_0$ and $\|\varphi\|_\Sigma$ is large, then we do not know whether $u_\pm \in \Sigma$.

(ii) Theorem 7.5.4 does not apply to the case $\lambda > 0$. In fact, if $\alpha < 4/(N+2)$, there are arbitrarily small initial values $\varphi \in \Sigma$ that do not have a scattering state, even in the sense of $L^2(\mathbb{R}^N)$. To see this, let $\varphi \in \Sigma$ be a nontrivial solution of the equation

$$-\Delta\varphi + \varphi = \lambda|\varphi|^\alpha\varphi$$

(see Chapter 8). Given $\omega > 0$, set $\varphi_\omega(x) = \omega^{\frac{1}{\alpha}}\varphi(x\sqrt{\omega})$. It follows that $-\Delta\varphi_\omega + \omega\varphi_\omega = \lambda|\varphi_\omega|^\alpha\varphi_\omega$. Therefore, $u_\omega(t,x) = e^{i\omega t}\varphi_\omega(x)$ satisfies (4.1.1) and $\mathcal{T}(-t)u_\omega(t) = e^{i\omega t}\mathcal{T}(-t)\varphi_\omega$ does not have any strong limit as $t \to \infty$ in $L^2(\mathbb{R}^N)$. On the other hand, one easily verifies that if $\alpha < 4/(N+2)$, then $\|u_\omega\|_\Sigma \to 0$ as $\omega \downarrow 0$. However, we will see below (Theorem 7.5.7) that if $\alpha > 4/(N+2)$, then small initial values in Σ have scattering states in Σ at $\pm\infty$.

PROOF OF THEOREM 7.5.4. By Proposition 7.5.1, we need only show that $v(t)$ has a strong limit in $L^2(\mathbb{R}^N)$ as $t \uparrow 1$. As observed above (Remark 7.5.5), there is a better result when $\alpha > \alpha_0$. Therefore, we may assume that $\alpha \le \alpha_0$, and in particular $\alpha < 4/N$. Therefore, it follows from (7.5.9) and (7.5.11) that

$$\|v(t)\|_{L^2} \le C, \tag{7.5.17}$$

$$\|v(t)\|_{L^{\alpha+2}} \le C, \tag{7.5.18}$$

$$\|\nabla v(t)\|_{L^2} \le C(1-t)^{\frac{N\alpha-4}{4}}, \tag{7.5.19}$$

for all $t \in [0,1)$. By using the embeddings $L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N) \hookrightarrow H^{-2}(\mathbb{R}^N)$ and equation (7.5.5), we obtain

$$\begin{aligned}\|v_t\|_{H^{-2}} &\le \|\Delta v\|_{H^{-2}} + C(1-t)^{\frac{N\alpha-4}{2}} \| |v|^\alpha v\|_{H^{-2}} \\ &\le C\|v\|_{L^2} + C(1-t)^{\frac{N\alpha-4}{2}} \|v\|_{L^{\alpha+2}}^{\alpha+1}.\end{aligned}$$

Therefore,

$$\|v_t\|_{H^{-2}} \le C + C(1-t)^{\frac{N\alpha-4}{2}},$$

by (7.5.17) and (7.5.18). It follows that $v_t \in L^1((0,1), H^{-2}(\mathbb{R}^N))$. Hence, there exists $w \in H^{-2}(\mathbb{R}^N)$ such that $v(t) \to w$ in $H^{-2}(\mathbb{R}^N)$ as $t \uparrow 1$. By using (7.5.17) again, we obtain that $w \in L^2(\mathbb{R}^N)$ and

$$v(t) \rightharpoonup w \quad \text{in } L^2(\mathbb{R}^N) \text{ as } t \uparrow 1. \tag{7.5.20}$$

Consider now $\psi \in H^1(\mathbb{R}^N)$, and let $0 \le t \le \tau < 1$. We deduce from (7.5.5) that

$$\begin{aligned}(v(\tau) - v(t), \psi)_{L^2} &= \int_t^\tau \langle v_t, \psi \rangle_{H^{-1},H^1}\, ds \\ &= \int_t^\tau (i\nabla v, \nabla \psi)_{L^2}\, ds + \int_t^\tau (1-s)^{\frac{N\alpha-4}{2}} \langle i\lambda |v|^\alpha v, \psi\rangle_{L^{\frac{\alpha+2}{\alpha+1}}, L^{\alpha+2}}\, ds,\end{aligned}$$

and so

$$\begin{aligned}|(v(\tau) - v(t), \psi)_{L^2}| &\le C\|\nabla \psi\|_{L^2} \int_t^\tau \|\nabla v\|_{L^2}\, ds \\ &\quad + C\|\psi\|_{L^{\alpha+2}} \int_t^\tau (1-s)^{\frac{N\alpha-4}{2}} \|v\|_{L^{\alpha+2}}^{\alpha+1}\, ds \\ &\le C\|\nabla\psi\|_{L^2} \int_t^\tau (1-s)^{\frac{N\alpha-4}{4}}\, ds + C\|\psi\|_{L^{\alpha+2}} \int_t^\tau (1-s)^{\frac{N\alpha-4}{2}}\, ds,\end{aligned}$$

by (7.5.19) and (7.5.18). Letting $\tau \uparrow 1$ and applying (7.5.20), we obtain

$$\begin{aligned}|(w - v(t), \psi)_{L^2}| &\le C\|\nabla\psi\|_{L^2} \int_t^1 (1-s)^{\frac{N\alpha-4}{4}}\, ds + C\|\psi\|_{L^{\alpha+2}} \int_t^1 (1-s)^{\frac{N\alpha-4}{2}}\, ds \\ &\le C(1-t)^{\frac{N\alpha}{4}} \|\nabla\psi\|_{L^2} + C(1-t)^{\frac{N\alpha-2}{2}} \|\psi\|_{L^{\alpha+2}}.\end{aligned}$$

We now let $\psi = v(t)$ and we apply again (7.5.19) and (7.5.18). It follows that

$$\begin{aligned}|(w - v(t), v(t))_{L^2}| &\le C(1-t)^{\frac{N\alpha}{4}} (1-t)^{\frac{N\alpha-4}{4}} + C(1-t)^{\frac{N\alpha-2}{2}} \\ &\le C(1-t)^{\frac{N\alpha-2}{2}} \underset{t\uparrow 1}{\longrightarrow} 0.\end{aligned} \tag{7.5.21}$$

Finally,

$$\|v(t) - w\|_{L^2}^2 = -(w - v(t), v(t))_{L^2} + (w - v(t), w)_{L^2} \underset{t\uparrow 1}{\longrightarrow} 0,$$

by (7.5.21) and (7.5.20). This completes the proof. □

Besides the fact that Theorem 7.5.4 does not apply to the case $\lambda > 0$, neither does it allow us to construct the wave and scattering operators, since the initial value φ and the scattering states $u_\pm$ do not belong to the same space. We will improve this result under more restrictive assumptions on α by solving the initial-value problem for the nonautonomous equation (7.5.5) and by applying Proposition 7.5.1 which relates the behavior of u at infinity and the behavior of v at $t = 1$.

However, we want to solve the Cauchy problem for (7.5.5) starting from any time $t \in [0, 1]$, including $t = 1$ where the nonautonomous term might be singular. In order to do this, define the function

$$f(t) = \begin{cases} \lambda(1-t)^{\frac{N\alpha-4}{2}} & \text{if } -\infty < t < 1 \\ \lambda & \text{if } t \geq 1, \end{cases} \tag{7.5.22}$$

and consider the equation

$$iv_t + \Delta u + f(t)|v|^\alpha v = 0. \tag{7.5.23}$$

Under appropriate assumptions on α, the initial-value problem for (7.5.23) can be solved starting from any time $t \in \mathbb{R}$, and we have the following result.

THEOREM 7.5.6. *Assume* (7.5.1)–(7.5.2) *and let* Σ *be defined by* (6.7.3)–(6.7.4). *Assume further that*

$$\alpha > \frac{4}{N+2} \quad (\alpha > 2 \text{ if } N = 1). \tag{7.5.24}$$

It follows that for every $t_0 \in \mathbb{R}$ *and* $\psi \in \Sigma$, *there exist* $T_m(t_0, \psi) < t_0 < T_M(t_0, \psi)$ *and a unique, maximal solution* $v \in C((T_m, T_M), \Sigma)$ *of equation* (7.5.23). *The solution* v *is maximal in the sense that if* $T_M < \infty$ (*respectively,* $T_m > -\infty$), *then* $\|u(t)\|_{H^1} \to \infty$ *as* $t \uparrow T_M$ (*respectively,* $t \downarrow T_m$). *In addition, the solution* v *has the following properties.*

(i) *If* $T_M = 1$, *then* $\liminf_{t\uparrow 1}\{(1-t)^\delta \|v(t)\|_{H^1}\} > 0$ *with* $\delta = \frac{N+2}{4} - \frac{1}{\alpha}$ *if* $N \geq 3$, $\delta < 1 - \frac{1}{\alpha}$ *if* $N = 2$, *and* $\delta = \frac{1}{2} - \frac{1}{\alpha}$ *if* $N = 1$.

(ii) *The solution* v *depends continuously on* ψ *in the following way. The mapping* $\psi \mapsto T_M$ *is lower semicontinuous* $\Sigma \to (0, \infty]$, *and the mapping* $\psi \mapsto T_m$ *is upper semicontinuous* $\Sigma \to [-\infty, 0)$. *In addition, if* $\psi_n \to \psi$ *in* Σ *as* $n \to \infty$ *and if* $[S, T] \in (T_m, T_M)$, *then* $v_n \to v$ *in* $C([S, T], \Sigma)$, *where* v_n *denotes the solution of* (7.5.23) *with initial value* ψ_n.

PROOF. The result follows by applying Theorems 4.11.1 and 4.11.2 with $h(t) = f(t - t_0)$. □

We now give some applications of Theorem 7.5.6 to the scattering theory in Σ for (4.1.1).

THEOREM 7.5.7. *Assume* (7.5.1)–(7.5.2) *and* (7.5.24). *Let* Σ *be the space defined by* (6.7.3)–(6.7.4). *With the notation of Section* 7.1 (*corresponding to the* Σ *topology*), *the following properties hold:*

(i) *The sets* $\mathcal{R}_\pm$ *and* $\mathcal{U}_\pm$ *are open subsets of* Σ *containing* 0. *The operators* $U_\pm : \mathcal{R}_\pm \to \mathcal{U}_\pm$ *are bicontinuous bijections* (*for the* Σ *topology*) *and the operators* $\Omega_\pm : \mathcal{U}_\pm \to \mathcal{R}_\pm$ *are bicontinuous bijections* (*for the* Σ *topology*).

(ii) *The sets* $\mathcal{O}_\pm$ *are open subsets of* Σ *containing* 0, *and the scattering operator* $\mathbf{S}$ *is a bicontinuous bijection* $\mathcal{O}_- \to \mathcal{O}_+$ (*for the* Σ *topology*).

REMARK 7.5.8. Theorem 7.5.7 implies that there is a "low energy" scattering theory (i.e., scattering theory for small initial data) in Σ for the equation (4.1.1) with $g(u) = \lambda|u|^\alpha u$, provided $4/(N+2) < \alpha < 4/(N-2)$ ($2 < \alpha < \infty$, if $N = 1$). As observed before (see Remark 7.5.5), if $\alpha < 4/(N+2)$ and $\lambda > 0$, then there is no low energy scattering.

PROOF OF THEOREM 7.5.7. Let $\varphi \in \Sigma$ and let u be the corresponding solution of (4.1.1) with $g(u) = \lambda|u|^\alpha u$. Let v be the solution of (7.5.5) with the initial value ψ defined by $\psi(x) = \varphi(x)e^{-i\frac{|x|^2}{4}}$ (see Theorem 7.5.6). The solution v is defined by (7.5.4), as long as (7.5.4) makes sense. Therefore, it follows from Proposition 7.5.1 and Theorem 7.5.6 that $\varphi \in \mathcal{R}_+$ if and only if $T_{\mathrm{M}}(0,\psi) > 1$, and that in this case $u_+ = e^{i\frac{|x|^2}{4}}\mathcal{T}(-1)v(1)$. Therefore, the open character of $\mathcal{R}_+$ and the continuity of the operator U_+ follow from the continuous dependence of v on ψ (property (ii) of Theorem 7.5.6).

Let now $y \in \Sigma$, and set $w = \mathcal{T}(1)(e^{-i\frac{|x|^2}{4}}y)$, so that $y = e^{i\frac{|x|^2}{4}}\mathcal{T}(-1)w$. It follows from Proposition 7.5.1 and Theorem 7.5.6 that $y = U_+\varphi$ for some $\varphi \in \Sigma$ (i.e., $y \in \mathcal{U}_+$) if and only if $T_m(1,w) < 0$. In this case, $\varphi = e^{i\frac{|x|^2}{4}}z(0)$, where z is the solution of (7.5.5) with the initial value $z(1) = w$. Thus φ is uniquely determined and the operator U_+ is injective. Furthermore, the open character of $\mathcal{U}_+$ and the continuity of the operator $\Omega_+ = (U_+)^{-1}$ follow as above from the continuous dependence of z on w.

As observed in Remark 7.1.3, the similar statements for $\mathcal{R}_-$, $\mathcal{U}_-$, U_-, and Ω_- are equivalent, by changing t to $-t$ and $u(t)$ to $\overline{u}(-t)$. Therefore, we have proved part (i) of the theorem. Part (ii) now follows from part (i) and the definitions of $\mathcal{O}_\pm$ and $\mathbf{S}$ (formulae (7.1.8) and (7.1.9)). ☐

We now establish further properties of the wave operators $\Omega_\pm$.

THEOREM 7.5.9. *Assume* (7.5.1)–(7.5.2) *and* (7.5.24). *Let Σ be the space defined by* (6.7.3)–(6.7.4). *With the notation of Section* 7.1 (*corresponding to the Σ topology*), *the following properties hold:*

(i) *If $\lambda < 0$, then $\mathcal{U}_\pm = \Sigma$. Therefore, the wave operators $\Omega_\pm$ are bicontinuous bijections $\Sigma \to \mathcal{R}_\pm$.*

(ii) *If $\lambda > 0$ and $\alpha < 4/N$, then $\mathcal{U}_\pm = \Sigma$. Therefore, the wave operators $\Omega_\pm$ are bicontinuous bijections $\Sigma \to \mathcal{R}_\pm$.*

PROOF. Assume $\lambda < 0$, or $\lambda > 0$ and $\alpha < 4/N$. Let $w \in \Sigma$, and let z be the solution of (7.5.5) with the initial value $z(1) = w$. By Theorem 7.5.6, z is defined on some interval $[1-\varepsilon, 1]$ with $\varepsilon > 0$. Set

$$\phi(y) = \varepsilon^{\frac{N}{2}} e^{i\frac{\varepsilon|y|^2}{4}} z(1-\varepsilon, \varepsilon y) \in \Sigma\,.$$

Let u be the solution of equation (4.1.1) with the initial value

$$u\Big(\frac{1-\varepsilon}{\varepsilon}\Big) = \phi. \tag{7.5.25}$$

Since $\lambda < 0$, or $\lambda > 0$ and $\alpha < 4/N$, we obtain that u is global. Therefore, we may define $\varphi = u(0)$. We claim that $\varphi \in \mathcal{R}_+$ and that $u_+ = e^{i\frac{|x|^2}{4}}\mathcal{T}(-1)w$. Indeed, consider v defined by (7.5.4). We see by applying (7.5.4) with $t = 1-\varepsilon$ and (7.5.25) that

$$v(1-\varepsilon) = z(1-\varepsilon)\,,$$

so that by uniqueness $v \equiv z$, and so the claim follows from Proposition 7.5.1. This completes the proof. □

We now study the asymptotic completeness. We first recover the results of Section 7.4, with a different proof, though.

THEOREM 7.5.10. *Assume* (7.5.1)–(7.5.2) *and let* Σ *be defined by* (6.7.3)–(6.7.4). *If* $\lambda < 0$ *and* $\alpha > \alpha_0$ *with* α_0 *defined by* (6.3.3), *then* $\mathcal{U}_\pm = \mathcal{R}_\pm = \Sigma$. *In particular,* $U_\pm$, $\Omega_\pm$, *and* $\mathbf{S}$ *are bicontinuous bijections* $\Sigma \to \Sigma$. *Here, we use the notation of Section* 7.1 (*corresponding to the* Σ *topology*).

PROOF. By Theorem 7.5.9 and Remark 7.1.3, we need only show that $\mathcal{R}_+ = \Sigma$. Let $\varphi \in \Sigma$, let u be the solution of (4.1.1), and let v be defined by (7.5.4). If $\alpha \ge 4/N$, then it follows from (7.5.10) that $E_1(t)$ is nonincreasing, which implies that $\|\nabla v(t)\|_{L^2}$ is bounded as $t \uparrow 1$. Since $\|v(t)\|_{L^2}$ is also bounded by (7.5.9), we deduce that $\|v(t)\|_{H^1}$ is bounded as $t \uparrow 1$. By Theorem 7.5.6, this implies that $v(t)$ has a limit as $t \uparrow 1$, and so $\varphi \in \mathcal{R}_+$ by Proposition 7.5.1. We now assume $\alpha < 4/N$. It follows from (7.5.11) that $\|v(t)\|_{L^{\alpha+2}}$ remains bounded as $t \uparrow 1$. Set $r = \alpha + 2$, and let (q,r) be the corresponding admissible pair. Given $0 \le t_0 \le t < 1$, it follows from equation (7.5.23) and Strichartz's estimates that

$$\|v\|_{L^\infty((t_0,t),H^1)} + \|v\|_{L^q((t_0,t),W^{1,r})} \le C\|v(t_0)\|_{H^1} + C\|f|v|^\alpha v\|_{L^{q'}((t_0,t),W^{1,r'})}. \tag{7.5.26}$$

Since

$$\||v|^\alpha v\|_{W^{1,r'}} \le C\|v\|_{L^{\alpha+2}}^\alpha \|v\|_{W^{1,r}} \le C\|v\|_{W^{1,r}}\,,$$

we deduce from Hölder's inequality that

$$\|f|v|^\alpha v\|_{L^{q'}((t_0,t),W^{1,r'})} \le C\|f\|_{L^{\frac{q}{q-2}}(t_0,t)}\|v\|_{L^q((t_0,t),W^{1,r})}\,.$$

Since $\alpha > \alpha_0$, $f \in L^{\frac{q}{q-2}}(0,1)$. Therefore, if we choose t_0 sufficiently close to 1 so that $C\|f\|_{L^{\frac{q}{q-2}}(t_0,1)} \le 1/2$, then we deduce from (7.5.26) that

$$\|v\|_{L^\infty((t_0,t),H^1)} + \|v\|_{L^q((t_0,t),W^{1,r})} \le 2C\|v(t_0)\|_{H^1} \quad \text{for all } t_0 < t < 1.$$

Therefore, v remains bounded in $H^1(\mathbb{R}^N)$ as $t \uparrow 1$, and one concludes as above. □

Finally, we extend the asymptotic completeness result to the case $\alpha = \alpha_0$ (see Cazenave and Weissler [72]).

THEOREM 7.5.11. *Assume* $N = 1$ *or* $N \ge 3$, *let* Σ *be defined by* (6.7.3)–(6.7.4), *and let* α_0 *be defined by* (6.3.3). *If* $g(u) = \lambda|u|^\alpha u$ *with* $\lambda < 0$ *and* $\alpha = \alpha_0$, *then* $\mathcal{U}_\pm = \mathcal{R}_\pm = \Sigma$. *In particular,* $U_\pm$, $\Omega_\pm$, *and* $\mathbf{S}$ *are bicontinuous bijections* $\Sigma \to \Sigma$. *Here, we use the notation of Section* 7.1 (*corresponding to the* Σ *topology*).

PROOF. By Theorem 7.5.9 and Remark 7.1.3, we need only show that $\mathcal{R}_+ = \Sigma$. Let $\varphi \in \Sigma$, let $u \in C(\mathbb{R},\Sigma)$ be the solution of (4.1.1) with $g(u) = \lambda|u|^\alpha u$, and let v

be defined by (7.5.4). Note that, since u is defined on $[0,\infty)$, v is defined on $[0,1)$. By Proposition 7.5.1, $\varphi \in \mathcal{R}_+$ if $v(t)$ has a limit in Σ as $t \uparrow 1$. Therefore, in view of Theorem 7.5.6, we need only show that

$$\sup_{t\in[0,1)} \|v(t)\|_{H^1} < \infty. \tag{7.5.27}$$

We argue by contradiction and we assume that (7.5.27) does not hold; i.e.,

$$\limsup_{t\uparrow 1} \|v(t)\|_{H^1} = \infty. \tag{7.5.28}$$

We consider separately the cases $N \geq 3$ and $N = 1$.

CASE $N \geq 3$. By (7.5.28) and property (i) of Theorem 7.5.6,

$$\|\nabla v(t)\|_{L^2}^2 \geq \frac{a}{(1-t)^{\frac{N+2}{2}-\frac{2}{\alpha}}}$$

for some constant $a > 0$ and all $t \in [0,1)$. By applying (7.5.11), we obtain

$$\frac{d}{dt}E_2(t) \leq -\frac{b}{(1-t)^{\frac{N\alpha-2}{2}+\frac{N+2}{2}-\frac{2}{\alpha}}}$$

for some constant $b > 0$. Since $\alpha = \alpha_0$, the above inequality means

$$\frac{d}{dt}E_2(t) \leq -\frac{b}{(1-t)},$$

which implies that $E_2(t) \underset{t\uparrow 1}{\longrightarrow} -\infty$. This is absurd, since $E_2(t) \geq 0$. This completes the proof in the case $N \geq 3$.

CASE $N = 1$. The argument is the same as above, except that we first need to improve the lower estimate of the blowup given by property (i) of Theorem 7.5.6. We claim that

$$\|v(t)\|_{H^1} \geq \frac{a}{(1-t)^{\frac{(\alpha-2)(\alpha+4)}{4\alpha}}} \tag{7.5.29}$$

for some constant $a > 0$ and all $t \in [0,1)$. Indeed, note first that by (7.5.11),

$$\frac{d}{dt}E_2(t) \leq 0,$$

and so

$$\sup_{t\in[0,1)} \|v(t)\|_{L^{\alpha+2}} < \infty.$$

Fix $t_0 \in [0,1)$. It follows from equation (7.5.23) and Strichartz's estimates that

$$\|v\|_{L^\infty((t_0,t),H^1)} \leq C\|v(t_0)\|_{H^1} + C\|f|v|^\alpha v\|_{L^1((t_0,t),H^1)} \quad \text{for all } t \in (t_0,1).$$

On the other hand,

$$\||v|^\alpha v\|_{H^1} \leq C\|v\|_{L^\infty}^\alpha \|v\|_{H^1}$$

and, by Gagliardo-Nirenberg's inequality,

$$\|v\|_{L^\infty} \leq C\|v\|_{H^1}^{\frac{2}{\alpha+4}} \|v\|_{L^{\alpha+2}}^{\frac{\alpha+2}{\alpha+4}}.$$

Therefore, we deduce from the above four inequalities that there exists a constant K independent of t_0 and t such that

$$\|v\|_{L^\infty((t_0,t),H^1)} \le K\|v(t_0)\|_{H^1} + K\|f\|_{L^1(t_0,t)}\|v\|_{L^\infty((t_0,t),H^1)}^{\frac{3\alpha+4}{\alpha+4}}. \tag{7.5.30}$$

Now, by (7.5.28), there exists $t_1 \in (t_0, 1)$ such that $\|v\|_{L^\infty((t_0,t_1),H^1)} = (K+1)\|v(t_0)\|_{H^1}$. Letting $t = t_1$ in (7.5.30), we obtain

$$\|v(t_0)\|_{H^1} \le K\left((K+1)\|v(t_0)\|_{H^1}\right)^{\frac{3\alpha+4}{\alpha+4}} \|f\|_{L^1(t_0,t_1)},$$

hence

$$1 \le K(K+1)^{\frac{3\alpha+4}{\alpha+4}} \|v(t_0)\|_{H^1}^{\frac{2\alpha}{\alpha+4}} \|f\|_{L^1(t_0,t_1)}.$$

Since $\|f\|_{L^1(t_0,t_1)} \le \|f\|_{L^1(t_0,1)} \le C(1-t_0)^{\frac{\alpha-2}{2}}$, we obtain (7.5.29). We now conclude exactly as in the case $N \ge 3$. This completes the proof. □

REMARK 7.5.12. Here are some comments concerning Theorem 7.5.11.

(i) The conclusion of Theorem 7.5.11 holds in the case $N = 2$, but the result was established by a different method. See Nakanishi and Ozawa [263].

(ii) If $4/(N+2) < \alpha < \alpha_0$, then we do not know whether $\mathcal{R}_+ = \mathcal{R}_- = \Sigma$. Showing this property amounts to showing that no solution of (7.5.5) can blow up at $t = 1$.

REMARK 7.5.13. Ginibre and Velo [130] extended the construction of the wave operators (Theorem 7.5.7) to a wider range of α's by working in the space $H^s(\mathbb{R}^N) \cap \mathcal{F}(H^s(\mathbb{R}^N))$, where $0 < s < 2$. The lower bound on α for that method is given by

$$\alpha > \max\left\{s-1, \frac{4}{N+2s}, \frac{2}{N}\right\}.$$

If $N \le 3$, one obtains the lower bound $\alpha > 2/N$ by letting $s = 3/2$. If $N \ge 4$, there is still a gap between the admissible values of α and the lower bound $\alpha > 2/N$ for the scattering theory given by Theorem 7.5.2. See also Nakanishi and Ozawa [263] for related results.

7.6. Morawetz's Estimate

This section is devoted to the proof of Morawetz's estimate, which is essential for constructing the scattering operator on the energy space. See Lin and Strauss [230], and Ginibre and Velo [137, 138, 143]. We begin with the following generalized Sobolev's estimates.

LEMMA 7.6.1. *Let $1 \le p < \infty$. If $q < N$ is such that $0 \le q \le p$, then $\frac{|u(\cdot)|^p}{|\cdot|^q} \in L^1(\mathbb{R}^N)$ for every $u \in W^{1,p}(\mathbb{R}^N)$. Furthermore,*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^q}\,dx \le \left(\frac{p}{N-q}\right)^q \|u\|_{L^p}^{p-q}\|\nabla u\|_{L^p}^q \quad \text{for every } u \in W^{1,p}(\mathbb{R}^N). \tag{7.6.1}$$

PROOF. By density and Fatou's lemma, we need only establish (7.6.1) for $u \in \mathcal{D}(\mathbb{R}^N)$. Let $z(x) = |x|^{-q}x$. We have $\nabla \cdot z = (N-q)|x|^{-q}$. Integrating the formula

$$|u|^p \nabla \cdot z = \nabla \cdot (|u|^p z) - p|u|^{p-1}\nabla|u|$$

over the set $\{x \in \mathbb{R}^N; |x| > r > 0\}$, we obtain that

$$(N-q)\int_{\{|x|>r\}} \frac{|u(x)|^p}{|x|^q}\,dx \le p\int_{\{|x|>r\}} \frac{|u(x)|^{p-1}|\nabla u(x)|}{|x|^{q-1}}\,dx\,. \tag{7.6.2}$$

Applying Hölder's inequality, it follows that

$$(N-q)\int_{\{|x|>r\}} \frac{|u(x)|^p}{|x|^q}\,dx \le p\Big(\int_{\{|x|>r\}} \frac{|u(x)|^p}{|x|^q}\,dx\Big)^{\frac{q-1}{q}} \|u\|_{L^p}^{\frac{p-q}{q}} \|\nabla u\|_{L^p}\,.$$

Since $r > 0$ is arbitrary, (7.6.1) follows. □

COROLLARY 7.6.2. *If $N \ge 4$, then $\frac{|u(\cdot)|^2}{|\cdot|^3} \in L^1(\mathbb{R}^N)$ for every $u \in H^2(\mathbb{R}^N)$. Furthermore, there exists C such that*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^3}\,dx \le C\|u\|_{H^2}^2 \quad \textit{for every } u \in H^2(\mathbb{R}^N). \tag{7.6.3}$$

PROOF. Note that it suffices to establish (7.6.3) for $u \in \mathcal{D}(\mathbb{R}^N)$. Applying (7.6.2) with $q = 3$ and $p = 2$, we obtain

$$\begin{aligned}(N-3)\int_{\{|x|>r\}} \frac{|u(x)|^2}{|x|^3}\,dx &\le 2\int_{\{|x|>r\}} \frac{|u(x)|}{|x|}\,\frac{|\nabla u(x)|}{|x|}\,dx \\ &\le 2\Big(\int_{\{|x|>r\}} \frac{|u(x)|^2}{|x|^2}\,dx\Big)^{\frac{1}{2}}\Big(\int_{\{|x|>r\}} \frac{|\nabla u(x)|^2}{|x|^2}\,dx\Big)^{\frac{1}{2}}.\end{aligned}$$

Applying (7.6.1) with $p = q = 2$ to both u and ∇u, we obtain

$$(N-3)\int_{\{|x|>r\}} \frac{|u(x)|^2}{|x|^3}\,dx \le C\|u\|_{H^1}\|\nabla u\|_{H^1} \le C\|u\|_{H^2}^2\,.$$

The result follows by letting $r \downarrow 0$. □

We now assume that $N \ge 3$, and we consider

$$g(u) = Vu + f(u(\cdot)) + (W \star |u|^2)u\,,$$

where V, f, and W are as follows. The potential V is real valued such that $V, \nabla V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $p > N/2$. The function $f : [0,\infty) \to \mathbb{R}$ is continuous and $f(0) = 0$. We assume that there exist constants C and $\alpha \in [0, \frac{4}{N-2})$ such that

$$|f(v) - f(u)| \le C(1 + |u|^\alpha + |v|^\alpha)|v-u| \quad \text{for all } u, v \in \mathbb{R}\,,$$

and we extend f to $\mathbb{C}$ by setting

$$f(z) = \frac{z}{|z|}f(|z|) \quad \text{for all } z \in \mathbb{C},\ z \ne 0\,.$$

We set

$$F(z) = \int_0^{|z|} f(s)ds \quad \text{for all } z \in \mathbb{C}\,.$$

Finally, $W : \mathbb{R}^N \to \mathbb{R}$ is an even, real-valued potential such that $W, \nabla W \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $q \ge 1$, $q > N/4$.

We know (see Section 3.3) that g is the gradient of the potential G defined by

$$G(u) = \int_{\mathbb{R}^N} \left\{ \frac{1}{2} V(x)|u(x)|^2 + F(x, u(x)) + \frac{1}{4}(W \star |u|^2)(x)|u(x)|^2 \right\} dx .$$

Finally, we set

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx - G(u) \quad \text{for all } u \in H^1(\mathbb{R}^N) .$$

For $u \in H^1(\mathbb{R}^N)$, we set

$$h(u) = \frac{1}{2} V_r |u|^2 + \frac{N-1}{2r}\{2F(u) - \overline{u} f(u)\} + \frac{1}{2}|u|^2 \frac{x}{|x|} \cdot (\nabla W \star |u|^2) . \tag{7.6.4}$$

If u is such that $h(u) \in L^1(\mathbb{R}^N)$, we set

$$H(u) = \int_{\mathbb{R}^N} h(u) dx . \tag{7.6.5}$$

We will use the following estimate.

LEMMA 7.6.3. *Let $N \geq 3$, let g be as above, and set $\rho = \max\{\alpha + 2, \frac{4q}{2q-1}\}$. If h and H are defined by* (7.6.4) *and* (7.6.5)*, then $h(u) \in L^1(\mathbb{R}^N)$ for every $u \in H^1(\mathbb{R}^N) \cap W^{1,\rho}(\mathbb{R}^N)$. Furthermore, there exists C such that*

$$\begin{aligned} |H(v) - H(u)| &\leq C(1 + \|u\|_{H^1} + \|v\|_{H^1})^{\alpha+2} \\ &\quad \times (\|u\|_{H^1} + \|u\|_{W^{1,\rho}} + \|v\|_{H^1} + \|v\|_{W^{1,\rho}})\|v - u\|_{H^1} \end{aligned} \tag{7.6.6}$$

for all $u, v \in H^1(\mathbb{R}^N) \cap W^{1,\rho}(\mathbb{R}^N)$.

PROOF. Let us write $V_r = V_1 + V_2$, where $V_1 \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and $V_2 \in L^\infty(\mathbb{R}^N)$; $\nabla W = Z_1 + Z_2$, where $Z_1 \in L^q(\mathbb{R}^N)$ and $Z_2 \in L^\infty(\mathbb{R}^N)$; and $f = f_1 + f_2$, where f_1 is globally Lipschitz continuous and $|f_2(v) - f_2(u)| \leq C(|u| + |v|)^\alpha |v - u|$ for all $u, v \in \mathbb{C}$. Set

$$\phi_i(u) = |u|^2 \frac{x}{|x|} \cdot (Z_i \star |u|^2) \quad \text{and} \quad \psi_i(u) = \frac{1}{r}\{2F_i(u) - \overline{u} f_i(u)\} \quad \text{for } i = 1, 2.$$

Consider $u, v \in \mathcal{D}(\mathbb{R}^N)$. We have

$$\begin{aligned} \int_{\mathbb{R}^N} |V_1(|v|^2 - |u|^2)| &\leq \int_{\mathbb{R}^N} |V_1|(|v| + |u|)|v - u| \\ &\leq C\|V_1\|_{L^{\frac{N}{2}}} (\|v\|_{L^{\frac{2N}{N-2}}} + \|u\|_{L^{\frac{2N}{N-2}}})\|v - u\|_{L^{\frac{2N}{N-2}}} \\ &\leq C(\|u\|_{H^1} + \|v\|_{H^1})\|v - u\|_{H^1} . \end{aligned}$$

Also,

$$\int_{\mathbb{R}^N} |V_2(|v|^2 - |u|^2)| \leq C(\|u\|_{L^2} + \|v\|_{L^2})\|v - u\|_{L^2} .$$

Finally,

$$\int_{\mathbb{R}^N} |\psi_1(v) - \psi_1(u)| \le C \int_{\mathbb{R}^N} \frac{|v| + |u|}{r} |v - u|$$

and

$$\int_{\mathbb{R}^N} |\psi_2(v) - \psi_2(u)| \le C \int_{\mathbb{R}^N} \frac{(|v| + |u|)^{\alpha+1}}{r} |v - u| .$$

Applying Hölder's inequality and (7.6.1), we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\psi_2(v) - \psi_2(u)| \\
&\quad \le C(\|v\|_{L^{\alpha+2}} + \|u\|_{L^{\alpha+2}})^{\alpha}(\|\nabla v\|_{L^{\alpha+2}} + \|\nabla u\|_{L^{\alpha+2}})\|v - u\|_{L^{\alpha+2}} \\
&\quad \le C(1 + \|u\|_{H^1} + \|v\|_{H^1})^{\alpha+2}(\|u\|_{H^1} + \|u\|_{W^{1,\rho}} + \|v\|_{H^1} + \|v\|_{W^{1,\rho}})\|v - u\|_{H^1} .
\end{aligned}$$

Also,

$$\int_{\mathbb{R}^N} |\psi_1(v) - \psi_1(u)| \le C(\|\nabla v\|_{L^2} + \|\nabla u\|_{L^2})\|v - u\|_{L^2} .$$

One obtains the same inequalities for ϕ_1 and ϕ_2 by applying Young's and Hölder's inequalities. Therefore, (7.6.6) holds for all $u, v \in \mathcal{D}(\mathbb{R}^N)$. The result now follows easily by density. □

We are now in a position to state and prove the main result of this section.

THEOREM 7.6.4. (Morawetz's estimate) *Assume $N \ge 3$ and let g be as in Lemma* 7.6.3. *If $\varphi \in H^1(\mathbb{R}^N)$ and if $u \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R}^N))$ is the maximal solution of* (4.1.1), *then*

$$\int_s^t H(u(\tau))d\tau \le \frac{1}{2}\{(iu(t), u_r(t))_{L^2} - (iu(s), u_r(s))_{L^2}\} \tag{7.6.7}$$

for all $-T_{\min} < s < t < T_{\max}$, where $H(u)$ is defined by (7.6.5).

REMARK 7.6.5. Note that the inequality (7.6.7) makes sense. Indeed, it follows from Remark 4.4.3 or Theorem 4.4.6 that $u \in L^q((s,t), W^{1,r}(\mathbb{R}^N))$ for every admissible pair (q,r). Applying Lemma 7.6.3, we deduce easily that $H(u) \in L^1(s,t)$.

PROOF OF THEOREM 7.6.4. We proceed in two steps.

STEP 1. (7.6.7) holds, when $\varphi \in H^2(\mathbb{R}^N)$. Note that by Remark 5.3.3, $u \in C((-T_{\min}, T_{\max}), H^2(\mathbb{R}^N)) \cap C^1((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N))$. Therefore, the equation (4.1.1) makes sense in $L^2(\mathbb{R}^N)$ and we may multiply it by $u_r + (N-1)u/2r \in C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N))$ (by Lemma 7.6.1). Therefore,

$$\left(iu_t + \Delta u + g(u), u_r + \frac{N-1}{2r}u\right)_{L^2} = 0 \quad \text{on } (-T_{\min}, T_{\max}) . \tag{7.6.8}$$

We claim that

$$\left(iu_t, u_r + \frac{N-1}{2r}u\right)_{L^2} = \frac{1}{2}\frac{d}{dt}(iu, u_r)_{L^2} . \tag{7.6.9}$$

Indeed, by density, we need only establish the identity (7.6.9) for smooth functions u. In this case, it follows from integrating the identity

$$\operatorname{Re}\left\{iu_t\left(\overline{u}_r+\frac{N-1}{2r}\overline{u}\right)\right\}=\frac{1}{2}\partial_t\operatorname{Re}(iu\overline{u}_r)+\frac{1}{2}\nabla\cdot\left(\frac{x}{r}\operatorname{Re}(iu_t\overline{u})\right).$$

We also claim that

$$\left(\Delta u,u_r+\frac{N-1}{2r}u\right)_{L^2}\le 0\,. \tag{7.6.10}$$

Again by density, we need only establish (7.6.10) for $u\in\mathcal{D}(\mathbb{R}^N)$. Note that in this case,

$$\begin{aligned}\operatorname{Re}\left\{\Delta u\left(\overline{u}_r+\frac{N-1}{2r}\overline{u}\right)\right\}&=\nabla\cdot\operatorname{Re}\left\{\nabla u\left(\overline{u}_r+\frac{N-1}{2r}\overline{u}\right)-\frac{x}{2r}|\nabla u|^2\right\}\\&\quad+\nabla\cdot\left(\frac{N-1}{4r^3}x|u|^2\right)\\&\quad-\frac{1}{r}\{|\nabla u|^2-|u_r|^2\}-\frac{(N-1)(N-3)}{4r^3}|u|^2,\end{aligned}$$

and so

$$\left(\Delta u,u_r+\frac{N-1}{2r}u\right)_{L^2}=-\int_{\mathbb{R}^N}\frac{1}{r}\{|\nabla u|^2-|u_r|^2\}-a-b\,,$$

where

$$a=\begin{cases}2\pi|u(0)|^2 & \text{if } N=3\\ 0 & \text{if } N\ge 4,\end{cases}\qquad b=\begin{cases}0 & \text{if } N=3\\ \frac{(N-1)(N-3)}{4}\int_{\mathbb{R}^N}\frac{|u|^2}{r^3} & \text{if } N\ge 4.\end{cases}$$

Note that b is well defined by Corollary 7.6.2. Inequality (7.6.10) follows immediately. Furthermore,

$$\left(Vu,u_r+\frac{N-1}{2r}u\right)_{L^2}=-\frac{1}{2}\int_{\mathbb{R}^N}V_r|u|^2\,. \tag{7.6.11}$$

Also, we need only establish (7.6.11) for $u\in\mathcal{D}(\mathbb{R}^N)$. In this case, (7.6.11) follows from integrating the identity

$$\operatorname{Re}\left\{Vu\left(\overline{u}_r+\frac{N-1}{2r}\overline{u}\right)\right\}=\nabla\cdot\left(x\frac{V|u|^2}{2r}\right)-\frac{1}{2}V_r|u|^2\,.$$

Note also that

$$\left(f(u),u_r+\frac{N-1}{2r}u\right)_{L^2}=-\int_{\mathbb{R}^N}\frac{N-1}{2r}\{2F(u)-\overline{u}f(u)\}\,. \tag{7.6.12}$$

Note that we need only establish (7.6.12) for $u\in\mathcal{D}(\mathbb{R}^N)$. In this case, (7.6.12) follows from integrating the identity

$$\operatorname{Re}\left\{f(u)\left(\overline{u}_r+\frac{N-1}{2r}\overline{u}\right)\right\}=\nabla\cdot\left(x\frac{F(u)}{r}\right)-\frac{N-1}{2r}\{2F(u)-\overline{u}f(u)\}\,.$$

Finally, we claim that

$$\left((W\star|u|^2)u, u_r + \frac{N-1}{2r}u\right)_{L^2} = -\frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \frac{x}{|x|}\cdot(\nabla W\star|u|^2). \tag{7.6.13}$$

Also, we need only establish (7.6.13) for $u \in \mathcal{D}(\mathbb{R}^N)$. In this case,

$$\mathrm{Re}\left\{(W\star|u|^2)u\left(\overline{u}_r + \frac{N-1}{2r}\overline{u}\right)\right\} = \nabla\cdot\left(\frac{x}{2r}(W\star|u|^2)|u|^2\right) - \frac{1}{2}|u|^2\frac{x}{|x|}\cdot\nabla(W\star|u|^2).$$

Equation (7.6.13) is obtained by integrating the above equality. We deduce (7.6.7) from formulae (7.6.8)–(7.6.13).

STEP 2. Let $\varphi \in H^1(\mathbb{R}^N)$, let u be the corresponding maximal solution of (4.1.1), and let $-T_{\min} < s < t < T_{\max}$. Consider a sequence $\varphi^m \in H^2(\mathbb{R}^N)$ such that $\varphi^m \to \varphi$ is $H^1(\mathbb{R}^N)$. Let u^m be the corresponding solutions of (4.1.1). It follows from Theorem 3.3.9 that $u^m \to u$, in $C([s,t], H^1(\mathbb{R}^N))$. Furthermore, it follows from Remarks 4.4.3 and 4.4.4 that for every admissible pair (q,r), u^m is bounded in $L^q((s,t), W^{1,r}(\mathbb{R}^N))$, uniformly with respect to m. In particular,

$$(iu^m(\tau), u^m_r(\tau))_{L^2} \underset{m\to\infty}{\longrightarrow} (iu(\tau), u_r(\tau))_{L^2}\,,$$

uniformly on $[s,t]$, and by Lemma 7.6.3,

$$\int_s^t H(u^m(\tau))d\tau \underset{m\to\infty}{\longrightarrow} \int_s^t H(u(\tau))d\tau\,.$$

The result now follows by applying Step 1. □

COROLLARY 7.6.6. *Assume $N \geq 3$ and let $g(u) = -\eta|u|^\alpha u$ for some $\eta > 0$ and $0 < \alpha < 4/(N-2)$. For every $\varphi \in H^1(\mathbb{R}^N)$, the maximal solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ of (4.1.1) satisfies*

$$\int_{-\infty}^{+\infty}\int_{\mathbb{R}^N} \frac{|u(t,x)|^{\alpha+2}}{|x|}\,dx\,dt < \infty\,. \tag{7.6.14}$$

In addition, $u(t) \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ as $t \to \pm\infty$.

PROOF. It follows from Remark 6.8.1(i) that the solution u is global and bounded in $H^1(\mathbb{R}^N)$. Applying (7.6.7), we obtain

$$\int_{-t}^{t}\int_{\mathbb{R}^N} \frac{|u(s,x)|^{\alpha+2}}{|x|}\,dx\,ds \leq C(\|u(t)\|^2_{H^1} + \|u(-t)\|^2_{H^1}) \leq C\,.$$

Hence (7.6.14) follows by letting $t \uparrow \infty$. In order to show the weak convergence to 0, we need to verify that for every $\psi \in \mathcal{D}(\mathbb{R}^N)$, $\langle u(t), \psi\rangle \to 0$ as $t \to \pm\infty$. Note that

$$|\langle u(t),\psi\rangle| \leq \int_{\mathbb{R}^N} |u(t)||\psi| \leq \int_{\mathbb{R}^N} \frac{|u(t)|}{|x|^{\frac{1}{\alpha+2}}}|x|^{\frac{1}{\alpha+2}}|\psi| \leq C\left(\int_{\mathbb{R}^N} \frac{|u(t,x)|^{\alpha+2}}{|x|}\right)^{\frac{1}{\alpha+2}},$$

and so

$$\int_{-\infty}^{+\infty} |(u(t), \psi)|^{\alpha+2}\, dt < \infty .$$

Finally, u is bounded in $H^1(\mathbb{R}^N)$, and so by the equation, u_t is bounded in $H^{-1}(\mathbb{R}^N)$. We see in particular that the function $t \mapsto |(u(t), \psi)|$ is (uniformly) Lipschitz continuous $\mathbb{R} \to \mathbb{R}$. Hence the result follows. □

REMARK 7.6.7. Theorem 7.6.4 requires $N \geq 3$. (If $N = 1, 2$, then singular terms appear in the proof of (7.6.10).) This is the reason why the scattering theory in the energy space (the asymptotic completeness part) was developed only for $N \geq 3$. Recently, Nakanishi [259] obtained a substitute for Morawetz's estimate in any dimension. More precisely, in the model case $g(u) = -\eta |u|^\alpha u$ with $\eta > 0$ and $0 < \alpha < 4/(N-2)$ ($0 < \alpha < \infty$ if $N = 1, 2$),

$$\int_1^\infty \int_{\mathbb{R}^N} \frac{1}{(|x|^2+t^2)^{\frac{3}{2}}} \Big(|(x + 2it\nabla)u|^2 + \frac{2N\eta\alpha}{\alpha+2} t^2 |u|^{\alpha+2} \Big) \leq$$
$$\Big(1 + \frac{(N+5)(3-N)^+}{2} \Big) \|\varphi\|_{L^2}^2 + 4 \sup_{t \geq 0} \|u(t)\|_{H^1}^2 ;$$

in particular,

$$\int_1^\infty \int_{\mathbb{R}^N} \frac{t^2}{(|x|^2+t^2)^{\frac{3}{2}}} |u|^{\alpha+2} < \infty . \tag{7.6.15}$$

This is obtained by taking the scalar product of the equation with $\rho u + \Gamma \cdot \nabla u$, where

$$\Gamma(t,x) = \frac{2x}{\sqrt{|x|^2+t^2}}, \quad \rho(t,x) = \frac{N-1-it}{\sqrt{|x|^2+t^2}} + \frac{t^2}{(|x|^2+t^2)^{\frac{3}{2}}} .$$

Note that (7.6.15) is weaker than (7.6.14) particularly because of the time dependence in the factor of $|u|^{\alpha+2}$. It is sufficient, however, to deduce the asymptotic completeness in dimensions $N = 1$ and $N = 2$ under the assumption $\alpha > 4/N$, i.e., the analogue to Theorem 7.8.1. See [259]. Note that the proof, however, is much more delicate than the proof of Theorem 7.8.1.

7.7. Decay of Solutions in the Energy Space

Throughout this section we assume that $N \geq 3$. We apply Morawetz's estimate to the study of the asymptotic behavior of solutions. For simplicity, we restrict our attention to the model case

$$g(u) = -\eta |u|^\alpha u , \tag{7.7.1}$$

where

$$\eta > 0, \quad 0 < \alpha < \frac{4}{N-2} \quad (0 < \alpha < \infty \text{ if } N = 1) , \tag{7.7.2}$$

and we refer to Section 7.9 and Ginibre and Velo [137, 138] for more general results. Note that in this case, $T_{\min}(\varphi) = T_{\max}(\varphi) = \infty$ for all $\varphi \in H^1(\mathbb{R}^N)$ (see Remark 6.8.1(i)). Note also that we may apply Corollary 7.6.6. Our main result of this section is the following.

THEOREM 7.7.1. *Let $N \geq 3$ and assume* (7.7.1)–(7.7.2). *If*

$$\alpha > \frac{4}{N}, \tag{7.7.3}$$

then for every $\varphi \in H^1(\mathbb{R}^N)$, the maximal solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ of (4.1.1) *satisfies*

$$\|u(t)\|_{L^r} \underset{t\to\pm\infty}{\longrightarrow} 0 \quad \text{for every } 2 < r < \frac{2N}{N-2}. \tag{7.7.4}$$

REMARK 7.7.2. The above result is due to Ginibre and Velo [137]. However the proof that we present below follows closely an idea of Lin and Strauss [230].

PROOF OF THEOREM 7.7.1. The proof that we give below does not cover the case $N = 3$ and $\alpha \in (\frac{4}{3}, \frac{\sqrt{17}-1}{2}]$. (The proof in that case is slightly more complicated, and the result follows from Theorem 7.9.2 below.) We only consider $t > 0$, the case $t < 0$ being treated similarly. Note that we need only establish (7.7.4) for $r = \alpha + 2$, since the general case follows immediately from the boundedness of the solution in $H^1(\mathbb{R}^N)$ and Hölder's inequality. We proceed in several steps.

STEP 1. The estimate

$$\int_{\{|x|\geq t\log t\}} |u(t,x)|^{\alpha+2}\,dx \underset{t\to+\infty}{\longrightarrow} 0 \tag{7.7.5}$$

holds. Indeed, let $M > 0$ and let

$$\theta_M(x) = \begin{cases} \frac{|x|}{M} & \text{if } |x| \leq M \\ 1 & \text{if } |x| \geq M \end{cases}$$

so that $\theta_M \in W^{1,\infty}(\mathbb{R}^N)$ and $\|\nabla\theta_M\|_{L^\infty} \leq 1/M$. Thus $\theta_M u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ and

$$\langle iu_t + \Delta u + g(u), i\theta_M u\rangle_{H^{-1},H^1} = 0\,.$$

Note that

$$\langle iu_t, i\theta_M u\rangle_{H^{-1},H^1} = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \theta_M |u|^2\,,$$

$$\langle g(u), i\theta_M u\rangle_{H^{-1},H^1} = 0\,,$$

and

$$\begin{aligned} -\langle \Delta u, i\theta_M u\rangle_{H^{-1},H^1} = \langle \nabla u, i\nabla\theta_M u\rangle_{L^2} = \langle \nabla u, iu\nabla\theta_M\rangle_{L^2} \\ = -\operatorname{Re}\int_{\mathbb{R}^N} i\overline{u}\nabla u\cdot\nabla\theta_M \\ \leq \frac{1}{M}\|u(t)\|_{H^1}^2 \leq \frac{C}{M}\,. \end{aligned}$$

It follows that

$$\int_{\mathbb{R}^N} \theta_M |u(t,x)|^2 \, dx \le \frac{Ct}{M} + \int_{\mathbb{R}^N} \theta_M |\varphi|^2 \, dx \quad \text{for every } t \in \mathbb{R}.$$

Letting $M = t \log t$, we obtain

$$\int_{\{|x| \ge t \log t\}} |u(t,x)|^2 \, dx \le \frac{C}{\log t} + \int_{\mathbb{R}^N} \theta_{t \log t} |\varphi|^2 \, dx .$$

Applying the dominated convergence theorem to the last term in the right-hand side of the above estimate, we obtain that

$$\int_{\{|x| \ge t \log t\}} |u(t,x)|^2 \, dx \underset{t \to +\infty}{\longrightarrow} 0 .$$

The result now follows from Hölder's inequality and the boundedness of u in $H^1(\mathbb{R}^N)$.

STEP 2. For every $\varepsilon > 0$, $t > 1$, $\tau > 0$, there exists $t_0 > \max\{t, 2\tau\}$ such that

$$\int_{t_0 - 2\tau}^{t_0} \int_{\{|x| \le s \log s\}} |u(s,x)|^{\alpha+2} \, dx \, ds \le \varepsilon . \tag{7.7.6}$$

Indeed, by Morawetz's estimate (7.6.14),

$$\begin{aligned} +\infty &> \int_1^\infty \frac{1}{s \log s} \int_{\{|x| \le s \log s\}} |u(s,x)|^{\alpha+2} \\ &\ge \sum_{k=0}^\infty \int_{t+2k\tau}^{t+2(k+1)\tau} \frac{1}{s \log s} \int_{\{|x| \le s \log s\}} |u(s,x)|^{\alpha+2} \\ &\ge \sum_{k=0}^\infty \frac{1}{\gamma_k} \int_{t+2k\tau}^{t+2(k+1)\tau} \int_{\{|x| \le s \log s\}} |u(s,x)|^{\alpha+2} \end{aligned}$$

with $\gamma_k = (t + 2(k+1)\tau) \log(t + 2(k+1)\tau)$. Since

$$\sum_{k=0}^\infty \frac{1}{(t + 2(k+1)\tau) \log(t + 2(k+1)\tau)} = \infty ,$$

we see that there exists $k > 0$ for which

$$\int_{t+2k\tau}^{t+2(k+1)\tau} \int_{\{|x| \le s \log s\}} |u(s,x)|^{\alpha+2} \le \varepsilon .$$

Hence the result follows with $t_0 = t + 2(k+1)\tau$.

STEP 3. For every $\varepsilon > 0$, $a, b > 0$, there exists $t_0 > \max\{a, b\}$ such that

$$\sup_{s \in [t_0 - b, t_0]} \|u(s)\|_{L^{\alpha+2}} \le \varepsilon . \tag{7.7.7}$$

Consider $t > \tau > 0$, and write

$$
\begin{aligned}
u(t) &= \mathcal{T}(t)\varphi + i\int_0^{t-\tau} \mathcal{T}(t-s)g(u(s))ds + i\int_{t-\tau}^{t} \mathcal{T}(t-s)g(u(s))ds \\
&= v(t) + w(t,\tau) + z(t,\tau).
\end{aligned}
\tag{7.7.8}
$$

It follows from Corollary 2.3.7 that

$$
\|v(t)\|_{L^{\alpha+2}} \underset{t\to\infty}{\longrightarrow} 0\,. \tag{7.7.9}
$$

Let now

$$
p = \begin{cases} \infty & \text{if } \alpha \geq 1 \\ \frac{2}{1-\alpha} & \text{if } \alpha < 1. \end{cases}
$$

Observe that $N(\frac{1}{2} - \frac{1}{p}) = \frac{N}{2}\min\{\alpha, 1\} > 1$. Therefore, we deduce from (2.2.4) that

$$
\begin{aligned}
\|w(t,\tau)\|_{L^p} &\leq \int_0^{t-\tau} (t-s)^{-N(\frac{1}{2}-\frac{1}{p})} \|u(s)\|_{L^{(\alpha+1)p'}}^{\alpha+1}\, ds \\
&\leq C\tau^{-(\frac{N}{2}\min\{\alpha,1\}-1)} \sup_{s\in\mathbb{R}} \|u(s)\|_{L^{(\alpha+1)p'}}^{\alpha+1}\,.
\end{aligned}
$$

Since $2 \leq (\alpha+1)p' \leq \frac{2N}{N-2}$, and since u is bounded in $H^1(\mathbb{R}^N)$, there exists C such that

$$
\|w(t,\tau)\|_{L^p} \leq C\tau^{-(\frac{N}{2}\min\{\alpha,1\}-1)} \quad \text{for all } t > \tau > 0. \tag{7.7.10}
$$

On the other hand, note that

$$
w(t,\tau) = \mathcal{T}(\tau)u(t-\tau) - \mathcal{T}(t)\varphi\,,
$$

and so it follows from conservation of charge that

$$
\|w(t,\tau)\|_{L^2} \leq 2\|\varphi\|_{L^2}\,. \tag{7.7.11}
$$

Applying (7.7.10), (7.7.11), and Hölder's inequality, we deduce that there exists K such that

$$
\|w(t,\tau)\|_{L^{\alpha+2}} \leq K\tau^{-\frac{N\alpha - 2\max\{\alpha,1\}}{2(\alpha+2)}} \quad \text{for all } t > \tau > 0. \tag{7.7.12}
$$

Finally, by (2.2.4),

$$
\|z(t,\tau)\|_{L^{\alpha+2}} \leq \int_{t-\tau}^{t} (t-s)^{-\frac{N\alpha}{2(\alpha+2)}} \|u(s)\|_{L^{\alpha+2}}^{\alpha+1}\, ds\,. \tag{7.7.13}
$$

Note that $N\alpha < 2(\alpha+2)$, and let $p \in (1, \frac{2(\alpha+2)}{N\alpha})$. It follows in particular that $(\alpha+1)p' > \alpha+2$. Applying (7.7.13), Hölder's inequality, and the boundedness of u in $L^{\alpha+2}(\mathbb{R}^N)$, we obtain

$$
\begin{aligned}
\|z(t,\tau)\|_{L^{\alpha+2}} &\leq \Big(\int_{t-\tau}^{t} (t-s)^{\frac{N\alpha p}{2(\alpha+2)}}\, ds\Big)^{\frac{1}{p}} \Big(\int_{t-\tau}^{t} \|u(s)\|_{L^{\alpha+2}}^{(\alpha+1)p'}\Big)^{\frac{1}{p'}} \\
&\leq C\tau^{\delta}\Big(\int_{t-\tau}^{t} \|u(s)\|_{L^{\alpha+2}}^{\alpha+2}\Big)^{\mu}
\end{aligned}
$$

for some $\delta, \mu > 0$. In particular, there exists L such that

$$\begin{aligned}\|z(t,\tau)\|_{L^{\alpha+2}} \le L\tau^{\mu+\delta}\Big(\sup_{s\ge t-\tau} \int_{\{|x|\ge s\log s\}} |u(s,x)|^{\alpha+2}\,dx\Big)^{\mu} \\ + L\tau^{\delta}\Big(\int_{t-\tau}^{t} \int_{\{|x|\le s\log s\}} |u(s,x)|^{\alpha+2}\,dx\,ds\Big)^{\mu}.\end{aligned} \tag{7.7.14}$$

It follows from (7.7.9) that there exists $t_1 \ge \max\{a,b\}$ such that

$$\|v(t)\|_{L^{\alpha+2}} \le \frac{\varepsilon}{4} \quad \text{for } t \ge t_1 . \tag{7.7.15}$$

Next, let $\tau_1 > b$ such that

$$\|w(t,\tau_1)\|_{L^{\alpha+2}} \le \frac{\varepsilon}{4} \quad \text{for } t > 0 , \tag{7.7.16}$$

which exists by (7.7.12). By Step 1, there exists $t_2 \ge t_1$ such that

$$L\tau_1^{\mu+\delta}\Big(\sup_{s\ge t-\tau_1} \int_{\{|x|\ge s\log s\}} |u(s,x)|^{\alpha+2}\,dx\Big)^{\mu} \le \frac{\varepsilon}{4} \quad \text{for } t \ge t_2 . \tag{7.7.17}$$

Finally, by Step 2, there exists $t_0 \ge t_2$ such that

$$L\tau_1^{\delta}\Big(\int_{t_0-2\tau_1}^{t_0} \int_{\{|x|\le s\log s\}} |u(s,x)|^{\alpha+2}\,dx\,ds\Big)^{\mu} \le \frac{\varepsilon}{4} . \tag{7.7.18}$$

Note that $[t-\tau_1, t] \subset [t_0 - 2\tau_1, t_0]$ for all $t \in [t_0 - b, t_0]$. Therefore, it follows from (7.7.18) that

$$L\tau_1^{\delta}\Big(\int_{t-b}^{t} \int_{\{|x|\le s\log s\}} |u(s,x)|^{\alpha+2}\,dx\,ds\Big)^{\mu} \le \frac{\varepsilon}{4} . \tag{7.7.19}$$

Applying (7.7.8), (7.7.15), (7.7.16), (7.7.14), (7.7.17), and (7.7.19), we deduce that $\|u(t)\|_{L^{\alpha+2}} \le \varepsilon$ for every $t \in [t_0 - b, t_0]$. Hence the result follows.

STEP 4. We need to show that for every $\varepsilon > 0$, $\|u(t)\|_{L^{\alpha+2}} \le \varepsilon$ for t large. Let $t > \tau > 0$. It follows from (7.7.8) and (7.7.12) that

$$\|u(t)\|_{L^{\alpha+2}} \le \|v(t)\|_{L^{\alpha+2}} + K\tau^{-\frac{N\alpha - 2\max\{\alpha,1\}}{2(\alpha+2)}} + \|z(t,\tau)\|_{L^{\alpha+2}} . \tag{7.7.20}$$

Consider $\varepsilon > 0$, and let τ_ε be defined by

$$K\tau_\varepsilon^{-\frac{N\alpha - 2\max\{\alpha,1\}}{2(\alpha+2)}} = \frac{\varepsilon}{4} . \tag{7.7.21}$$

We deduce from (7.7.9) that there exists $t_1 > 0$ such that

$$\|v(t)\|_{L^{\alpha+2}} \le \frac{\varepsilon}{4} \quad \text{for } t \ge t_1 . \tag{7.7.22}$$

Applying (7.7.20), (7.7.21), and (7.7.22), we obtain

$$\|u(t)\|_{L^{\alpha+2}} \le \frac{\varepsilon}{2} + \|z(t,\tau_\varepsilon)\|_{L^{\alpha+2}} \quad \text{for } t \ge t_1. \tag{7.7.23}$$

Note also that by (2.2.4),

$$\begin{aligned}\|z(t,\tau_\varepsilon)\|_{L^{\alpha+2}} &\le \int_{t-\tau_\varepsilon}^t (t-s)^{-\frac{N\alpha}{2(\alpha+2)}}\|u(s)\|_{L^{\alpha+2}}^{\alpha+1}\,ds \\ &\le M\tau_\varepsilon^{1-\frac{N\alpha}{2(\alpha+2)}} \sup_{t-\tau_\varepsilon,t}\|u(s)\|_{L^{\alpha+2}}^{\alpha+1} \quad \text{for every } t\ge\tau_\varepsilon\,.\end{aligned} \tag{7.7.24}$$

By Step 3, there exists $t_0 \ge \max\{\tau_\varepsilon, t_1\}$ such that $\|u(t)\|_{L^{\alpha+2}} \le \varepsilon$ for $t\in[t_0-\tau_\varepsilon,t_0]$. Therefore, we can define

$$t_\varepsilon = \sup\{t\ge t_0 : \|u(s)\|_{L^{\alpha+2}}\le\varepsilon \text{ for all } s\in[t_0-\tau_\varepsilon,t]\}\,.$$

Assume that $t_\varepsilon<\infty$. It follows that

$$\|u(t_\varepsilon)\|_{L^{\alpha+2}} = \varepsilon\,. \tag{7.7.25}$$

Applying (7.7.23) and (7.7.24) with $t=t_\varepsilon$, we obtain that

$$\varepsilon \le \frac{\varepsilon}{2} + M\tau_\varepsilon^{1-\frac{N\alpha}{2(\alpha+2)}}\varepsilon^{\alpha+1}\,,$$

which implies

$$\tau_\varepsilon^{1-\frac{N\alpha}{2(\alpha+2)}}\varepsilon^{\alpha} \ge \frac{1}{2M}\,.$$

Applying (7.7.21), we see that

$$\tau_\varepsilon^{-\gamma} \ge \frac{1}{2M(4K)^\alpha}\,, \tag{7.7.26}$$

where

$$\gamma = \frac{\alpha(N\alpha-2-2\max\{\alpha,1\})+(N\alpha-4)}{2(\alpha+2)}\,.$$

Observe that when $\alpha\le1$, we have $\gamma>0$ (remember that $N\alpha>4$). Therefore, (7.7.26) implies that τ_ε is bounded by a positive number. This is a contradiction when ε is small, since $\tau_\varepsilon\to\infty$ as $\varepsilon\downarrow0$. When $\alpha>1$, one easily verifies that $\gamma>0$ when $N\ge4$, or when $N=3$ and $\alpha>\frac{\sqrt{17}-1}{2}$, in which case we obtain the same contradiction. Therefore, $t_\varepsilon=\infty$, which is the desired estimate. □

THEOREM 7.7.3. *Let $N\ge3$ and assume (7.7.1), (7.7.2), and (7.7.3). For every $\varphi\in H^1(\mathbb{R}^N)$, the maximal solution $u\in C(\mathbb{R},H^1(\mathbb{R}^N))$ of (4.1.1) satisfies*

$$u\in L^q(\mathbb{R},W^{1,r}(\mathbb{R}^N)) \quad \textit{for every admissible pair } (q,r). \tag{7.7.27}$$

For the proof, we will use the following elementary lemma.

LEMMA 7.7.4. *Let $a,b>0$ and $p>1$. Assume that b is small enough so that the function $f(x)\equiv a-x+bx^p$ is negative for some $x>0$, and let x_0 be the first (positive) zero of f. Let $I\subset\mathbb{R}$ be an interval and let $\phi\in C(I,\mathbb{R}_+)$ satisfy*

$$\phi(t)\le a+b\phi(t)^p \quad \textit{for all } t\in I.$$

If $\phi(t_0)=0$ (or more generally $\phi(t_0)\le x_0$) for some $t_0\in I$, then $\phi(t)\le x_0$ for all $t\in I$.

PROOF. By assumption, the set $J = \{x \geq 0; f(x) \geq 0\}$ is of the form $J = [0,y] \cup [z,\infty)$ for some $0 < y < x_0 < z$. Since $\{\phi(t) : t \in I\}$ is a connected set and $f(\phi(t)) \geq 0$, we must have either $\{\phi(t) : t \in I\} \subset [0,y]$, or else $\{\phi(t) : t \in I\} \subset [z,\infty)$. This proves the result. □

PROOF OF THEOREM 7.7.3. Let (γ,ρ) be the admissible pair such that $\rho = \alpha+2$. For every $S, t > 0$,

$$u(t+S) = \mathcal{T}(t)u(S) + i\int_0^t \mathcal{T}(t-s)g(u(S+s))ds\,.$$

It follows from Strichartz's estimates, Remark 1.3.1(vii), and Hölder's inequality that for every $t > S > 0$,

$$\begin{aligned}\|u\|_{L^\gamma((S,t),W^{1,\rho})} &\leq C\|u(S)\|_{H^1} + C\Big(\int_S^t \|u(s)\|_{L^\rho}^{\alpha\gamma'}\|u(s)\|_{W^{1,\rho}}^{\gamma'}\Big)^{1/\gamma'} \\ &\leq C\|u(S)\|_{H^1} \\ &\quad + C\Big(\int_S^t \|u(s)\|_{L^\rho}^{(\alpha+1)\gamma'-\gamma}\|u(s)\|_{L^\rho}^{\gamma-\gamma'}\|u(s)\|_{W^{1,\rho}}^{\gamma'}\Big)^{\frac{1}{\gamma'}},\end{aligned}$$

where C is independent of t, S. Note that $(\alpha+1)\gamma' > \gamma$, and so

$$\begin{aligned}&\Big(\int_S^t \|u(s)\|_{L^\rho}^{(\alpha+1)\gamma'-\gamma}\|u(s)\|_{L^\rho}^{\gamma-\gamma'}\|u(s)\|_{W^{1,\rho}}^{\gamma'}\Big)^{1/\gamma'} \\ &\qquad\qquad \leq \sup\{\|u(s)\|_{L^\rho} : s \geq S\}^{\alpha+1-\frac{\gamma}{\gamma'}}\|u\|_{L^\gamma((S,t),W^{1,\rho})}^{\frac{\gamma}{\gamma'}}\,.\end{aligned}$$

It follows from Theorem 7.7.1 that

$$\begin{aligned}\Big(\int_S^t \|u(s)\|_{L^\rho}^{(\alpha+1)\gamma'-\gamma}\|u(s)\|_{L^\rho}^{\gamma-\gamma'}\|u(s)\|_{W^{1,\rho}}^{\gamma'}\Big)^{1/\gamma'} &\leq \varepsilon(S)\|u\|_{L^\gamma((S,t),W^{1,\rho})}^{\frac{\gamma}{\gamma'}} \\ &= \varepsilon(S)\|u\|_{L^\gamma((S,t),W^{1,\rho})}^{\gamma-1},\end{aligned}$$

where $\varepsilon(S) \to 0$ as $S \to \infty$. Therefore,

$$\|u\|_{L^\gamma((S,t),W^{1,\rho})} \leq C\|\varphi\|_{H^1} + \varepsilon(S)\|u\|_{L^\gamma((S,t),W^{1,\rho})}^{\gamma-1}\,.$$

By Lemma 7.7.4, we see that if we fix S large enough, then $\|u\|_{L^\gamma((S,t),W^{1,\rho})} \leq K$ for some K independent of t. Therefore, $u \in L^\gamma((S,\infty),W^{1,\rho}(\mathbb{R}^N))$. One shows as well that for S large enough, $u \in L^\gamma((-\infty,-S),W^{1,\rho}(\mathbb{R}^N))$, and so $u \in L^\gamma(\mathbb{R},W^{1,\rho}(\mathbb{R}^N))$. This implies that $g(u) \in L^{\gamma'}(\mathbb{R},W^{1,\rho'}(\mathbb{R}^N))$, and the result follows from Strichartz's estimates. □

REMARK 7.7.5. One can add the following property to the statement of Theorem 7.7.3. If $\varphi \in H^2(\mathbb{R}^N)$, then $u_t \in L^q(\mathbb{R},L^r(\mathbb{R}^N))$ for every admissible pair (q,r). This is obtained by a similar argument, by applying the estimates used in the proof of Theorem 5.3.1. It is not difficult to see, using Sobolev's inequalities, that this implies $u \in L^q(\mathbb{R},W^{2,r}(\mathbb{R}^N))$ for every admissible pair (q,r); in particular, $u \in L^\infty(\mathbb{R},H^2(\mathbb{R}^N))$.

7.8. Scattering Theory in the Energy Space

We apply the results of Section 7.7 in order to construct the scattering operator in the energy space $H^1(\mathbb{R}^N)$. The results below are due to Ginibre and Velo [137]. We still assume that $N \geq 3$ and that g is given by (7.7.1)–(7.7.2). We refer to Section 7.9 and to Ginibre and Velo [137, 138] for more general results, and to Nakanishi [259] for the case $N = 1, 2$. We first construct the scattering states.

THEOREM 7.8.1. *Let $N \geq 3$ and assume (7.7.1), (7.7.2), and (7.7.3). If $\varphi \in H^1(\mathbb{R}^N)$ and if $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ is the maximal solution of (4.1.1), then there exist $u^+, u^- \in H^1(\mathbb{R}^N)$ such that $\|\mathcal{T}(-t)u(t) - u^\pm\|_{H^1} \underset{t\to\pm\infty}{\longrightarrow} 0$. In addition,*

$$\|u^+\|_{L^2} = \|u^-\|_{L^2} = \|\varphi\|_{L^2} \quad \textit{and} \quad \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u^+|^2 = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u^-|^2 = E(\varphi)\,.$$

PROOF. Let $v(t) = \mathcal{T}(-t)u(t)$. We have

$$v(t) = \varphi + i\int_0^t \mathcal{T}(-s)g(u(s))ds\,.$$

Therefore for $0 < t < \tau$,

$$v(t) - v(\tau) = i\int_\tau^t \mathcal{T}(-s)g(u(s))ds\,.$$

It follows from Strichartz's estimates that

$$\|v(t) - v(\tau)\|_{H^1} = \|\mathcal{T}(t)(v(t) - v(\tau))\|_{H^1} \leq C\|g(u)\|_{L^{q'}((t,\tau),W^{1,r'})}\,,$$

where (q, r) is the admissible pair such that $r = \alpha + 2$, and so

$$\|v(t) - v(\tau)\|_{H^1} \underset{t,\tau\to\infty}{\longrightarrow} 0$$

(see the end of the proof of Theorem 7.7.3). Therefore, there exists $u^+ \in H^1(\mathbb{R}^N)$ such that $v(t) \to u^+$ in H^1 as $t \to \infty$. One shows as well that there exists $u^- \in H^1(\mathbb{R}^N)$ such that $v(t) \to u^-$ in H^1 as $t \to -\infty$. The other properties follow from conservation of charge and energy. □

REMARK 7.8.2. The mappings $U_+ : \varphi \mapsto u^+$ and $U_- : \varphi \mapsto u^-$ defined by Theorem 7.8.1 map $H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$. In fact, one can show with similar estimates that U_+ and U_- are continuous $H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$.

REMARK 7.8.3. We deduce from Corollary 2.3.6 the following formula:

$$u^\pm = \varphi + i\int_0^{\pm\infty} \mathcal{T}(-s)g(u(s))ds\,;$$

in particular,

$$u(t) = \mathcal{T}(t)u^\pm - i\int_t^{\pm\infty} \mathcal{T}(t-s)g(u(s))ds \quad \text{for all } t \in \mathbb{R}\,. \tag{7.8.1}$$

We now construct the wave operators.

THEOREM 7.8.4. *Let $N \geq 3$ and assume (7.7.1), (7.7.2), and (7.7.3).*

(i) *For every $u^+ \in H^1(\mathbb{R}^N)$, there exists a unique $\varphi \in H^1(\mathbb{R}^N)$ such that the maximal solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ of (4.1.1) satisfies*

$$\|\mathcal{T}(-t)u(t) - u^+\|_{H^1} \to 0 \text{ as } t \to +\infty.$$

(ii) *For every $u^- \in H^1(\mathbb{R}^N)$, there exists a unique $\varphi \in H^1(\mathbb{R}^N)$ such that the maximal solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ of (4.1.1) satisfies*

$$\|\mathcal{T}(-t)u(t) - u^-\|_{H^1} \to 0 \text{ as } t \to -\infty.$$

PROOF. We prove (i), the proof of (ii) being similar. The idea of the proof is to solve equation (7.8.1) by a fixed-point argument. To that end, we introduce the function $\omega(t) = \mathcal{T}(t)u^+$. Let (q,r) be the admissible pair such that $r = \alpha + 2$. It follows from Strichartz's estimates and Corollary 2.3.7 that $\omega \in L^q(\mathbb{R}, W^{1,r}(\mathbb{R}^N))$ and that $\|\omega(t)\|_{L^r} \to 0$ as $t \to \infty$. Consider $S > 0$ and let

$$K_S = \|\omega\|_{L^q((S,\infty),W^{1,r})} + \sup_{t \geq S} \|\omega(t)\|_{L^r}. \tag{7.8.2}$$

Note that

$$K_S \underset{S \to \infty}{\longrightarrow} 0. \tag{7.8.3}$$

Let

$$E = \left\{u \in L^q((S,\infty), W^{1,r}(\mathbb{R}^N)) : \|u\|_{L^q((S,\infty),W^{1,r})} + \sup_{t \geq S} \|\omega(t)\|_{L^r} \leq 2K_S\right\},$$

and

$$\mathrm{d}(u,v) = \|v - u\|_{L^q((S,\infty),L^r)} \quad \text{for } u, v \in E.$$

It is easily checked that (E, d) is a complete metric space. Given $u \in E$, we have (see the proof of Theorem 7.7.3)

$$\|g(u)\|_{L^{q'}((S,\infty),W^{1,r'})} \leq C(2K_S)^{\alpha+1}.$$

By Corollary 2.3.6, $\mathcal{J}(u)$ defined by

$$\mathcal{J}(u)(t) = -i \int_t^\infty \mathcal{T}(t-s)g(u(s))ds \tag{7.8.4}$$

makes sense and

$$\mathcal{J}(u) \in C([S,\infty), H^1(\mathbb{R}^N)) \cap L^q((S,\infty), W^{1,r}(\mathbb{R}^N)), \tag{7.8.5}$$

and

$$\|\mathcal{J}(u)\|_{L^q((S,\infty),W^{1,r})} + \|\mathcal{J}(u)\|_{L^\infty((S,\infty),H^1)} \leq C(2K_S)^{\alpha+1}. \tag{7.8.6}$$

Applying (7.8.3), (7.8.6), and Sobolev's inequality, we deduce that

$$\|\mathcal{J}(u)\|_{L^q((S,\infty),W^{1,r})} + \|\mathcal{J}(u)\|_{L^\infty((S,\infty),L^r)} \leq K_S \quad \text{for } S \text{ sufficiently large.} \tag{7.8.7}$$

Putting together (7.8.2) and (7.8.7), we see that $\mathcal{A}$ defined by

$$\mathcal{A}(u)(t) = \mathcal{T}(t)u_+ + \mathcal{J}(u)(t) \quad \text{for } t \geq S$$

maps E to itself if S is large enough. One easily verifies with similar estimates that if S is large enough, one has

$$\mathrm{d}(\mathcal{A}(u), \mathcal{A}(v)) \leq \frac{1}{2}\,\mathrm{d}(u, v) \quad \text{for all } u, v \in E. \tag{7.8.8}$$

It follows from Banach's fixed-point theorem that $\mathcal{A}$ has a fixed point $u \in E$, which satisfies the equation (7.8.1) on $[S, \infty)$. Note that $u \in C([S, \infty), H^1(\mathbb{R}^N))$ by (7.8.5); in particular, $\psi = u(S) \in H^1(\mathbb{R}^N)$. Note also that

$$u(t + S) = \mathcal{T}(t)\psi + i \int_0^t \mathcal{T}(t - s) g(u(s + S))ds\,.$$

Therefore, u is the solution of the problem

$$\begin{cases} iu_t + \Delta u + g(u) = 0 \\ u(S) = \psi. \end{cases}$$

Note that, by Remark 6.8.1, the solution u is global. In particular, $u(0) \in H^1(\mathbb{R}^N)$ is well defined. It follows from the equation (7.8.1) that

$$\mathcal{T}(-t)u(t) - u^+ = -i \int_t^\infty \mathcal{T}(s) g(u(s))ds\,.$$

Since $u \in E$, it is not difficult to show with the above estimates that $\|\mathcal{T}(-t)u(t) - u^+\|_{H^1} \to 0$ as $t \to \infty$. Therefore, $\varphi = u(0)$ satisfies the conclusions of the theorem.

It remains to show uniqueness. Let $\varphi_1, \varphi_2 \in H^1(\mathbb{R}^N)$, let u_1 and u_2 be the corresponding solutions of (4.1.1), and assume that $\|\mathcal{T}(-t)u_j(t) - u^+\|_{H^1} \to 0$ as $t \to \infty$ for $j = 1, 2$. It follows from Remark 7.8.3 that u_j is a solution of (7.8.1). Furthermore, it follows from Theorems 7.7.1 and 7.7.3 that $u_j \in L^q(\mathbb{R}, W^{1,r}(\mathbb{R}^N))$ and that $\|u_j(t)\|_{L^r} \to 0$ as $t \to \infty$. In a similar way to the proof of (7.8.8), one obtains that $u_1(t) = u_2(t)$ for t sufficiently large. By uniqueness for the Cauchy problem at finite time, we conclude that $\varphi_1 = \varphi_2$. □

REMARK 7.8.5. Note that the above proof of the construction of φ for a given u_0 only uses a fixed-point argument. In particular, it still works for $N = 1, 2$. It is not difficult to see that it also works in the limiting case $\alpha = 4/N$. The proof of *uniqueness* is more delicate and uses the decay estimate of Theorem 7.7.3. This is where we use the assumption $N \geq 3$. As observed in Remark 7.8.9 below, uniqueness also holds in dimension $N = 1$ or 2.

REMARK 7.8.6. Nakanishi [260] has extended the existence part of Theorem 7.8.4 to the case $\alpha > 2/N$ when $N \geq 3$. The construction is by a compactness argument. Note that when $\alpha \leq 4/N$, uniqueness is an open problem (see Remark 7.8.5).

REMARK 7.8.7. The wave operators $\Omega_+ : u^+ \mapsto \varphi$ and $\Omega_- : u^- \mapsto \varphi$ defined by Theorem 7.8.4 map $H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$. In fact, one can show with similar estimates that Ω_+ and Ω_- are continuous. By Theorems 7.8.1 and 7.8.4, $U_\pm \Omega_\pm = \Omega_\pm U_\pm = I$ on $H^1(\mathbb{R}^N)$, where $U_\pm$ is defined by Remark 7.8.2. In particular, $\Omega_\pm : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ is one-to-one with continuous inverse $(\Omega_\pm)^{-1} = U_\pm$.

THEOREM 7.8.8. *Let $N \geq 3$ and assume* (7.7.1), (7.7.2), *and* (7.7.3). *For every $u^- \in H^1(\mathbb{R}^N)$, there exist a unique $u^+ \in H^1(\mathbb{R}^N)$ and a unique $\varphi \in H^1(\mathbb{R}^N)$, such* *that the maximal solution* $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ *of* (4.1.1) *satisfies* $\mathcal{T}(-t)u(t) \to u^\pm$ *in* $H^1(\mathbb{R}^N)$ *as* $t \to \pm\infty$. *The scattering operator*

$$\mathbf{S} : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N) \quad \textit{mapping} \quad u^- \mapsto u^+$$

is continuous, one-to-one, and its inverse is continuous $H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$. *In addition,* $\|u^+\|_{L^2} = \|u^-\|_{L^2}$ *and* $\|\nabla u^+\|_{L^2} = \|\nabla u^-\|_{L^2}$ *for every* $u^- \in H^1(\mathbb{R}^N)$.

PROOF. The result follows from Theorems 7.8.1 and 7.8.4, and Remark 7.8.7, by setting $\mathbf{S} = U_+\Omega_-$. Note that $\mathbf{S}^{-1} = U_-\Omega_+$. □

REMARK 7.8.9. We note that the conclusion of Theorems 7.8.1, 7.8.4, and 7.8.8 also hold if $N = 1$ or $N = 2$. See Remark 7.6.7 and Nakanishi [259].

7.9. Comments

The estimates of Theorem 7.3.1 hold for more general nonlinearities. In particular, consider $g(u) = f(u(\cdot))$, where f is as in the beginning of Section 7.2. Assume that $F(s) \le 0$ for all $s \ge 0$, and that there exists $0 < \delta \le 4/N$ such that $-s^{-2-\delta}F(s)$ is a nondecreasing function of $s \ge 0$. We have the following result.

PROPOSITION 7.9.1. *Let* g *be as above. If* $\varphi \in H^1(\mathbb{R}^N)$ *is such that* $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$, *and if* u *is the maximal solution of* (4.1.1), *then*

$$\int_{\mathbb{R}^N} F(u(t))dx \le C|t|^{-\frac{N\delta}{2}} \quad \textit{and} \quad \|u(t)\|_{L^r} \le C|t|^{-\frac{N^2\delta}{4}(\frac{1}{2}-\frac{1}{r})}$$

for all $t \in \mathbb{R}$ *and all* $2 \le r \le 2N/(N-2)$.

PROOF. It follows from Theorem 7.2.1 that v defined by (7.2.7) satisfies

$$t^2E(v) \le \|x\varphi\|_{L^2}^2 - \int_0^t s \int_{\mathbb{R}^N} (8(N+2)F(u) - 4N\,\mathrm{Re}(f(u)\overline{u}))dx\,ds\,.$$

By assumption, $-sf(s) \le -(2+\delta)F(s)$. Therefore,

$$\int_{\mathbb{R}^N} |\nabla v(t)|^2\,dx - 2t^2 \int_{\mathbb{R}^N} F(u(t))dx \le (4 - N\delta) \int_0^t \int_{\mathbb{R}^N} sF(u(s))dx\,ds\,.$$

One concludes as for Theorem 7.3.1 that

$$\int_{\mathbb{R}^N} F(u(t))dx \le C|t|^{-\frac{N\delta}{2}}\,,$$

and so $\|\nabla v(t)\|_{L^2} \le C|t|^{-\frac{N\delta}{4}}$. The result now follows from Gagliardo-Nirenberg's inequality and conservation of charge. □

Applying these estimates, one can extend the scattering theory in Σ to more general nonlinearities (see Ginibre and Velo [133, 132]).

The result of Theorem 7.7.1 can be generalized in the following way. Consider $g(u) = f(u(\cdot))$, where f is as in the beginning of Section 7.2. Assume that there exists $\delta < 4/N$ such that

$$F(s) \leq C(s^2 + s^{\delta+2}) \tag{7.9.1}$$

so that all solutions of (4.1.1) are global (see Section 6.1). Assume further that

$$|f(s)| \leq C(s^{\mu+1} + s^{\nu+1}) \quad \text{for all } s \geq 0 \tag{7.9.2}$$

for some $4/N < \mu \leq \nu < 4/(N-2)$. Assume finally that

$$2F(s) - sf(s) \geq c\min\{s^{\mu+2}, s^{\nu+2}\} \quad \text{for all } s \geq 0\,. \tag{7.9.3}$$

We have the following result.

THEOREM 7.9.2. *Assume $N \geq 3$ and let g be as above. For every $\varphi \in H^1(\mathbb{R}^N)$, the maximal solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ of (4.1.1) satisfies*

$$\|u(t)\|_{L^r} \underset{t\to\pm\infty}{\longrightarrow} 0\,,$$

for every $2 < r < 2N/(N-2)$.

PROOF. The proof is an adaptation of the proof of Theorem 7.7.1. We only prove the result for $t \to +\infty$, the case $t \to -\infty$ being similar. Note also that we need only establish the result for $r = \nu + 2$, the general case following immediately from the boundedness of the solution in $H^1(\mathbb{R}^N)$ and Hölder's inequality.

STEP 1. We have the estimate

$$\int_{\{|x|\geq t\log t\}} |u(t,x)|^r\,dx \underset{t\to+\infty}{\longrightarrow} 0 \quad \text{for all } 2 < r < \frac{2N}{N-2}\,. \tag{7.9.4}$$

The proof is the same as that of Step 1 of the proof of Theorem 7.7.1.

STEP 2. For every $\varepsilon > 0$, $t > 1$, $\tau > 0$, there exists $t_0 > \max\{t, 2\tau\}$ such that

$$\int_{t_0-2\tau}^{t_0}\int_{\{|x|\leq t\log t\}} \min\{|u|^{\mu+2}, |u|^{\nu+2}\}dx\,dt \leq \varepsilon\,. \tag{7.9.5}$$

This follows from Morawetz's estimate and (7.9.3). (See the proof of Theorem 7.7.1, Step 2.)

STEP 3. For every $\varepsilon > 0$, $t > 1$, $\tau > 0$ and $2 < r < 2N/(N-2)$, there exists $t_0 > \max\{t, 2\tau\}$ such that

$$\int_{t_0-2\tau}^{t_0}\int_{\{|x|\leq t\log t\}} |u(t,x)|^r\,dx\,dt \leq \varepsilon\,. \tag{7.9.6}$$

Note that we need only establish the result for $r = \mu + 2$, the general case following immediately from the boundedness of the solution in $H^1(\mathbb{R}^N)$ and Hölder's inequality. Consider $\varepsilon > 0$, $t > 1$, $\tau > 0$. Let

$$v(t,x) = \begin{cases} u(t,x) & \text{if } |u(t,x)| \leq 1 \\ 0 & \text{if } |u(t,x)| > 1, \end{cases}$$

and $w = u - v$. It follows from Step 2 that for every $\varepsilon' > 0$, there exists $t_0 > \max\{t, 2\tau\}$ such that

$$\int_{t_0-2\tau}^{t_0} \int_{\{|x|\le t\log t\}} |v(t,x)|^{\nu+2}\,dx\,dt + \int_{t_0-2\tau}^{t_0} \int_{\{|x|\le t\log t\}} |w(t,x)|^{\mu+2}\,dx\,dtb \le \varepsilon'. \tag{7.9.7}$$

Note that

$$\int_{t_0-2\tau}^{t_0} \int_{\{|x|\le t\log t\}} |u(t,x)|^{\mu+2}\,dx\,dt = \int_{t_0-2\tau}^{t_0} \int_{\{|x|\le t\log t\}} |v(t,x)|^{\mu+2}\,dx\,dt + \int_{t_0-2\tau}^{t_0} \int_{\{|x|\le t\log t\}} |w(t,x)|^{\mu+2}\,dx\,dt. \tag{7.9.8}$$

Applying Hölder's inequality in space and time, and conservation of charge, we obtain

$$\int_{t_0-2\tau}^{t_0} \int_{\{|x|\le t\log t\}} |w(t,x)|^{\mu+2} \le C\tau^{\frac{\nu-\mu}{\nu+2}} \Big(\int_{t_0-2\tau}^{t_0} \int_{\{|x|\le t\log t\}} |w(t,x)|^{\nu+2} \Big)^{\frac{\mu+2}{\nu+2}}. \tag{7.9.9}$$

Choosing ε' such that

$$\varepsilon' + C\tau^{\frac{\nu-\mu}{\nu+2}} (\varepsilon')^{\frac{\mu+2}{\nu+2}} \le \varepsilon, \tag{7.9.10}$$

the result follows from (7.9.8), (7.9.9), (7.9.7), and (7.9.10).

STEP 4. For every $\varepsilon > 0$ and $t, \tau > 0$, there exists $t_0 > \max\{t, \tau\}$ such that

$$\sup_{s\in[t_0-\tau, t_0]} \|u(s)\|_{L^{\nu+2}} \le \varepsilon. \tag{7.9.11}$$

Consider $t > \tau > 0$, and write

$$\begin{aligned} u(t) &= \mathcal{T}(t)\varphi + i\int_0^{t-\tau} \mathcal{T}(t-s)g(u(s))ds + i\int_{t-\tau}^{t} \mathcal{T}(t-s)g(u(s))ds \\ &= v(t) + w(t,\tau) + z(t,\tau). \end{aligned} \tag{7.9.12}$$

It follows from Corollary 2.3.7 that

$$\|v(t)\|_{L^{\nu+2}} \underset{t\to\infty}{\longrightarrow} 0. \tag{7.9.13}$$

Arguing as in the proof of Theorem 7.7.1, Step 3, one shows easily that there exists K such that

$$\|w(t,\tau)\|_{L^{\nu+2}} \le K\tau^{-\frac{\nu(N\mu-2\max\{\mu,1\})}{2\mu(\nu+2)}} \quad \text{for all } t > \tau > 0. \tag{7.9.14}$$

Let now

$$\rho = \frac{(\mu+1)(\nu+2)}{\nu+1} \in (2, \mu+2] \tag{7.9.15}$$

Arguing as in the proof of Theorem 7.7.1, Step 3, one shows easily that there exist $L < \infty$ and $a, b, c > 0$ such that

$$\|z(t,\tau)\|_{L^{\nu+2}} \le L(1+\tau^a)\Big[\Big(\int_{t-\tau}^t \|u\|_{L^{\nu+2}}^{\nu+2}\Big)^b + \Big(\int_{t-\tau}^t \|u\|_{L^\rho}^\rho\Big)^c\Big], \tag{7.9.16}$$

and one concludes as in the proof of Theorem 7.7.1, Step 3.

STEP 5. We need to show that for every $\varepsilon > 0$, we have $\|u(t)\|_{L^{\nu+2}} \le \varepsilon$ for t large. Consider $\varepsilon > 0$, and let τ_ε be defined by

$$K\tau_\varepsilon^{-\frac{\nu(N\mu-2\max\{\mu,1\})}{2\mu(\nu+2)}} = \frac{\varepsilon}{4}. \tag{7.9.17}$$

It follows from (7.9.13) that there exists $t_1 > 0$ such that

$$\|v(t)\|_{L^{\nu+2}} \le \frac{\varepsilon}{4} \quad \text{for } t \ge t_1. \tag{7.9.18}$$

Applying (7.9.12), (7.9.13), (7.9.14), and (7.9.17), we obtain

$$\|u(t)\|_{L^{\nu+2}} \le \frac{\varepsilon}{2} + \|z(t,\tau_\varepsilon)\|_{L^{\nu+2}} \quad \text{for } t \ge \max\{t_1, \tau_\varepsilon\}. \tag{7.9.19}$$

Note also that, given $t \ge \tau_\varepsilon$, we deduce from (2.2.4) that

$$\|z(t,\tau_\varepsilon)\|_{L^{\nu+2}} \le C\tau_\varepsilon^{1-\frac{N\nu}{2(\nu+2)}} \sup_{t-\tau_\varepsilon,t} \big(\|u\|_{L^\rho}^{\mu+1} + \|u\|_{L^{\nu+2}}^{\nu+1}\big), \tag{7.9.20}$$

where ρ is given by (7.9.15) (compare the proof of Theorem 7.7.1, Step 4). By Hölder's inequality and conservation of charge,

$$\|u\|_{L^\rho}^{\mu+1} \le C\|u\|_{L^{\nu+2}}^{\frac{\mu(\nu+2)-\nu}{\nu}}. \tag{7.9.21}$$

Note that

$$\frac{\mu(\nu+2)-\nu}{\nu} \le \frac{\mu(\nu+1)}{\nu} \le \nu+1,$$

and so it follows from (7.9.20), (7.9.21), and the boundedness of u in $H^1(\mathbb{R}^N)$ that there exists M such that

$$\|z(t,\tau_\varepsilon)\|_{L^{\nu+2}} \le M\tau_\varepsilon^{1-\frac{N\nu}{2(\nu+2)}} \sup_{t-\tau_\varepsilon,t} \big(\|u\|_{L^{\nu+2}}^{\frac{\mu(\nu+2)-\nu}{\nu}}\big). \tag{7.9.22}$$

By Step 4, there exists $t_0 \ge \max\{\tau_\varepsilon, t_1\}$ such that $\|u(t)\|_{L^{\nu+2}} \le \varepsilon$ for $t \in [t_0 - \tau_\varepsilon, t_0]$. Therefore, we may define

$$t_\varepsilon = \sup\{t \ge t_0 : \|u(s)\|_{L^{\nu+2}} \le \varepsilon \text{ for all } s \in [t_0-\tau_\varepsilon, t]\}.$$

Assume that $t_\varepsilon < \infty$. It follows that

$$\|u(t_\varepsilon)\|_{L^{\nu+2}} = \varepsilon. \tag{7.9.23}$$

Applying (7.9.22), (7.9.23), and (7.9.19) with $t = t_\varepsilon$, we deduce that

$$\varepsilon \le \frac{\varepsilon}{2} + M\tau_\varepsilon^{1-\frac{N\nu}{2(\nu+2)}} \varepsilon^{\frac{\mu(\nu+2)-\nu}{\nu}},$$

which implies $\tau_\varepsilon^{1-\frac{N\nu}{2(\nu+2)}}\varepsilon^{\frac{\mu(\nu+2)-2\nu}{\nu}} \geq 1/2M$. Applying (7.9.17), we obtain that

$$\tau_\varepsilon^{-\gamma} \geq \frac{1}{2M(4K)^{\frac{\mu(\nu+2)-2\nu}{\nu}}}, \tag{7.9.24}$$

where

$$\gamma = \frac{(\mu(\nu+2)-2\nu)(N\mu-2-2\max\{\mu,1\})+\nu(N\mu-4)}{2\mu(\nu+2)}.$$

One can conclude as in the proof of Theorem 7.7.1, Step 4, provided $\gamma > 0$. If $\mu \leq 1$, then

$$\gamma = \frac{(\mu(\nu+2)-\nu)(N\mu-4)}{2\mu(\nu+2)}.$$

Note that $\mu > 2/N > \nu/(\nu+2)$, so that $\mu(\nu+2)-\nu > 0$. Since also $N\mu > 4$, we see that $\gamma > 0$. If $\mu > 1$, then $\gamma = \frac{N-2}{2}(\mu - \phi(\nu))$, where

$$\phi(x) = \frac{(N-2)x+4}{(N-2)(x+2)}.$$

When $N \geq 4$, ϕ is nondecreasing, and so $\phi(\nu) \leq \phi(4/(N-2)) = 4/N$. This implies again that $\gamma > 0$. When $N = 3$, $\phi(x)$ is decreasing and $\phi(4/(N-2)) = 4/N$. Since $N\mu > 4$, there exists $\nu \leq \overline{\nu} < 4/(N-2)$ such that $\mu - \phi(\overline{\nu}) > 0$. Observe that f satisfies as well assumptions (7.9.2) and (7.9.3) with ν replaced by $\overline{\nu}$. Therefore, in this case also, $\gamma > 0$. This completes the proof. □

REMARK 7.9.3. It is not difficult to extend the results of Theorems 7.7.3, 7.8.1, 7.8.4, and 7.8.8 to the case where g is as in Theorem 7.9.2. Therefore, one can construct a scattering theory in $H^1(\mathbb{R}^N)$ for such nonlinearities (see Ginibre and Velo [137, 138]).

REMARK 7.9.4. Concerning the decay of solutions in L^∞, see Ginibre and Velo [132], Dong and Li [107] (one-dimensional case), Cazenave [57] (two-dimensional case), and Lin and Strauss [320] (three-dimensional case).

REMARK 7.9.5. When $g(u) = \lambda|u|^{\frac{4}{N}}u$, $\lambda \in \mathbb{R}$, a scattering theory can be constructed in a subset of $L^2(\mathbb{R}^N)$, containing, for example, all functions with small L^2 norm and also all functions in $u \in L^2(\mathbb{R}^N)$ such that $xu \in L^2(\mathbb{R}^N)$ in the case $\lambda < 0$ (see Cazenave and Weissler [71] and also M. Weinstein [359] for a related result). A low energy scattering theory can also be constructed in $H^s(\mathbb{R}^N)$; see Nakamura and Ozawa [255].

REMARK 7.9.6. When $g(u) = \lambda|u|^\alpha u$, with $\lambda > 0$ and $\alpha \geq 4/N$, it is not difficult to adapt the proofs of Theorems 7.8.1, 7.8.4, and 7.8.8 (by using Theorem 6.2.1) in order to construct the scattering operator S on the set $\{u \in H^1(\mathbb{R}^N) : \|u\|_{H^1} \leq \varepsilon\}$ for ε small enough. Obviously, the scattering operator cannot be defined on the whole space $H^1(\mathbb{R}^N)$, since some solutions blow up in finite time (see Remark 6.8.1). The assumption $\alpha \geq 4/N$ is optimal (see Cazenave and Weissler [72], Remark 4.4).

REMARK 7.9.7. The results of Sections 7.3 and 7.4 can be extended to Hartree-type nonlinearities. See Cazenave, Dias, and Figueira [61], Chadam and Glassey [76], Dias [103], Dias and Figueira [104], Ginibre and Velo [134], Hayashi [159], Hayashi and Ozawa [185, 188, 189, 186], Hayashi and Tsutsumi [194], Lange [221, 220], P.-L. Lions [233, 234], Nawa and Ozawa [271], and Pecher and Von Wahl [296]. The results of Sections 7.6, 7.7, and 7.8 can also be extended; see Ginibre and Velo [143] and Nakanishi [258].

REMARK 7.9.8. It follows from Theorem 7.5.2 that if $g(u) = \lambda|u|^\alpha u$ with $\lambda \in \mathbb{R}$ and $\alpha \leq 2/N$, then no solution of (4.1.1) has a scattering state, even for the L^2 topology. This means that no solution behaves as $t \to \pm\infty$ like a solution of the Schrödinger equation $iu_t + \Delta u = 0$. However, it may happen that some solutions behave as $t \to \pm\infty$ like a solution of a different, linear Schrödinger-type equation. This is the theory of modified wave operators. See Ginibre and Ozawa [129], Ginibre and Velo [141, 142, 144], Hayaski, Kaikina, and Naumkin [172], Hayashi and Naumkin [181], Hayashi, Naumkin, and Ozawa [184], Nakanishi [262, 261], and Ozawa [287].

REMARK 7.9.9. In the case $N = 3$ and $g(u) = \lambda|u|^2 u$ with $\lambda < 0$, Colliander et al. [90] have shown that the Cauchy problem is globally well-posed in $H^s(\mathbb{R}^N)$ for $s > 4/5$ and constructed the scattering operator on all of $H^s(\mathbb{R}^N)$. The results are based on a new form of Morawetz's estimate.

CHAPTER 8

Stability of Bound States in the Attractive Case

In this chapter we study the stability of standing waves of the nonlinear Schrödinger equation for a class of attractive nonlinearities. Throughout the chapter, we consider the problem (4.1.1) in the model case $g(u) = \lambda|u|^\alpha u$ where $\lambda > 0$ and $0 < \alpha < 4/(N-2)$ ($0 < \alpha < \infty$ if $N = 1, 2$), and we indicate references concerning more general nonlinearities. Without loss of generality, we may assume that $\lambda = 1$. We have seen in the preceding chapter that when $\lambda < 0$, all solutions converge weakly to 0, as $t \to \pm\infty$. When $\lambda > 0$, we have a completely different situation. Indeed, in the case $\alpha \geq 4/N$, all solutions with small initial data converge weakly to 0 as $t \to \pm\infty$, see Theorem 6.2.1; and, on the other hand, it follows from Remark 6.8.1 that solutions with "large" initial data blow up in finite time. In fact, in both the case $\alpha \geq 4/N$ and the case $\alpha < 4/N$, we show in Section 8.1 the existence of a third type of solutions, that are global but do not converge weakly to 0. More precisely, we construct solutions of (4.1.1) of the form

$$u(t,x) = e^{i\omega t}\varphi(x),$$

where $\omega \in \mathbb{R}$ and $\varphi \in H^1(\mathbb{R}^N)$, $\varphi \neq 0$. Such solutions are called standing waves, or stationary states, or localized solutions. In Section 8.2, we show that when $\alpha \geq 4/N$ a class of standing waves is unstable, and in Section 8.3, we show that when $\alpha < 4/N$ a class of standing waves is stable. We apply purely variational methods, and we refer to Section 8.4 for other methods.

8.1. Nonlinear Bound States

Throughout this section, we consider g of the form

$$g(u) = |u|^\alpha u \tag{8.1.1}$$

with

$$0 < \alpha < \frac{4}{N-2} \quad (0 < \alpha < \infty \text{ if } N = 1, 2). \tag{8.1.2}$$

We look for solutions of (4.1.1) of the form

$$u(t,x) = e^{i\omega t}\varphi(x), \tag{8.1.3}$$

where $\omega \in \mathbb{R}$ is a given parameter and $\varphi \in H^1(\mathbb{R}^N)$, $\varphi \neq 0$. It is clear that φ must solve the problem

$$\begin{cases} \varphi \in H^1(\mathbb{R}^N), \varphi \neq 0, \\ -\Delta\varphi + \omega\varphi = |\varphi|^\alpha\varphi. \end{cases} \tag{8.1.4}$$

We refer to Strauss [323], Berestycki and Lions [25], Berestycki, Gallouët, and Kavian [24], Berestycki, Lions, and Peletier [26], and Jones and Küpper [199] for a complete study of (8.1.4) as well as for similar problems with more general nonlinearities. In particular, it is known that if $\omega \le 0$, then (8.1.4) does not have any solution. Therefore, from now on we assume that

$$\omega > 0\,. \tag{8.1.5}$$

We begin with a regularity result.

THEOREM 8.1.1. *Assume (8.1.2), $a > 0$, and $b \in \mathbb{R}$. If $u \in H^1(\mathbb{R}^N)$ satisfies $-\Delta u + au = b|u|^\alpha u$ in $H^{-1}(\mathbb{R}^N)$, then the following properties hold:*

(i) *$u \in W^{3,p}(\mathbb{R}^N)$ for every $2 \le p < \infty$. In particular, $u \in C^2(\mathbb{R}^N)$ and $|D^\beta u(x)| \underset{|x|\to\infty}{\longrightarrow} 0$ for all $|\beta| \le 2$.*

(ii) *There exists $\varepsilon > 0$ such that $e^{\varepsilon|x|}(|u(x)| + |\nabla u(x)|) \in L^\infty(\mathbb{R}^N)$.*

PROOF. Changing $u(x)$ to $(|b|/\sqrt{a})^{-\frac{1}{\alpha}} u(x/\sqrt{a})$, we may assume that u satisfies

$$-\Delta u + u = b|u|^\alpha u \tag{8.1.6}$$

with $|b| = 1$. Note that (8.1.6) can be written in the form

$$\mathcal{F}^{-1}((1 + 4\pi^2|\xi|^2)\mathcal{F}u) = b|u|^\alpha u\,, \tag{8.1.7}$$

where $\mathcal{F}$ is the Fourier transform and (8.1.7) makes sense in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^N)$.

(i) Note that if $u \in L^p(\mathbb{R}^N)$ for some $\alpha+1 < p < \infty$, then $|u|^\alpha u \in L^{\frac{p}{\alpha+1}}(\mathbb{R}^N)$. It follows that $u \in H^{2,\frac{p}{\alpha+1}}(\mathbb{R}^N) = W^{2,\frac{p}{\alpha+1}}(\mathbb{R}^N)$ (see Remark 1.4.1). Applying Sobolev's embedding theorem, this implies that

$$u \in L^q(\mathbb{R}^N) \quad \text{for all } q \ge \frac{p}{\alpha+1} \text{ such that } \frac{1}{q} \ge \frac{\alpha+1}{p} - \frac{2}{N}\,. \tag{8.1.8}$$

Consider the sequence q_j defined by

$$\frac{1}{q_j} = (\alpha+1)^j \left(\frac{1}{\alpha+2} - \frac{2}{N\alpha} + \frac{2}{N\alpha(\alpha+1)^j} \right).$$

Since $(N-2)\alpha < 4$, we see that $\frac{\alpha}{\alpha+2} - \frac{2}{N} = -\delta$ with $\delta > 0$. We have

$$\frac{1}{q_{j+1}} - \frac{1}{q_j} = -(\alpha+1)^j \delta \le -\delta\,,$$

and so $\frac{1}{q_j}$ is decreasing and $\frac{1}{q_j} \underset{j\to\infty}{\longrightarrow} -\infty$. Since $q_0 = \alpha + 2$, it follows that there exists $k \ge 0$ such that

$$\frac{1}{q_\ell} > 0 \quad \text{for } 0 \le \ell \le k; \qquad \frac{1}{q_{k+1}} \le 0\,.$$

We claim that $u \in L^{q_k}(\mathbb{R}^N)$. Indeed, $u \in H^1(\mathbb{R}^N)$ so that $u \in L^{q_0}(\mathbb{R}^N)$; and if $u \in L^{q_\ell}(\mathbb{R}^N)$ for some $\ell \le k-1$, then by (8.1.8),

$$u \in L^q(\mathbb{R}^N) \quad \text{for all } q \ge \frac{q_\ell}{\alpha+1} \text{ such that } \frac{1}{q} \ge \frac{\alpha+1}{q_\ell} - \frac{2}{N} = \frac{1}{q_{\ell+1}}\,.$$

In particular, $u \in L^{q_{\ell+1}}(\mathbb{R}^N)$. Hence the claim follows. Applying once again (8.1.8), we deduce that $u \in L^q(\mathbb{R}^N)$ for all $q \ge q_k/(\alpha+1)$ such that $1/q \ge 1/q_{k+1}$. In particular, we may let $q = \infty$. Therefore, $|u|^\alpha u \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, so that $u \in W^{2,p}(\mathbb{R}^N)$ for all $2 \le p < \infty$. Applying Remark 1.3.1(vii), we obtain that $|u|^\alpha u \in W^{1,p}(\mathbb{R}^N)$ for all $2 \le p < \infty$. In particular, it follows from (8.1.6) that for every $j \in \{1, \dots, N\}$, $(-\Delta + I)\partial_j u \in L^p(\mathbb{R}^N)$; i.e., $\mathcal{F}^{-1}((1 + 4\pi^2|\xi|^2)\mathcal{F}\partial_j u) \in L^p(\mathbb{R}^N)$. Thus $\partial_j u \in H^{2,p}(\mathbb{R}^N) = W^{2,p}(\mathbb{R}^N)$ (see Remark 1.4.1). Therefore, $u \in W^{3,p}(\mathbb{R}^N)$ for all $2 \le p < \infty$. By Sobolev's embedding, $u \in C^{2,\delta}(\mathbb{R}^N)$ for all $0 < \delta < 1$, so that $|D^\beta u(x)| \to 0$ as $|x| \to \infty$ for all $|\beta| \le 2$.

(ii) Let $\varepsilon > 0$ and $\theta_\varepsilon(x) = e^{\frac{|x|}{1+\varepsilon|x|}}$. θ_ε is bounded, Lipschitz continuous, and $|\nabla \theta_\varepsilon| \le \theta_\varepsilon$ a.e. Taking the scalar product of the equation with $\theta_\varepsilon u \in H^1(\mathbb{R}^N)$, we obtain

$$\text{Re} \int_{\mathbb{R}^N} \nabla u \cdot \nabla(\theta_\varepsilon \overline{u}) + \int_{\mathbb{R}^N} \theta_\varepsilon |u|^2 \le \int_{\mathbb{R}^N} \theta_\varepsilon |u|^{\alpha+2}. \tag{8.1.9}$$

Note that $\nabla(\theta_\varepsilon \overline{u}) = \overline{u}\nabla\theta_\varepsilon + \theta_\varepsilon \nabla \overline{u}$. Therefore,

$$\text{Re}\Big(\nabla u \cdot \nabla(\theta_\varepsilon \overline{u})\Big) \ge \theta_\varepsilon |\nabla u|^2 - \theta_\varepsilon |u||\nabla u|.$$

Applying (8.1.9) and Cauchy-Schwarz's inequality, we obtain easily

$$\int_{\mathbb{R}^N} \theta_\varepsilon |u|^2 \le 2 \int_{\mathbb{R}^N} \theta_\varepsilon |u|^{\alpha+2}. \tag{8.1.10}$$

By (i), there exists $R > 0$ such that $|u(x)|^\alpha \le 1/4$ for $|x| \ge R$. Therefore,

$$2\int_{\mathbb{R}^N} \theta_\varepsilon |u|^{\alpha+2} \le 2 \int_{\{|x| \le R\}} e^{|x|} |u|^{\alpha+2} + \frac{1}{2} \int_{\mathbb{R}^N} \theta_\varepsilon |u|^2. \tag{8.1.11}$$

Putting together (8.1.10) and (8.1.11), we obtain

$$\int_{\mathbb{R}^N} \theta_\varepsilon |u|^2 \le 4 \int_{\{|x| \le R\}} e^{|x|} |u|^{\alpha+2}.$$

Letting $\varepsilon \downarrow 0$, we deduce that

$$\int_{\mathbb{R}^N} e^{|x|} |u|^2 < \infty. \tag{8.1.12}$$

Since u is globally Lipschitz continuous by (i), we deduce easily from (8.1.12) that $|u(x)|^{N+2} e^{|x|}$ is bounded. Next, applying ∂_j to equation (8.1.6) and multiplying the resulting equation by $\theta_\varepsilon \partial_j \overline{u}$ for $j = 1, \dots, N$, we obtain by the same calculations as above that

$$\int_{\mathbb{R}^N} e^{|x|} |\nabla u|^2 < \infty.$$

Since ∇u is globally Lipschitz continuous by (i), we deduce that $|\nabla u(x)|^{N+2} e^{|x|}$ is bounded, as above. □

LEMMA 8.1.2. *Assume (8.1.2), $a > 0$, and $b \in \mathbb{R}$. If $u \in H^1(\mathbb{R}^N)$ satisfies $-\Delta u + au = b|u|^\alpha u \in H^{-1}(\mathbb{R}^N)$, then the following properties hold:*

(i) $\int_{\mathbb{R}^N} |\nabla u|^2 + a \int_{\mathbb{R}^N} |u|^2 = b \int_{\mathbb{R}^N} |u|^{\alpha+2}$.

(ii) (Pohozaev's identity)

$$(N-2)\int_{\mathbb{R}^N} |\nabla u|^2 + Na\int_{\mathbb{R}^N} |u|^2 = \frac{2Nb}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2}.$$

PROOF. Equality (i) is obtained by multiplying the equation by $\overline{u}$, taking the real part, and integrating by parts.

The identity (ii) is obtained by multiplying the equation by $x\cdot\nabla\overline{u}$ and taking the real parts. Indeed, one obtains

$$\operatorname{Re}(-\Delta u(x\cdot\nabla\overline{u})) + a\operatorname{Re}(u(x\cdot\nabla\overline{u})) = b\operatorname{Re}(|u|^\alpha u(x\cdot\nabla\overline{u})).$$

Applying the identities

$$\begin{aligned}
\operatorname{Re}(-\Delta u(x\cdot\nabla\overline{u})) &= -\frac{N-2}{2}|\nabla u|^2 + \nabla\cdot\Big(-\operatorname{Re}(\nabla u(x\cdot\nabla\overline{u})) + \frac{1}{2}x|\nabla u|^2\Big),\\
\operatorname{Re}(u(x\cdot\nabla\overline{u})) &= -\frac{N}{2}|u|^2 + \frac{1}{2}\nabla\cdot(x|u|^2),\\
\operatorname{Re}(|u|^\alpha u(x\cdot\nabla\overline{u})) &= -\frac{N}{\alpha+2}|u|^{\alpha+2} + \frac{1}{\alpha+2}\nabla\cdot(x|u|^{\alpha+2}),
\end{aligned}$$

and integrating over $\mathbb{R}^N$ yields the result. Note that these calculations are justified by the regularity properties of Theorem 8.1.1. □

Before stating the main result of this section we need to introduce some notation. Assuming (8.1.2) and $\omega > 0$, we introduce the following functionals on $H^1(\mathbb{R}^N)$.

(8.1.13) $$T(u) = \int_{\mathbb{R}^N} |\nabla u|^2\,dx,$$

(8.1.14) $$V(u) = \frac{1}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2}\,dx - \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2\,dx,$$

(8.1.15) $$S(u) = \frac{1}{2}T(u) - V(u),$$

(8.1.16) $$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2}\,dx = S(u) - \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2\,dx.$$

One easily verifies that these functionals are in $C^1(H^1(\mathbb{R}^N),\mathbb{R})$, and that $T'(u) = -2\Delta u$, $V'(u) = |u|^\alpha u - \omega u$. We introduce the sets A and G defined by

(8.1.17) $$A = \{u\in H^1(\mathbb{R}^N) : u\neq 0 \text{ and } -\Delta u + \omega u = |u|^\alpha u\},$$

(8.1.18) $$G = \{u\in A : S(u)\le S(v) \text{ for all } v\in A\}.$$

We have the following result.

COROLLARY 8.1.3. *Assume* (8.1.2) *and* $\omega > 0$. *If* $u \in H^1(\mathbb{R}^N)$ *satisfies* (8.1.4), *then*

(8.1.19)
$$S(u) = \frac{1}{N} T(u),$$

(8.1.20)
$$(N-2)T(u) = 2NV(u),$$

(8.1.21)
$$E(u) = \frac{N\alpha - 4}{2N\alpha} T(u),$$

(8.1.22)
$$\frac{\omega}{2} \int_{\mathbb{R}^N} |u|^2 = \frac{4-(N-2)\alpha}{2N\alpha} T(u).$$

PROOF. These identities follow immediately from Lemma 8.1.2. □

Our goal is to show that A and G are nonempty and to characterize G. For technical reasons, we consider separately the cases $N \ge 3$, $N = 2$, and $N = 1$.

THEOREM 8.1.4. *Assume* $N \ge 3$, (8.1.2), *and* $\omega > 0$.

(i) *A and G are nonempty.*

(ii) *$u \in G$ if and only if u solves the minimization problem*

(8.1.23)
$$\begin{cases} V(u) = \Lambda^{\frac{N}{2}}, \\ S(u) = \min\{S(w) : V(w) = \Lambda^{\frac{N}{2}}\}, \end{cases}$$

where $\Lambda = \frac{N-2}{2N} \inf\{T(v) : V(v) = 1\}$. *In addition,* $\min\{S(w) : V(w) = \Lambda^{\frac{N}{2}}\} = \frac{2}{N-2}\Lambda^{\frac{N}{2}}$.

(iii) *There exists a real-valued, positive, spherically symmetric, and decreasing function $\varphi \in G$ such that* $G = \bigcup\{e^{i\theta}\varphi(\cdot - y) : \theta \in \mathbb{R}, y \in \mathbb{R}^N\}$.

THEOREM 8.1.5. *Assume* $N = 2$, (8.1.2), *and* $\omega > 0$.

(i) *A and G are nonempty.*

(ii) *$u \in G$ if and only if u solves the minimization problem*

(8.1.24)
$$\begin{cases} u \in N \text{ and } \int_{\mathbb{R}^N} |u|^2 = \gamma, \\ S(u) = \min\{S(w) : w \in N\}, \end{cases}$$

where $N = \{u \in H^1(\mathbb{R}^N) : V(u) = 0 \text{ and } u \neq 0\}$ *and* $\gamma = \frac{4}{\omega\alpha} \min_{w \in N} S(w)$.

(iii) *There exists a real-valued, positive, spherically symmetric, and decreasing function $\varphi \in G$ such that* $G = \bigcup\{e^{i\theta}\varphi(\cdot - y) : \theta \in \mathbb{R}, y \in \mathbb{R}^N\}$.

THEOREM 8.1.6. *Assume* $N = 1$, (8.1.2), *and* $\omega > 0$.

(i) *A and G are nonempty.*

(ii) $A = G$.

(iii) *There exists a real-valued, positive, spherically symmetric, and decreasing function $\varphi \in G$ such that* $G = \bigcup\{e^{i\theta}\varphi(\cdot - y) : \theta \in \mathbb{R}, y \in \mathbb{R}\}$.

Let us first consider the case $N = 1$, which is especially simple.

PROOF OF THEOREM 8.1.6. Note that (8.1.4) is the ordinary differential equation

$$-u'' + \omega u = |u|^\alpha u. \tag{8.1.25}$$

Define $c = (\omega(\alpha+2)/2)^{\frac{1}{\alpha}}$, and let φ be the maximal, real-valued solution of (8.1.25) such that $\varphi(0) = c$ and $\varphi'(0) = 0$. It is clear that φ is an even function of x. Furthermore, on multiplying the equation by φ', we obtain

$$\frac{d}{dx}\Big(\frac{1}{2}\varphi'^2 - \frac{\omega}{2}\varphi^2 + \frac{1}{\alpha+2}|\varphi|^{\alpha+2}\Big) = 0,$$

and so

$$\frac{1}{2}\varphi'^2 - \frac{\omega}{2}\varphi^2 + \frac{1}{\alpha+2}|\varphi|^{\alpha+2} = 0 \tag{8.1.26}$$

throughout the existence interval. It follows easily that φ is bounded and therefore exists for all $x \in \mathbb{R}$. Furthermore, $\varphi''(0) = -\omega\alpha c/2 < 0$. Therefore, there exists $a > 0$ such that $\varphi' < 0$ on $(0, a)$. We claim that $\varphi' < 0$ on $(0, \infty)$. Otherwise, there would exist $b > 0$ such that $\varphi' < 0$ on $(0, b)$ and $\varphi'(b) = 0$. Applying (8.1.26), this would imply that $\varphi(b) = -c$. Therefore, there would exist $d \in (0, b)$ such that $\varphi(d) = 0$. Applying again (8.1.26), we would obtain $\varphi'(d) = 0$, which would imply that $\varphi \equiv 0$. Therefore, φ decreases to a limit $\ell \in [0, c)$. In particular, there exists $x_m \to \infty$ such that $\varphi'(x_m) \to 0$. Passing to the limit in (8.1.26), we obtain that

$$\ell^2\Big(\frac{\ell^\alpha}{\alpha+2} - \frac{1}{2}\Big) = 0,$$

which implies $\ell = 0$. Therefore φ decreases to 0, as $x \to +\infty$, and we deduce easily that the decay is exponential. Therefore φ'' and hence φ' also decay exponentially to 0. Therefore, $\varphi \in A$, which proves (i). Let now $v \in A$. On multiplying the equation by $\overline{v}'$, we obtain

$$\frac{1}{2}|v'|^2 - \frac{\omega}{2}|v|^2 + \frac{1}{\alpha+2}|v|^{\alpha+2} = K. \tag{8.1.27}$$

Since $v \in H^1(\mathbb{R})$, it follows that $v(x) \to 0$ as $|x| \to \infty$. Therefore, by the equation, $v''(x) \to 0$ as $|x| \to \infty$, and so $v'(x) \to 0$ as $|x| \to \infty$. Letting $|x| \to \infty$ in (8.1.27), we deduce that $K = 0$, and so

$$\frac{1}{2}|v'|^2 - \frac{\omega}{2}|v|^2 + \frac{1}{\alpha+2}|v|^{\alpha+2} = 0. \tag{8.1.28}$$

In particular, $|v| > 0$, for if v would vanish, then by (8.1.28) v' would vanish at the same time and we would have $v \equiv 0$. Therefore, we may write $v = \rho e^{i\theta}$, where $\rho > 0$ and $\rho, \theta \in C^2(\mathbb{R})$. Writing down the system of equations satisfied by ρ, θ, we see in particular that $\rho\theta'' + 2\rho'\theta' \equiv 0$, which implies that there exists $K \in \mathbb{R}$ such that $\rho^2\theta' \equiv K$, and so $\theta' \equiv K/\rho^2$. On the other hand, since $|v'|$ is bounded, it follows that $\rho^2\theta'^2$ is bounded. This means that K^2/ρ^2 is bounded. Since $\rho(x) \to 0$ as $|x| \to \infty$, we must have $K = 0$. Therefore (remember that $\rho > 0$) $\theta \equiv \theta_0$ for some $\theta_0 \in \mathbb{R}$. Thus $v = e^{i\theta_0}\rho$. Since $\rho \in H^1(\mathbb{R}^N)$, there must exist $x_0 \in \mathbb{R}$ such that $\rho'(x_0) = 0$; and, by (8.1.28), $\rho(x_0) = c$. Let now $w(x) = \rho(x - x_0)$. It follows that w satisfies (8.1.25), $w(0) = c$, and $w'(0) = 0$. By uniqueness of the initial-value problem for (8.1.25), we have $w \equiv \varphi$, and so $v(x) = e^{i\theta_0}\varphi(x + x_0)$, which completes the proof. □

We next consider the case $N \geq 3$, and we begin with the following lemma.

LEMMA 8.1.7. *Assume $N \geq 3$, (8.1.2), and $\omega > 0$. It follows that the minimization problem*

$$\begin{cases} V(u) = 1 \\ T(u) = \min\{T(w) : V(w) = 1\} \end{cases} \tag{8.1.29}$$

has a solution. Every solution u of (8.1.29) satisfies the equation

$$-\Delta u + \Lambda\omega u = \Lambda|u|^\alpha u\,,$$

where

$$\Lambda = \frac{N-2}{2N}\inf\{T(v) : V(v) = 1\}\,. \tag{8.1.30}$$

PROOF. We repeat the proof of Berestycki and Lions [25]. We recall the definition of the Schwarz symmetrization. If $u \in L^2(\mathbb{R}^N)$ is a nonnegative function, we denote by u^* the unique spherically symmetric, nonnegative, nonincreasing function such that

$$|\{x \in \mathbb{R}^N : u^*(x) > \lambda\}| = |\{x \in \mathbb{R}^N : u(x) > \lambda\}| \quad \text{for all } \lambda > 0.$$

We refer to Berestycki and Lions [25], appendix A.III for the main properties of the Schwarz symmetrization. In particular,

$$\int_{\mathbb{R}^N} |u^*|^p = \int_{\mathbb{R}^N} |u|^p \tag{8.1.31}$$

for all $1 \leq p < \infty$ such that $u \in L^p(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} |\nabla u^*|^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2 \quad \text{if } u \in H^1(\mathbb{R}^N). \tag{8.1.32}$$

The proof proceeds in four steps.

STEP 1. Selection of a minimizing sequence. Let $u \in H^1(\mathbb{R}^N)$. One can easily find $\lambda > 0$ such that $V(\lambda u) = 1$. Therefore, the set $\{u \in H^1(\mathbb{R}^N) : V(u) = 1\}$ is nonempty. Let $(v_m)_{m\in\mathbb{N}}$ be a minimizing sequence of (8.1.29). Let $u_m = |v_m|^*$. It follows from (8.1.31) and (8.1.32) that $(u_m)_{m\in\mathbb{N}}$ is also a minimizing sequence of (8.1.29).

STEP 2. Estimates of $(u_m)_{m\in\mathbb{N}}$. By definition, $\|\nabla u_m\|_{L^2}$ is bounded, and by Sobolev's inequality, $(u_m)_{m\in\mathbb{N}}$ is bounded in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. On the other hand, $V(u_m) = 1$ implies that

$$\frac{\omega}{2}\int_{\mathbb{R}^N} |u_m|^2 \leq \frac{1}{\alpha+2}\int_{\mathbb{R}^N} |u_m|^{\alpha+2}\,.$$

By Hölder's inequality, this implies that

$$\frac{\omega}{2}\|u_m\|_{L^2}^2 \leq \frac{1}{\alpha+2}\|u_m\|_{L^{\frac{2N}{N-2}}}^{\frac{N\alpha}{2}}\|u_m\|_{L^2}^{\alpha+2-N\alpha/2}\,.$$

Since $\alpha + 2 - N\alpha/2 < 2$, it follows that $(u_m)_{m\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$, hence in $H^1(\mathbb{R}^N)$.

STEP 3. Passage to the limit. By Step 2 and Proposition 1.7.1, there exist $u \in H^1(\mathbb{R}^N)$ and a subsequence, which we still denote by $(u_m)_{m\in\mathbb{N}}$, such that $u_m \to u$ as $m \to \infty$, weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^{\alpha+2}(\mathbb{R}^N)$. By the weak lower semicontinuity of the L^2 norm,

$$V(u) \geq 1 \quad \text{and} \quad T(u) \leq \liminf_{m\to\infty} T(u_m) = \frac{2N}{N-2}\Lambda, \tag{8.1.33}$$

where Λ is defined by (8.1.30). Since $V(u) \geq 1$, it follows that $u \neq 0$. We claim that in fact $V(u) = 1$. Indeed, if $V(u) > 1$, then there exists $\lambda > 1$ such that $v(x) = u(\lambda x)$ satisfies $V(v) = 1$. It follows that

$$T(v) = \lambda^{2-N} T(u) < T(u) \leq \frac{2N}{N-2}\Lambda,$$

which contradicts the definition of Λ. Thus, $V(u) = 1$, which implies by definition of Λ that $T(u) \geq 2N\Lambda/(N-2)$. Comparing with (8.1.33), we see that $T(u) = 2N\Lambda/(N-2)$. Therefore, u satisfies (8.1.29).

STEP 4. Conclusion. Let u be any solution of (8.1.29). There exists a Lagrange multiplier λ such that

$$-\Delta u = \lambda(|u|^\alpha u - \omega u). \tag{8.1.34}$$

Taking the L^2-scalar product of (8.1.34) with u, we obtain

$$T(u) = \lambda\Big((\alpha+2)V(u) + \frac{\alpha\omega}{2}\int_{\mathbb{R}^N} |u|^2\Big) = \lambda\mu$$

with $\mu > 0$. Therefore, $\lambda > 0$. Applying Lemma 8.1.2(ii), we deduce that

$$T(u) = \frac{2N}{N-2}\lambda V(u) = \frac{2N}{N-2}\lambda.$$

Since $T(u) = 2N\Lambda/(N-2)$, it follows that $\lambda = \Lambda$. This completes the proof. □

COROLLARY 8.1.8. *Assume $N \geq 3$, (8.1.2), and $\omega > 0$. If Λ is defined by (8.1.30), then the minimization problem*

$$\begin{cases} V(u) = \Lambda^{\frac{N}{2}} \\ T(u) = \min\{T(w) : V(w) = \Lambda^{\frac{N}{2}}\} \end{cases} \tag{8.1.35}$$

has a solution. Every solution u of (8.1.35) satisfies the equation (8.1.4). In addition,

$$\min\{T(w) : V(w) = \Lambda^{\frac{N}{2}}\} = \frac{2N}{N-2}\Lambda^{\frac{N}{2}}. \tag{8.1.36}$$

PROOF. Given $u \in H^1(\mathbb{R}^N)$, let $Au \in H^1(\mathbb{R}^N)$ be defined by

$$u(x) = Au(\Lambda^{\frac{1}{2}} x).$$

One quite easily verifies that u satisfies (8.1.29) if and only if Au satisfies (8.1.35). Therefore, it follows from Lemma 8.1.7 that (8.1.35) has a solution. Finally, given a solution u of (8.1.35), let v be defined by $Av = u$. It follows that v satisfies (8.1.29), and so $T(v) = 2N\Lambda/(N-2)$, by (8.1.30). This implies that $T(u) = \Lambda^{\frac{N}{2}-1}T(v) = 2N\Lambda^{\frac{N}{2}}/(N-2)$. Hence (8.1.36) follows. Furthermore, since v satisfies

$$-\Delta u + \Lambda\omega u = \Lambda|u|^{\alpha}u\,,$$

it follows that u satisfies (8.1.4). This completes the proof. □

COROLLARY 8.1.9. *Assume* $N \ge 3$, (8.1.2), *and* $\omega > 0$. *If* Λ *is defined by* (8.1.30), *then the minimization problem*

$$\begin{cases} V(u) = \Lambda^{\frac{N}{2}} \\ S(u) = \min\{S(w) : V(w) = \Lambda^{\frac{N}{2}}\} \end{cases} \tag{8.1.37}$$

has a solution. Every solution u *of* (8.1.37) *satisfies the equation* (8.1.4). *In addition,*

$$\min\{S(w) : V(w) = \Lambda^{\frac{N}{2}}\} = \frac{2}{N-2}\Lambda^{\frac{N}{2}}\,. \tag{8.1.38}$$

Finally, u *satisfies* (8.1.37) *if and only if* u *satisfies* (8.1.35).

PROOF. Let $u \in H^1(\mathbb{R}^N)$ be such that $V(u) = \Lambda^{\frac{N}{2}}$. We have

$$S(u) = \frac{1}{2}T(u) - \Lambda^{\frac{N}{2}}\,,$$

so that u satisfies (8.1.35) if and only if u satisfies (8.1.37). Therefore, (8.1.37) has a solution by Corollary 8.1.8. Finally, let u satisfy (8.1.37). It follows that u satisfies (8.1.35), and by Corollary 8.1.8, u satisfies (8.1.4). Furthermore, (8.1.38) is a consequence of (8.1.36) and (8.1.19). □

COROLLARY 8.1.10. *Assume* $N \ge 3$, (8.1.2), *and* $\omega > 0$. *It follows that* G *is nonempty. Furthermore,* $u \in G$ *if and only if* u *satisfies* (8.1.37).

PROOF. Consider a solution u of (8.1.37). It follows from Corollary 8.1.9 that u satisfies (8.1.35) and (8.1.4). In particular, we deduce from (8.1.36) and (8.1.38) that

$$V(u) = \Lambda^{\frac{N}{2}}\,, \quad T(u) = \frac{2N}{N-2}\Lambda^{\frac{N}{2}}\,, \quad S(u) = \frac{2}{N-2}\Lambda^{\frac{N}{2}}\,. \tag{8.1.39}$$

Applying Corollary 8.1.9, we deduce that A is nonempty. Consider any $v \in A$. It follows from Corollary 8.1.3 that if

$$V(v) = \gamma^{\frac{N}{2}}\,, \tag{8.1.40}$$

then

$$T(v) = \frac{2N}{N-2}\gamma^{\frac{N}{2}} \quad \text{and} \quad S(v) = \frac{2}{N-2}\gamma^{\frac{N}{2}}\,. \tag{8.1.41}$$

Let $\sigma = \Lambda/\gamma$, and let $v(x) = w(\sigma^{\frac{1}{2}}x)$. We have $V(w) = \Lambda^{\frac{N}{2}}$, and so by (8.1.36),

$$T(w) \ge \frac{2N}{N-2}\Lambda^{\frac{N}{2}}\,. \tag{8.1.42}$$

By (8.1.41),

$$T(w) = \sigma^{\frac{N}{2}-1}T(v) = \frac{2N}{N-2}\Lambda^{\frac{N}{2}}\frac{\gamma}{\Lambda}.$$

Applying (8.1.42), we deduce that $\gamma \geq \Lambda$. By (8.1.39) and (8.1.41), this implies that

$$S(v) \geq S(u), \tag{8.1.43}$$

and so $u \in G$. In particular, G is nonempty. If we assume further that $v \in G$, then we must have $S(v) \leq S(u)$, since u satisfies (8.1.4). In view of (8.1.43), this means that

$$S(v) = S(v).$$

Applying (8.1.39), (8.1.40), and (8.1.41), we obtain that

$$V(v) = \Lambda^{\frac{N}{2}} \quad \text{and} \quad S(u) = \frac{2}{N-2}\Lambda^{\frac{N}{2}}.$$

By Corollary 8.1.9, v satisfies (8.1.38), which completes the proof. □

Finally, before completing the proof of Theorem 8.1.4, we need the following lemma.

LEMMA 8.1.11. *Let $a : \mathbb{R}^N \to \mathbb{R}$ be continuous and assume that $a(x) \to 0$ as $|x| \to \infty$. If there exists $v \in H^1(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} (|\nabla v|^2 - a|v|^2)\, dx < 0, \tag{8.1.44}$$

then there exist $\lambda > 0$ and a positive solution $u \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ of the equation

$$-\Delta u + \lambda u = au. \tag{8.1.45}$$

In addition, if $w \in H^1(\mathbb{R}^N)$ is nonnegative, $w \neq 0$, and if there exists $\nu \in \mathbb{R}$ such that $-\Delta w + \nu w = aw$, then there exists $c > 0$ such that $w = cu$. In particular, $\mu = \lambda$.

PROOF. We claim that the minimization problem

$$\begin{cases} \|u\|_{L^2} = 1 \\ J(u) = \min\{J(v) : \|v\|_{L^2} = 1\}, \end{cases} \tag{8.1.46}$$

where

$$J(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 - a|u|^2) dx,$$

has a nonnegative solution. Indeed, let $(v_m)_{m\in\mathbb{N}}$ be a minimizing sequence of (8.1.45), and let $u_m = |v_m|$. Since $|u_m| = |v_m|$ and $|\nabla u_m| \leq |\nabla v_m|$, we see that $(u_m)_{m\in\mathbb{N}}$ is also a minimizing sequence. Since $a \in L^\infty(\mathbb{R}^N)$ by assumption, we deduce easily that $(u_m)_{m\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Therefore, there exists a subsequence, which we still denote by $(u_m)_{m\in\mathbb{N}}$, and there exists $u \in H^1(\mathbb{R}^N)$ such that $u_m \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Note that $u \geq 0$ and let us show that u satisfies (8.1.46). For every $r > 0$,

$$\int_{\mathbb{R}^N} |a||u_m^2 - u^2| \leq \int_{\{|x|\leq r\}} |a|(u_m + u)|u_m - u| + \sup\{|a(x)| : |x| \geq r\} \int_{\{|x|\geq r\}} (u_m^2 + u^2)\,.$$

It follows that

$$\int_{\mathbb{R}^N} |a||u_m^2 - u^2| \leq 2\|a\|_{L^\infty}\Big(\int_{\{|x|\leq r\}} |u_m - u|^2\Big)^{\frac{1}{2}} + 2\sup\{|a(x)| : |x| \geq r\}\,.$$

Consider $\varepsilon > 0$. There exists $r > 0$ such that

$$2\sup\{|a(x)| : |x| \geq r\} \leq \varepsilon/2\,.$$

Since the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(B_r)$ is compact, we deduce that, for m large enough,

$$2\|a\|_{L^\infty}\Big(\int_{\{|x|\leq r\}} |u_m - u|^2\Big)^{\frac{1}{2}} \leq \varepsilon/2\,.$$

Therefore,

$$\int_{\mathbb{R}^N} |a||u_m^2 - u^2| \leq \varepsilon \quad \text{for } m \text{ large enough.}$$

It follows that

$$\int_{\mathbb{R}^N} |a|u_m^2 \underset{m\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} |a|u^2\,.$$

Using the weak lower semicontinuity of the L^2 norm, we obtain that

$$J(u) \leq -\mu \quad \text{and} \quad \|u\|_{L^2} \leq 1\,,$$

where $-\mu = \inf\{J(v) : \|v\|_{L^2} = 1\}$. Note that by (8.1.44), $\mu > 0$, and so $u \neq 0$. We have $\|u\|_{L^2} = 1$, since, otherwise, there would exist $k > 1$ such that $w = ku$ satisfies $\|w\|_{L^2} = 1$. We would obtain $J(w) = k^2 J(u) < -\mu$, which is a contradiction by definition of μ. Therefore, $\|u\|_{L^2} = 1$, and, again by definition of μ, we must have $J(u) = -\mu$. This proves the claim. Therefore, there exists a Lagrange multiplier λ such that

$$-\Delta u + \lambda u = au\,. \tag{8.1.47}$$

On taking the L^2-scalar product of the equation with u, we obtain

$$\lambda = \mu > 0\,. \tag{8.1.48}$$

It follows easily from (8.1.47) that $u \in H^2(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ (see the proof of Theorem 8.1.1); and since $u \geq 0$, we deduce from the strong maximum principle (Gilbarg and Trudinger [127], corollary 8.21, p. 199) that

$$u > 0\,. \tag{8.1.49}$$

So far, we have proved the first part of the statement of the lemma. Let now $\nu \in \mathbb{R}$ be such that there exists a solution $w \in H^1(\mathbb{R}^N)$, $w \geq 0$, of the equation

$$-\Delta w + \nu w = aw\,. \tag{8.1.50}$$

We may assume that $w \neq 0$. On multiplying (8.1.47) by w, (8.1.50) by u, and computing the difference, we obtain

$$(\lambda - \nu) \int_{\mathbb{R}^N} wu = 0\,.$$

Since $wu \geq 0$ and $wu \not\equiv 0$ by (8.1.49), this implies that $\nu = \lambda$. We now claim that there exists $c > 0$ such that $w = cu$, for if this were not the case, there would exist $c > 0$ such that $z = w - cu$ takes both positive and negative values. Note that

$$-\Delta z + \lambda z = az\,.$$

On multiplying the equation by z, we see that

$$J(z) = -\lambda \|z\|_{L^2}^2\,.$$

Therefore, y defined by

$$y = \frac{z}{\|z\|_{L^2}}$$

satisfies (8.1.46). It follows that $|y|$ also satisfies (8.1.46). Repeating the argument that we made for u, we deduce that $|y|$ satisfies (8.1.47), and that $|y| > 0$. Therefore, z has a constant sign, which is a contradiction. This completes the proof. □

PROOF OF THEOREM 8.1.4. Parts (i) and (ii) follow immediately from Corollary 8.1.10. It remains to show (iii). Consider $u \in G$, so that u satisfies (8.1.37). Let $f = |\operatorname{Re} u|$, $g = |\operatorname{Im} u|$, and $v = f + ig$. We have $|v| = |u|$ and $|\nabla v| = |\nabla u|$. It follows that v also satisfies (8.1.37). Applying Corollary 8.1.10, this implies that

$$-\Delta v + \omega v = |v|^\alpha v\,,$$

and so

$$\begin{cases} -\Delta f + \omega f = af \\ -\Delta g + \omega g = ag, \end{cases}$$

where $a = |v|^\alpha$. Applying Theorem 8.1.1, we deduce that a satisfies the assumption of Lemma 8.1.11. Furthermore,

$$J(v) = -\omega \|v\|_{L^2}^2 < 0\,.$$

It follows from Lemma 8.1.11 that there exist a positive function z and two nonnegative constants μ, ν such that $f = \mu z$ and $g = \nu z$. In particular, $\operatorname{Re} u$ and $\operatorname{Im} u$ do not change sign, and so there exist $c, d \in \mathbb{R}$ such that $u = cz + i\,dz$. This implies that there exist a positive function ψ and $\theta \in \mathbb{R}$ such that $u = e^{i\theta}\psi$. Therefore, ψ also satisfies (8.1.37), hence (8.1.4) follows by Corollary 8.1.10. By Theorem 8.1.1, $\psi \in C^2(\mathbb{R}^N)$ and $\psi(x) \to 0$ as $|x| \to \infty$. Applying Gidas, Ni, and Nirenberg [125, theorem 2, p. 370], we obtain that there exist a positive, spherically symmetric solution φ of (8.1.4) and $y \in \mathbb{R}^N$ such that $\psi(\cdot) = \varphi(\cdot - y)$. Therefore, $u(\cdot) = e^{i\theta}\varphi(\cdot - y)$. Note that φ, being radially symmetric, satisfies the ordinary differential equation

$$\varphi'' + \frac{N-1}{r}\varphi' + \varphi^{\alpha+1} - \omega\varphi = 0\,.$$

It follows from Kwong [219] that such a solution φ is unique. This completes the proof. □

REMARK 8.1.12. Note that we gave a self-contained proof of statements (i) and (ii). On the contrary, the proof of (iii) relies on the two difficult results of Gidas, Ni, and Nirenberg [125] and of Kwong [219]. We use property (iii) to prove a strong version of the stability property (cf. Section 8.3).

We finally consider the case $N = 2$. Note that the method for $N \geq 3$ does not apply to this case, since by Corollary 8.1.3, $V(u) = 0$ for every $u \in A$.

PROOF OF THEOREM 8.1.5. We proceed in four steps. We define

$$N = \{u \in H^1(\mathbb{R}^N) : V(u) = 0 \text{ and } u \neq 0\}, \tag{8.1.51}$$

$$c = \inf\{S(w) : w \in N\}, \tag{8.1.52}$$

and

$$\gamma = \frac{4}{\omega\alpha} \inf\{S(w) : w \in N\}. \tag{8.1.53}$$

Let us first observe that $\gamma > 0$. Indeed, consider $u \in N$. We have

$$\int_{\mathbb{R}^N} |u|^2 \leq \frac{2}{\omega(\alpha+2)} \int_{\mathbb{R}^N} |u|^{\alpha+2}.$$

On the other hand, it follows from Gagliardo-Nirenberg's inequality that there exists C independent of u such that

$$\int_{\mathbb{R}^N} |u|^{\alpha+2} \leq C(T(u))^{\frac{\alpha}{2}} \int_{\mathbb{R}^N} |u|^2.$$

This implies that there exists $\sigma > 0$ such that $T(u) \geq \sigma$, and so $S(u) \geq \sigma/2$ for all $u \in N$, which implies $\gamma > 0$.

STEP 1. The minimization problem (8.1.24) has a solution. We repeat the proof of Berestycki, Gallouët, and Kavian [24]. It is clear that $N \neq \varnothing$. Let $(v_m)_{m\in\mathbb{N}}$ be a minimizing sequence. In other words, $v_m \neq 0$, $V(v_m) = 0$, and $S(v_m) \to c$. Let $w_m = |v_m|^*$ (see the beginning of the proof of Lemma 8.1.7), so that $(w_m)_{m\in\mathbb{N}}$ has the same properties as $(v_m)_{m\in\mathbb{N}}$. Define now $(u_m)_{m\in\mathbb{N}}$ by $u_m(x) = w_m(\lambda_m^{1/2} x)$, where

$$\lambda_m = \frac{\|w_m\|_{L^2}^2}{\gamma}.$$

We have

$$\int_{\mathbb{R}^N} u_m^2 = \gamma, \tag{8.1.54}$$

$$V(u_m) = 0, \tag{8.1.55}$$

and

$$S(u_m) = S(w_m) \underset{m\to\infty}{\longrightarrow} c. \tag{8.1.56}$$

In particular, $(u_m)_{m\in\mathbb{N}}$ is also a minimizing sequence. It follows from (8.1.54), (8.1.55), and (8.1.56) that $(u_m)_{m\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Therefore, there exists a subsequence, which we still denote by $(u_m)_{m\in\mathbb{N}}$, and there exists $u \in H^1(\mathbb{R}^N)$ such that $u_m \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ as $m \to \infty$. In particular (see the proof of Lemma 8.1.7),

$$\int_{\mathbb{R}^N} u_m^{\alpha+2} \underset{m\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} u^{\alpha+2}, \quad \int_{\mathbb{R}^N} u^2 \le \liminf_{m\to\infty} \int_{\mathbb{R}^N} u_m^2 = \gamma,$$

and

$$T(u) \le \liminf_{m\to\infty} T(u_m).$$

Therefore,

$$V(u) \ge 0 \quad \text{and} \quad S(u) \le c.$$

We claim that $V(u) = 0$. To see this, we argue by contradiction. If $V(u) > 0$, then in particular $u \neq 0$, so that there exists $\lambda \in (0,1)$ such that $v = \lambda u$ satisfies $V(v) = 0$. Thus $v \in N$. Furthermore, $T(v) = \lambda^2 T(u) < T(u)$, so that $S(v) < S(u)$, which implies that $S(v) < c$. This contradicts the definition of c. Therefore, $V(u) = 0$. It follows that $V(u_m) \to V(u)$, which implies that

$$\int_{\mathbb{R}^N} u^2 = \lim_{m\to\infty} \int_{\mathbb{R}^N} u_m^2 = \gamma,$$

and so u satisfies (8.1.24).

STEP 2. Every solution of (8.1.24) belongs to A. Indeed, consider a solution u of (8.1.24) (which exists by Step 1). There exists a Lagrange multiplier λ such that

$$-\Delta u = \lambda(|u|^\alpha u - \omega u).$$

On taking the L^2-scalar product of the equation with u, we obtain

$$T(u) = \lambda\Big(\int_{\mathbb{R}^N} |u|^{\alpha+2} - \omega \int_{\mathbb{R}^N} |u|^2\Big).$$

Since u satisfies (8.1.24), this implies that

$$2c = \frac{\lambda\omega\alpha\gamma}{2},$$

and so $\lambda = 1$. Therefore, u satisfies (8.1.4).

STEP 3. u satisfies (8.1.24) if and only if $u \in G$. Consider any solution u of (8.1.24) and any $v \in A$ ($A \neq \varnothing$, by Step 2). It follows from Corollary 8.1.3 that $v \in N$ and

$$\int_{\mathbb{R}^N} |v|^2 = \frac{4}{\omega\alpha} S(v) = \gamma \frac{S(v)}{S(u)}. \tag{8.1.57}$$

Since $v \in N$, we deduce that $S(v) \ge S(u)$, and so $u \in G \neq \varnothing$.

Assume further that $v \in G$. Since $u \in G$ also, we have $S(u) = S(v)$. It follows from (8.1.57) that

$$\int_{\mathbb{R}^N} |v|^2 = \gamma,$$

which means that v satisfies (8.1.24). Hence the result is established.

STEP 4. Conclusion. Properties (i) and (ii) follow from Step 3. We establish (iii) by following the argument from the proof of Theorem 8.1.4. □

DEFINITION 8.1.13. A function $u \in A$ is called a *bound state* of (8.1.4). A function $u \in G$ is called a *ground state* of (8.1.4). By definition, this is a bound state that minimizes the action S among all other bound states.

REMARK 8.1.14. Note that the ground state is unique, modulo space translations and multiplication by $e^{i\theta}$, as follows from Theorems 8.1.4 to 8.1.6.

REMARK 8.1.15. In the literature, one sometimes calls any positive solution of (8.1.4) a ground state. It follows from Theorems 8.1.4 to 8.1.6 that these two definitions are equivalent, modulo multiplication by $e^{i\theta}$.

REMARK 8.1.16. In the case $N = 1$, every $u \in A$ is a ground state, since $A = G$. This is not true anymore when $N \geq 2$. Indeed, in this case, it follows from Berestycki and Lions [25] and Berestycki, Gallouët, and Kavian [24] that there exists a sequence $(u_m)_{m\in\mathbb{N}} \subset A$ such that $S(u_m) \to \infty$ as $m \to \infty$. This implies that for m large, $u_m \notin G$.

REMARK 8.1.17. Let u be the (unique) positive, spherically symmetric ground state of (8.1.4) with $\omega = 1$. For $\omega > 0$, let $u_\omega(x) = \omega^{1/\alpha}u(\omega^{\frac{1}{2}}x)$. It follows that u_ω satisfies (8.1.4), and so u_ω is the unique positive, spherically symmetric ground state of (8.1.4). We have

$$\|u_\omega\|_{H^1}^2 = \omega^{\frac{2}{\alpha}-\frac{N}{2}} \int_{\mathbb{R}^N} u^2 + \omega^{\frac{2}{\alpha}-\frac{N-2}{2}} \int_{\mathbb{R}^N} |\nabla u|^2 .$$

Therefore, if $\alpha \geq 4/N$, there exists $\sigma > 0$ such that $\|u_\omega\|_{H^1} \geq \sigma$ for all $\omega > 0$. On the other hand, if $\alpha < 4/N$, then $\|u_\omega\|_{H^1} \to 0$ as $\omega \to 0$. In particular, there exist ground states of (8.1.4) of arbitrarily small H^1 norm (when ω varies).

8.2. An Instability Result

We begin with the following result of M. Weinstein [356].

THEOREM 8.2.1. *Assume* (8.1.1) *with* $\alpha = 4/N$ *and let* $\omega > 0$. *If* $\varphi \in A$ *(cf. Theorems* 8.1.4, 8.1.5, *and* 8.1.6*), then* $u(t,x) = e^{i\omega t}\varphi(x)$ *is an unstable solution of* (4.1.1) *in the following sense. There exists* $(\varphi_m)_{m\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ *such that*

$$\varphi_m \underset{m\to\infty}{\longrightarrow} \varphi \quad in\ H^1(\mathbb{R}^N),$$

and such that the corresponding maximal solution u_m *of* (4.1.1) *blows up in finite time for both* $t > 0$ *and* $t < 0$.

PROOF. We have $E(\varphi) = 0$, by Corollary 8.1.3. Therefore, $E(\lambda\varphi) < 0$ for every $\lambda > 1$. On the other hand, it follows from Theorem 8.1.1 that $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$. Applying Theorem 6.5.4, we deduce that the maximal solution of (4.1.1) with the initial value $\lambda\varphi$ blows up in finite time for both $t > 0$ and $t < 0$. The result follows by letting, for example, $\varphi_m = (1 + \frac{1}{m})\varphi$. □

In the case $\alpha > 4/N$, we have the following result of Berestycki and Cazenave [23] (see also Cazenave [59, 60]).

THEOREM 8.2.2. *Assume* (8.1.1), (8.1.2), *and* $\omega > 0$. *Suppose further that* $\alpha > 4/N$. *If* $\varphi \in G$ (*cf. Theorems* 8.1.4, 8.1.5, *and* 8.1.6), *then* $u(t,x) = e^{i\omega t}\varphi(x)$ *is an unstable solution of* (4.1.1) *in the following sense. There exists* $(\varphi_m)_{m\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ *such that*

$$\varphi_m \underset{m\to\infty}{\longrightarrow} \varphi \text{ in } H^1(\mathbb{R}^N),$$

and such that the corresponding maximal solution u_m *of* (4.1.1) *blows up in finite time for both* $t > 0$ *and* $t < 0$.

REMARK 8.2.3. As we will see, the proof of Theorem 8.2.2 is much more complicated than the proof of Theorem 8.2.1. On the other hand, the result is much weaker (except when $N = 1$), since it only concerns the ground states (see Remark 8.1.16). It is presently unknown whether the other stationary states are unstable.

Let us define the functional $Q \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ by

$$Q(u) = \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N\alpha}{2(\alpha+2)} \int_{\mathbb{R}^N} |u|^{\alpha+2} \quad \text{for } u \in H^1(\mathbb{R}^N), \tag{8.2.1}$$

and let

$$M = \{u \in H^1(\mathbb{R}^N) : u \neq 0 \text{ and } Q(u) = 0\}. \tag{8.2.2}$$

The proof of Theorem 8.2.2 relies on the following result.

PROPOSITION 8.2.4. *Let* α, ω *be as in Theorem* 8.2.2. *If* $u \in H^1(\mathbb{R}^N)$, *then* $u \in G$ *if and only if* u *solves the following minimization problem:*

$$\begin{cases} u \in M, \\ S(u) = \min\{S(v) : v \in M\}. \end{cases} \tag{8.2.3}$$

For the proof of Proposition 8.2.4, we will use the following lemma.

LEMMA 8.2.5. *Given* $u \in H^1(\mathbb{R}^N)$, $u \neq 0$, *and* $\lambda > 0$, *set* $\mathcal{P}(\lambda, u)(x) = \lambda^{\frac{N}{2}} u(\lambda x)$. *The following properties hold:*

(i) *There exists a unique* $\lambda^*(u) > 0$ *such that* $\mathcal{P}(\lambda^*(u), u) \in M$.
(ii) *The function* $\lambda \mapsto S(\mathcal{P}(\lambda, u))$ *is concave on* $(\lambda^*(u), \infty)$.
(iii) $\lambda^*(u) < 1$ *if and only if* $Q(u) < 0$.
(iv) $\lambda^*(u) = 1$ *if and only if* $u \in M$.
(v) $S(\mathcal{P}(\lambda, u)) < S(\mathcal{P}(\lambda^*(u), u))$ *for every* $\lambda > 0$, $\lambda \neq \lambda^*(u)$.
(vi) $\frac{d}{d\lambda} S(\mathcal{P}(\lambda, u)) = \frac{1}{\lambda} Q(\mathcal{P}(\lambda, u))$ *for every* $\lambda > 0$.
(vii) $|\mathcal{P}(\lambda, u)|^* = \mathcal{P}(\lambda, |u|^*)$ *for every* $\lambda > 0$, *where* * *is the Schwarz symmetrization.*
(viii) *If* $u_m \to u$ *in* $H^1(\mathbb{R}^N)$ *weakly and in* $L^{\alpha+2}(\mathbb{R}^N)$ *strongly, then* $\mathcal{P}(\lambda, u_m) \to \mathcal{P}(\lambda, u)$ *in* $H^1(\mathbb{R}^N)$ *weakly and in* $L^{\alpha+2}(\mathbb{R}^N)$ *strongly for every* $\lambda > 0$.

PROOF. Let $u \in H^1(\mathbb{R}^N)$, $u \neq 0$, and let $u_\lambda = \mathcal{P}(\lambda, u)$. We have

$$S(u_\lambda) = \frac{\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\omega}{2} \int_{\mathbb{R}^N} |u|^2 - \frac{\lambda^{\frac{N\alpha}{2}}}{\alpha + 2} \int_{\mathbb{R}^N} |u|^{\alpha+2}. \tag{8.2.4}$$

Property (vi) follows easily. Let $\lambda^*(u)$ be defined by

$$\lambda^*(u)^{\frac{N\alpha-4}{2}} = \frac{2(\alpha+2)}{N\alpha} \Big(\int_{\mathbb{R}^N} |\nabla u|^2 \Big) \Big(\int_{\mathbb{R}^N} |u|^{\alpha+2} \Big)^{-1}.$$

Elementary calculations show that with $\lambda^*(u)$ defined as above, properties (i), (ii), (iii), (iv), and (v) are satisfied. Property (vii) follows easily from the definition of Schwarz's symmetrization (see the beginning of the proof of Lemma 8.1.7). Finally, given $\lambda > 0$, the operator $u \mapsto \mathcal{P}(\lambda, u)$ is linear and strongly continuous $H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$. Therefore, it is also weakly continuous. The $L^{\alpha+2}$ continuity is immediate. Hence (viii) follows. □

COROLLARY 8.2.6. *The set M is nonempty. If we set*

$$m = \inf\{S(u) : u \in M\}, \tag{8.2.5}$$

then $Q(u) \le S(u) - m$ for every $u \in H^1(\mathbb{R}^N)$ such that $Q(u) < 0$.

PROOF. It follows from Lemma 8.2.5(i) that M is nonempty. Let $u \in H^1(\mathbb{R}^N)$ be such that $Q(u) < 0$, and let $f(\lambda) = S(\mathcal{P}(\lambda, u))$. By Lemma 8.2.4(iii), $\lambda^*(u) < 1$, and, by (ii), f is concave on $(\lambda^*(u), 1)$. Therefore,

$$f(1) \ge f(\lambda^*(u)) + (1 - \lambda^*(u)) f'(1).$$

Applying Lemma 8.2.5(vi), we obtain

$$S(u) \ge f(\lambda^*(u)) + (1 - \lambda^*(u)) Q(u) \ge f(\lambda^*(u)) + Q(u).$$

Since by Lemma 8.2.5(i) $\mathcal{P}(\lambda^*(u), u) \in M$, we deduce that $f(\lambda^*(u)) \ge m$, and so

$$S(u) \ge m + Q(u),$$

which completes the proof. □

PROOF OF PROPOSITION 8.2.4. We proceed in three steps.

STEP 1. The minimization problem (8.2.3) has a solution. We know that $M \neq \varnothing$ by Corollary 8.2.6, so that (8.2.3) has a minimizing sequence $(v_m)_{m\in\mathbb{N}}$. In particular, $Q(v_m) = 0$ and $S(v_m) \to m$, where m is defined by (8.2.5). Let $w_m = |v_m|^*$, and $u_m = \mathcal{P}(\lambda^*(w_m), w_m)$. It follows from Lemma 8.2.5(i) that $u_m \in M$. Furthermore, it follows from Lemma 8.2.5(vii) that $u_m = |\mathcal{P}(\lambda^*(w_m), v_m)|^*$. Therefore,

$$S(u_m) \le S(\mathcal{P}(\lambda^*(w_m), v_m)) \le S(\mathcal{P}(\lambda^*(v_m), v_m)) \le S(v_m),$$

where the last two inequalities follow from Lemma 8.2.5(v) and (i). In particular, $(u_m)_{m\in\mathbb{N}}$ is a nonnegative, spherically symmetric, nonincreasing minimizing sequence of (8.2.3). Furthermore, note that

$$\begin{aligned} S(u_m) &= \frac{2}{N\alpha}Q(u_m) + \frac{N\alpha-4}{2N\alpha}\int_{\mathbb{R}^N} |\nabla u_m|^2 + \frac{\omega}{2}\int_{\mathbb{R}^N} u_m^2 \\ &= \frac{N\alpha-4}{2N\alpha}\int_{\mathbb{R}^N} |\nabla u_m|^2 + \frac{\omega}{2}\int_{\mathbb{R}^N} u_m^2 \,. \end{aligned}$$

It follows that $(u_m)_{m\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Since $Q(u_m)=0$, we deduce from Gagliardo-Nirenberg's inequality and the boundedness of $(u_m)_{m\in\mathbb{N}}$ in $L^2(\mathbb{R}^N)$ that there exists C such that

$$\|\nabla u_m\|_{L^2} \le C\|\nabla u_m\|_{L^2}^{\frac{N\alpha}{4}} \,.$$

Since $N\alpha > 4$, we obtain that $\|\nabla u_m\|_{L^2}$ is bounded from below, and since $Q(u_m) = 0$, there exists $\sigma > 0$ such that

$$\|u_m\|_{L^{\alpha+2}} \ge \sigma \quad \text{for all } m \ge 0\,. \tag{8.2.6}$$

By Proposition 1.7.1, there exist $v \in H^1(\mathbb{R}^N)$ and a subsequence, which we still denote by $(u_m)_{m\in\mathbb{N}}$, such that $u_m \to v$ as $m \to \infty$, in $H^1(\mathbb{R}^N)$ weakly and in $L^{\alpha+2}(\mathbb{R}^N)$ strongly, and so by (8.2.6), $v \ne 0$. Therefore, we may define $u = \mathcal{P}(\lambda^*(v), v)$. By Lemma 8.2.5(i), $u \in M$ and Lemma 8.2.5(vii), $\mathcal{P}(\lambda^*(v), u_m) \to u$ in $H^1(\mathbb{R}^N)$ weakly and in $L^{\alpha+2}(\mathbb{R}^N)$ strongly. Therefore,

$$\begin{aligned} S(u) \le \liminf_{m\to\infty} S(\mathcal{P}(\lambda^*(v), u_m)) &\le \liminf_{m\to\infty} S(\mathcal{P}(\lambda^*(u_m), u_m)) \\ &= \liminf_{m\to\infty} S(u_m) = m\,, \end{aligned}$$

where the last three inequalities follow from (v), (iv), and (8.2.5), and so u satisfies (8.2.3).

STEP 2. Every solution of (8.2.3) satisfies (8.1.4). Consider any solution u of (8.2.3). For $\sigma > 0$, let $u(x) = \sigma^{\frac{2}{\alpha}} u_\sigma(\sigma x)$. One easily verifies that

$$Q(u_\sigma) = \sigma^{N-2-\frac{4}{\alpha}}Q(u) = 0\,,$$

and so $u_\sigma \in M$. Since $u = u_1$ satisfies (8.2.3), we deduce that $f(\sigma) = S(u_\sigma)$ satisfies $f'(1) = 0$. One computes easily, by using the property $u_\sigma \in M$, that

$$f'(1) = \langle S'(u), u\rangle_{H^{-1},H^1}\,,$$

where S' is the gradient of the C^1 functional S (i.e., $S'(u) = -\Delta u + \omega u - |u|^\alpha u$). It follows that

$$\langle S'(u), u\rangle_{H^{-1},H^1} = 0\,. \tag{8.2.7}$$

On the other hand, $Q'(u) = -2\Delta u - \frac{N\alpha}{2}|u|^\alpha u$, and so since $u \in M$, we obtain

$$\langle Q'(u), u\rangle_{H^{-1},H^1} = -\alpha T(u) < 0\,. \tag{8.2.8}$$

Finally, since u satisfies (8.2.3), there exists a Lagrange multiplier λ such that $S'(u) = \lambda Q'(u)$. Applying (8.2.7) and (8.2.8), we deduce that $\lambda = 0$, and so $S'(u) = 0$, which means that u satisfies (8.1.4).

STEP 3. Conclusion. Consider

$$\ell = \min\{S(u) : u \in A\}. \tag{8.2.9}$$

Let $u \in G$. In particular, $S(u) = \ell$. Applying Corollary 8.1.3, one obtains easily that $u \in M$. Therefore, $S(u) \geq m$, where m is defined by (8.2.5). In particular,

$$\ell \geq m. \tag{8.2.10}$$

Consider now a solution u of (8.2.3). By Step 2, $u \in A$. Since $S(u) = m$, it follows from (8.2.9) that $m \geq \ell$. Comparing with (8.2.10), we obtain $m = \ell$. The equivalence of the two problems follows easily. □

PROOF OF THEOREM 8.2.2. Let $\varphi \in G$, and let $\varphi_\lambda = \mathcal{P}(\lambda, \varphi)$ for $\lambda > 0$. It follows from Proposition 8.2.4 and Lemma 8.2.5 that

$$Q(\varphi_\lambda) < 0 \tag{8.2.11}$$

and

$$S(\varphi_\lambda) < m = S(\varphi) \tag{8.2.12}$$

for all $\lambda > 1$. Let u_λ be the maximal solution of (4.1.1) with the initial value φ_λ. By conservation of charge and energy,

$$S(u_\lambda(t)) = S(\varphi_\lambda) \quad \text{for all } t \in (-T_{\min}(\varphi_\lambda), T_{\max}(\varphi_\lambda)). \tag{8.2.13}$$

By continuity, we deduce from (8.2.11) that $Q(u_\lambda(t)) < 0$ for $|t|$ small. On the other hand, if t is such that $Q(u_\lambda(t)) < 0$, then it follows from Corollary 8.2.6, (8.2.13), and (8.2.12) that

$$Q(u_\lambda(t)) \leq S(\varphi_\lambda) - m = -\delta < 0. \tag{8.2.14}$$

By continuity, (8.2.25) holds for all $t \in (-T_{\min}(\varphi_\lambda), T_{\max}(\varphi_\lambda))$. Applying Proposition 6.5.1, we deduce that f defined by (6.5.15) satisfies

$$f''(t) = 8Q(u_\lambda(t)) \leq -8\delta \quad \text{for all } t \in (-T_{\min}(\varphi_\lambda), T_{\max}(\varphi_\lambda)).$$

It follows easily that both $T_{\min}(\varphi_\lambda)$ and $T_{\max}(\varphi_\lambda)$ are finite (see the proof of Theorem 6.5.4). Hence the result follows, since $\varphi_\lambda \to \varphi$ in $H^1(\mathbb{R}^N)$ as $\lambda \downarrow 1$ (apply Theorem 8.1.1). □

REMARK 8.2.7. Theorems 8.2.1 and 8.2.2 show the instability of ground states when $\alpha \geq 4/N$. When $\alpha < 4/N$, it follows from the results of Section 8.3 that the ground states are, to the contrary, stable.

REMARK 8.2.8. The method of proof of Theorem 8.2.2 can be adapted to more general nonlinearities. See Berestycki and Cazenave [23], Fukuizumi and Ohta [118], and Ohta [282].

8.3. A Stability Result

Our goal in this section is to establish the following result of Cazenave and Lions [66] (see also P.-L. Lions [235, 236] and Cazenave [58])

THEOREM 8.3.1. *Assume* (8.1.1), (8.1.2), *and* $\omega > 0$. *Suppose further that* $\alpha < 4/N$. *If* $\varphi \in G$ (*cf. Theorems* 8.1.4, 8.1.5, *and* 8.1.6), *then* $u(t,x) = e^{i\omega t}\varphi(x)$ *is a stable solution of* (4.1.1) *in the following sense. For every* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that if* $\psi \in H^1(\mathbb{R}^N)$ *satisfies* $\|\varphi - \psi\|_{H^1} \le \delta(\varepsilon)$, *then the corresponding maximal solution* v *of* (4.1.1) *satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \inf_{y\in\mathbb{R}^N} \|v(t,\cdot) - e^{i\theta}\varphi(\cdot - y)\|_{H^1} \le \varepsilon . \tag{8.3.1}$$

In other words, there exist functions $\theta(t) \in \mathbb{R}$ *and* $y(t) \in \mathbb{R}^N$ *such that*

$$\sup_{t\in\mathbb{R}} \|v(t,\cdot) - e^{i\theta(t)}\varphi(\cdot - y(t))\|_{H^1} \le \varepsilon . \tag{8.3.2}$$

REMARK 8.3.2. Theorem 8.3.1 means that if ψ is close to φ in $H^1(\mathbb{R}^N)$, then the solution of (4.1.1) with initial value ψ remains close to the orbit of φ, modulo space translations. Note that $\alpha < 4/N$, which implies that all solutions of (4.1.1) are global (see Remark 6.8.1).

REMARK 8.3.3. The space translations appearing in (8.3.1) and (8.3.2) are necessary. Indeed, let $\varphi \in G$. Given $\varepsilon > 0$ and $y \in \mathbb{R}^N$ such that $|y| = 1$, let

$$\varphi_\varepsilon(x) = e^{i\varepsilon x\cdot y}\varphi(x) \quad \text{and} \quad u_\varepsilon(t,x) = e^{i\varepsilon(x\cdot y - \varepsilon t)}e^{i\omega t}\varphi(x - 2\varepsilon t y) .$$

One easily verifies that u_ε is the solution of (4.1.1) with initial value φ_ε. Furthermore, $\varphi_\varepsilon \to \varphi$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \downarrow 0$, but one easily verifies that for every $\varepsilon > 0$,

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \|u_\varepsilon(t) - e^{i\theta}\varphi\|_{H^1} = 2\|\varphi\|_{H^1} .$$

On the other hand, it is clear that if $\varphi \in G$ is spherically symmetric and if ψ is also spherically symmetric, one can remove the space translations in (8.3.1) and (8.3.2). In other words,

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \|v(t) - e^{i\theta}\varphi\|_{H^1} \le \varepsilon .$$

This follows from a trivial adaptation of the proof of the stability theorem in the subspace of $H^1(\mathbb{R}^N)$ of spherically symmetric function. Alternatively, this follows from the observation that if $f, g \in H^1(\mathbb{R}^N)$ are spherically symmetric, then

$$\inf_{y\in\mathbb{R}^N} \|f(\cdot) - g(\cdot - y)\|_{H^1} = \|f - g\|_{H^1} .$$

REMARK 8.3.4. The rotations $e^{i\theta}$ appearing in (8.3.1) and (8.3.2) are necessary. Indeed, let $\varphi \in G$ and let $u(t,x) = e^{i\omega t}\varphi(x)$. Given $\varepsilon > 0$, let

$$\varphi_\varepsilon(x) = (1+\varepsilon)^{1/\alpha}\varphi((1+\varepsilon)^{\frac{1}{2}}x) \quad \text{and} \quad u_\varepsilon(t,x) = e^{i\omega(1+\varepsilon)t}(1+\varepsilon)^{\frac{1}{\alpha}}\varphi((1+\varepsilon)^{\frac{1}{2}}x) .$$

One easily verifies that u_ε is the solution of (4.1.1) with initial value φ_ε. Furthermore, $\varphi_\varepsilon \to \varphi$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \downarrow 0$, but one easily verifies that for every $\varepsilon > 0$,

$$\sup_{t\in\mathbb{R}} \inf_{y\in\mathbb{R}^N} \|u_\varepsilon(t,\cdot) - \varphi(\cdot - y)\|_{H^1} = \sup_{t\in\mathbb{R}} \inf_{y\in\mathbb{R}^N} \|u_\varepsilon(t,\cdot) - u(t,\cdot - y)\|_{H^1} \ge \|\varphi\|_{H^1} .$$

REMARK 8.3.5. Theorem 8.3.1 only asserts the stability of ground states. Except when $N = 1$, where $A = G$, one does not know whether the other standing waves are stable.

The proof of Theorem 8.3.1 relies on the following result.

PROPOSITION 8.3.6. *Assume* $0 < \alpha < 4/N$ *and* $\omega > 0$. *Let* $\tau > 0$, *and let* E *be defined by* (8.1.16). *If*

$$\Gamma = \Big\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 = \tau \Big\} \tag{8.3.3}$$

and

$$-\nu = \inf\{E(v) : v \in \Gamma\}, \tag{8.3.4}$$

then the following properties hold:

(i) *The minimization problem*

$$\begin{cases} u \in \Gamma \\ E(u) = \min\{E(v) : v \in \Gamma\} \end{cases} \tag{8.3.5}$$

has a solution.

(ii) *If* $(u_m)_{m\in\mathbb{N}}$ *satisfies* $\|u_m\|_{L^2} \to \sqrt{\tau}$ *and* $E(u_m) \to -\nu$, *then there exist a subsequence* $(u_{m_k})_{k\in\mathbb{N}}$ *and a family* $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}^N$ *such that* $(u_{m_k}(\cdot - y_k))_{k\in\mathbb{N}}$ *has a strong limit* u *in* $H^1(\mathbb{R}^N)$. *In particular,* u *satisfies* (8.3.5).

PROOF. The proof relies on the concentration-compactness method introduced by P.-L. Lions [235, 236] in the form of Proposition 1.7.6. We proceed in three steps.

STEP 1. $0 < \nu < \infty$. It is clear that $\Gamma \neq \varnothing$. Let $u \in \Gamma$ and $\lambda > 0$; set

$$u_\lambda(x) = \lambda^{\frac{N}{2}} u(\lambda x).$$

It follows easily that $u_\lambda \in \Gamma$ and that

$$E(u_\lambda) = \frac{\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\lambda^{\frac{N\alpha}{2}}}{\alpha+2} \int_{\mathbb{R}^N} |u|^{\alpha+2}.$$

Since $N\alpha < 4$, we have $E(u_\lambda) < 0$ for λ small, and so $\nu > 0$. Next, we claim that there exist $\delta > 0$ and $K < \infty$ such that

$$E(u) \geq \delta \|u\|_{H^1}^2 - K \quad \text{for all } u \in \Gamma. \tag{8.3.6}$$

This follows immediately from Gagliardo-Nirenberg's inequality

$$\int_{\mathbb{R}^N} |u|^{\alpha+2} \leq C \Big(\int_{\mathbb{R}^N} |\nabla u|^2 \Big)^{\frac{N\alpha}{4}} \Big(\int_{\mathbb{R}^N} |u|^2 \Big)^{\frac{4-(N-2)\alpha}{4}}$$

and the property $N\alpha < 4$. Therefore, $\nu \geq -K > -\infty$.

STEP 2. Every minimizing sequence of (8.3.5) is bounded in $H^1(\mathbb{R}^N)$ and bounded from below in $L^{\alpha+2}(\mathbb{R}^N)$. Let $(u_n)_{n\geq 0}$ be a minimizing sequence. Since $u_n \in \Gamma$, $(u_n)_{n\geq 0}$ is bounded in $L^2(\mathbb{R}^N)$, then by (8.3.6) $(u_n)_{\geq 0}$ is bounded in

$H^1(\mathbb{R}^N)$. This proves the first part of the statement. Furthermore since $\nu > 0$, we have $E(u_n) \le -\nu/2$ for n large enough. It follows that

$$\int_{\mathbb{R}^N} |u_n|^{\alpha+2} \ge \frac{\alpha+2}{2}\nu. \tag{8.3.7}$$

Hence the result is established.

STEP 3. Conclusion. We need only prove (ii). Let $(\widetilde{u}_n)_{n\ge 0}$ satisfy $\|\widetilde{u}_n\|_{L^2} \to \sqrt{\mu}$ and $E(\widetilde{u}_n) \to -\nu$. Setting

$$u_n = \frac{\sqrt{\tau}}{\|\widetilde{u}_n\|_{L^2}}\widetilde{u}_n\,,$$

we deduce that $(u_n)_{n\ge 0}$ is a minimizing sequence of (8.3.5). Note that by rescaling, we may assume that $\tau = 1$. We now apply Proposition 1.7.6 to the minimizing sequence $(u_n)_{n\ge 0}$ (note that $a = 1$). We claim that

$$\mu = 1\,, \tag{8.3.8}$$

where μ is defined by (1.7.6). Note first that, since $(u_n)_{n\ge 0}$ is bounded from below in $L^{\alpha+2}(\mathbb{R}^N)$, we have $\mu > 0$ by Proposition 1.7.6(ii). Suppose now by contradiction that

$$0 < \mu < 1\,. \tag{8.3.9}$$

We use the sequences $(v_k)_{k\ge 0}$ and $(w_k)_{k\ge 0}$ introduced in Proposition 1.7.6(iii). It follows from (1.7.15)–(1.7.16) that

$$\liminf_{k\to\infty}(E(u_{n_k}) - E(v_k) - E(w_k)) \ge 0\,,$$

so that

$$\limsup_{k\to\infty}(E(v_k) + E(w_k)) \le -\nu\,. \tag{8.3.10}$$

Next, observe that, given $u \in H^1(\mathbb{R}^N)$ and $a > 0$, we have

$$E(u) = \frac{1}{a^2}E(au) + \frac{a^\alpha - 1}{\alpha+2}\int_{\mathbb{R}^N} |u|^{\alpha+2}\,.$$

Applying the above inequality with and $a_k = 1/\|v_k\|_{L^2}$, and since $a_k v_k \in \Gamma$, we obtain that

$$E(v_k) \ge \frac{-\nu}{a_k^2} + \frac{a_k^\alpha - 1}{\alpha+2}\int_{\mathbb{R}^N} |v_k|^{\alpha+2}\,.$$

Similarly,

$$E(w_k) \ge \frac{-\nu}{b_k^2} + \frac{b_k^\alpha - 1}{\alpha+2}\int_{\mathbb{R}^N} |w_k|^{\alpha+2}$$

with $b_k = 1/\|w_k\|_{L^2}$, and so

$$E(v_k) + E(w_k) \ge -\nu\big(a_k^{-2} + b_k^{-2}\big) + \frac{a_k^\alpha - 1}{\alpha+2}\int_{\mathbb{R}^N} |v_k|^{\alpha+2} + \frac{b_k^\alpha - 1}{\alpha+2}\int_{\mathbb{R}^N} |w_k|^{\alpha+2}\,.$$

Finally, note that $a_k^{-2} \to \mu$ and $b_k^{-2} \to 1-\mu$ by (1.7.14). In particular, by (8.3.9)

$$\theta := \min\{\mu^{-\frac{\alpha}{2}}, (1-\mu)^{-\frac{\alpha}{2}}\} > 1\,.$$

Therefore, using (1.7.16), then (8.3.7) we deduce that

$$\begin{aligned}\liminf_{k\to\infty}(E(v_k)+E(w_k)) &\geq -\nu + \frac{\theta-1}{\alpha+2}\liminf_{k\to\infty}\int_{\mathbb{R}^N}|u_{n_k}|^{\alpha+2}\\ &\geq -\nu + \frac{\theta-1}{2} > -\nu,\end{aligned}$$

which contradicts (8.3.10). Therefore, the proof of the claim (8.3.8) is complete. We finally apply Proposition 1.7.6(i), and we deduce that for some sequence $(y_k)_{k\geq 0}\subset\mathbb{R}^N$ and some $u\in H^1(\mathbb{R}^N)$, $u_{n_k}(\cdot - y_k)\to u$ in $L^2(\mathbb{R}^N)$ (and in particular, $u\in\Gamma$) and in $L^{\alpha+2}(\mathbb{R}^N)$. Together with the weak lower semicontinuity of the H^1 norm, this implies

$$E(u) \leq \lim_{k\to\infty} E(u_{n_k}) = -\nu\,.$$

By definition of ν, we have $E(u) = -\nu$. In particular, $E(u_{n_k}) \to E(u)$, and it follows that $\|\nabla u_{n_k}\|_{L^2}\to\|\nabla u\|_{L^2}$, which implies that $u_{n_k}(\cdot - y_k)\to u$ strongly in $H^1(\mathbb{R}^N)$. □

LEMMA 8.3.7. *Let $0<\alpha<4/N$ and $\omega>0$. There exists $\mu>0$ such that*

$$\int_{\mathbb{R}^N}|u|^2 = \mu \quad \text{for every ground state } u \text{ of (8.1.4)}. \tag{8.3.11}$$

PROOF. The result follows from uniqueness of the ground state up to translations and rotations (cf. Theorems 8.1.4, 8.1.5, and 8.1.6). Alternatively, when $N\geq 2$ the result follows from (8.1.19), (8.1.22), and property (ii) of Theorems 8.1.4 and 8.1.5. □

COROLLARY 8.3.8. *Let $0<\alpha<4/N$, $\omega>0$, and let μ be defined by* (8.3.11). *If $u\in H^1(\mathbb{R}^N)$, then u is a ground state of* (8.1.4) *if and only if u solves the minimization problem*

$$\begin{cases} u\in\Gamma\\ S(u)=\min\{S(v): v\in\Gamma\},\end{cases} \tag{8.3.12}$$

where Γ is defined by (8.3.3). *In addition, the problems* (8.3.12) *and* (8.3.5) *are equivalent.*

PROOF. We proceed in four steps.

STEP 1. Problem (8.3.12) is equivalent to problem (8.3.5), which has a solution by Proposition 8.3.6. Indeed, if $u\in\Gamma$, then $S(u) = E(u)+\omega\mu/2$, and so problem (8.3.12) is equivalent to problem (8.3.5).

STEP 2. We have $k\leq\ell$, where ℓ is defined by (8.2.9) and k is defined by

$$k = \inf\{S(v): v\in\Gamma\}\,. \tag{8.3.13}$$

Indeed, consider $u \in G$. We have $S(u) = \ell$, and by Lemma 8.3.7, $u \in \Gamma$. By definition of k, this implies $k \leq \ell$.

STEP 3. Every solution of (8.3.12) belongs to A. Consider a solution u of (8.3.12), and let

$$u_\lambda(x) = \lambda^{\frac{N}{2}} u(\lambda x) \quad \text{for } \lambda > 0.$$

We have $u_\lambda \in \Gamma$ and $u_1 = u$. It follows from (8.3.12) that

$$\frac{d}{d\lambda} S(u_\lambda)|_{\lambda=1} = 0\,,$$

which means that

$$T(u) = \frac{N\alpha}{2(\alpha+2)} \int_{\mathbb{R}^N} |u|^{\alpha+2}\,. \tag{8.3.14}$$

Now, since u satisfies (8.3.12), there exists a Lagrange multiplier λ such that $S'(u) = \lambda u$, and so there exists δ such that

$$-\Delta u + \delta\omega u = |u|^\alpha u\,. \tag{8.3.15}$$

On taking the L^2-scalar product of (8.3.15) with u and applying (8.3.14), we obtain

$$\delta\omega\mu = \frac{4-(N-2)\alpha}{N\alpha} T(u)\,,$$

from which it follows that $\delta > 0$. Define now v by

$$u(x) = \delta^{\frac{1}{\alpha}} v(\delta^{\frac{1}{2}} x)\,.$$

We deduce from (8.3.15) that $v \in A$, which implies

$$S(v) \geq \ell\,. \tag{8.3.16}$$

One computes easily that

$$S(u) = \delta^{\frac{4-(N-2)\alpha}{2\alpha}} S(v) + \frac{\omega\mu}{2}(1-\delta)\,.$$

Applying (8.3.16) and Step 2, we obtain that

$$\ell \geq \delta^{\frac{4-(N-2)\alpha}{2\alpha}} \ell + \frac{\omega\mu}{2}(1-\delta)\,.$$

On the other hand, it follows from Corollary 8.1.3 that $\ell > 0$, and by (8.3.11) and Corollary 8.1.3,

$$\frac{\omega\mu}{2} = \frac{4-(N-2)\alpha}{2\alpha}\ell\,,$$

and so

$$1 \geq \delta^{\frac{4-(N-2)\alpha}{2\alpha}} + \frac{4-(N-2)\alpha}{2\alpha}(1-\delta)\,.$$

This means that $f(\delta) \leq 0$, where

$$f(s) = s^{\frac{4-(N-2)\alpha}{2\alpha}} - \frac{4-(N-2)\alpha}{2\alpha} s + \frac{4-N\alpha}{2\alpha}\,.$$

One checks easily that $f(s) > 0$, if $s \neq 1$. Therefore, $\delta = 1$, which implies in view of (8.3.15) that $u \in A$.

STEP 4. Conclusion. It follows in particular from Steps 2 and 3 that $\ell = k$. Therefore, if $u \in G$, then $u \in \Gamma$ and $S(u) = k$, which implies that u satisfies (8.3.12). Conversely, let u be a solution of (8.3.12). We have $u \in A$ by Step 3, and since $S(u) = k = \ell$, it follows that $u \in G$. □

PROOF OF THEOREM 8.3.1. Assume by contradiction that there exist a sequence $(\psi_m)_{m\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$, a sequence $(t_m)_{m\in\mathbb{N}} \subset \mathbb{R}$, and $\varepsilon > 0$ such that

$$\|\psi_m - \varphi\|_{H^1} \underset{m\to\infty}{\longrightarrow} 0, \tag{8.3.17}$$

and such that the maximal solution u_m of (4.1.1) with initial value ψ_m (which is global, cf. Remark 8.3.2) satisfies

$$\inf_{\theta\in\mathbb{R}} \inf_{y\in\mathbb{R}^N} \|u_m(t_m,\cdot) - e^{i\theta}\varphi(\cdot - y)\|_{H^1} \geq \varepsilon. \tag{8.3.18}$$

Let us set

$$v_m = u_m(t_m). \tag{8.3.19}$$

It follows from Corollary 8.3.8; Theorems 8.1.4, 8.1.5, and 8.1.6; and (8.3.19) that (8.3.18) is equivalent to

$$\inf_{u\in G} \|v_m - u\|_{H^1} \geq \varepsilon. \tag{8.3.20}$$

Applying Corollary 8.3.8, we deduce from (8.3.17) that

$$\int_{\mathbb{R}^N} |\psi_m|^2 \underset{m\to\infty}{\longrightarrow} \mu \quad \text{and} \quad S(\psi_m) \underset{m\to\infty}{\longrightarrow} k,$$

where k is defined by (8.3.13). From conservation of charge and energy, we deduce that

$$\int_{\mathbb{R}^N} |v_m|^2 \underset{m\to\infty}{\longrightarrow} \mu, \quad \text{and} \quad S(v_m) \underset{m\to\infty}{\longrightarrow} k$$

as well. Therefore, $(v_m)_{m\in\mathbb{N}}$ is a minimizing sequence for the problem (8.3.12), hence of the problem (8.3.5) (see Corollary 8.3.8). From Proposition 8.3.6(ii) it follows that there exist $(y_m)_{m\in\mathbb{N}} \subset \mathbb{R}^N$ and a solution u of the problem (8.3.5) such that $\|v_m - u(\cdot - y_m)\|_{H^1} \to 0$. But $u \in G$ by Corollary 8.3.8, and so $u(\cdot - y_m) \in G$, which contradicts (8.3.20). □

REMARK 8.3.9. Note that the proof of Theorem 8.3.1 only makes use of the following two properties. The conservation laws of (4.1.1) (charge and energy), and the compactness of any minimizing sequence. Therefore, the method is quite general and may be applied to many situations. See, e.g., Cazenave [58], Cazenave and Lions [66], P.-L. Lions [235, 236]), and Ohta [282, 283, 284].

REMARK 8.3.10. One does not know in general about the functions $\theta(t)$ and $y(t)$ of (8.3.2). If both φ and ψ are spherically symmetric, one may let $y(t) \equiv 0$ (see Remark 8.3.4). Remarks 8.3.3 and 8.3.4 display examples for which one may let θ and y be linear in t. One does not know whether this is true in general. Concerning this question, see the remarkable papers of Soffer and Weinstein [316, 317]. They consider in particular a one-dimensional equation with a potential. In this case, $y \equiv 0$, but they also show that one may let θ be linear in t.

8.4. Comments

REMARK 8.4.1. There are other methods to study the stability of standing waves, based on the study of a linearized operator. See Shatah and Strauss [311], Grillakis, Shatah, and Strauss [154, 155]. See also Gonçalves Ribeiro [151], Blanchard, Stubbe, and Vazquez [33], M. Weinstein [358, 357], Rose and Weinstein [303], and Cid and Felmer [81]. The stability of excited states has also been studied, in particular by Jones [198] and Grillakis [153].

By using the techniques of Section 8.1, one can establish the following useful result of M. Weinstein [356] relating the ground states of (8.1.4) with the best constant in a Gagliardo-Nirenberg inequality.

LEMMA 8.4.2. *Let R be the (unique) spherically symmetric, positive ground state of the elliptic equation* (6.6.3), *i.e.,*

$$-\Delta R + R = |R|^\alpha R \quad \textit{in } \mathbb{R}^N$$

with $\alpha = 4/N$ *(see Definition* 8.1.13 *and Theorems* 8.1.4, 8.1.5, *and* 8.1.6). *It follows that the best constant in the Gagliardo-Nirenberg inequality*

$$\frac{1}{\alpha+2}\|\psi\|_{L^{\alpha+2}}^{\alpha+2} \le \frac{C}{2}\|\nabla\psi\|_{L^2}^2\|\psi\|_{L^2}^\alpha ,$$

is $C = \|R\|_{L^2}^{-\alpha}$.

PROOF. We follow the argument of M. Weinstein [356]. We need to show that

$$\inf_{u\in H^1, u\neq 0} J(u) = \frac{2\|R\|_{L^2}^\alpha}{\alpha+2}, \tag{8.4.1}$$

where

$$J(u) = \frac{\|\nabla u\|_{L^2}^2\|u\|_{L^2}^\alpha}{\|u\|_{L^{\alpha+2}}^{\alpha+2}} .$$

We set

$$\sigma = \inf_{u\in H^1, u\neq 0} J(u) ,$$

and we consider a minimizing sequence $(u_n)_{n\ge 0}$. We observe that by Gagliardo-Nirenberg's inequality, $\sigma > 0$. We consider v_n defined by $v_n(x) = \mu_n u_n(\lambda_n x)$ with

$$\lambda_n = \frac{\|u_n\|_{L^2}}{\|\nabla u_n\|_{L^2}} \quad \text{and} \quad \mu_n = \frac{\|u_n\|_{L^2}^{\frac{N-2}{2}}}{\|\nabla u_n\|_{L^2}^{\frac{N}{2}}} ,$$

so that $\|v_n\|_{L^2} = \|\nabla v_n\|_{L^2} = 1$ and

$$\|v_n\|_{L^{\alpha+2}}^{-(\alpha+2)} = J(v_n) = J(u_n) \underset{n\to\infty}{\longrightarrow} \sigma > 0 .$$

By symmetrization (see the proof of Lemma 8.1.7), we may assume that v_n is spherically symmetric, and so there exist a subsequence, which we still denote by $(v_n)_{n\ge 0}$, and $v \in H^1(\mathbb{R}^N)$ such that $v_n \to v$ in $H^1(\mathbb{R}^N)$ weakly and in $L^{\alpha+2}(\mathbb{R}^N)$ strongly (see Proposition 1.7.1). Since $\|v\|_{L^{\alpha+2}} = \lim_{n\to\infty}\|v_n\|_{L^{\alpha+2}} = \sigma^{-\frac{1}{\alpha+2}} > 0$, it follows that $v \neq 0$. This implies that

$$J(v) = \sigma \quad \text{and} \quad \|v\|_{L^2} = \|\nabla v\|_{L^2} = 1 . \tag{8.4.2}$$

In particular, $\frac{d}{dt}J(v+tw)|_{t=0} = 0$ for all $w \in H^1(\mathbb{R}^N)$ and, taking into account (8.4.2), we obtain

$$-\Delta v + \frac{\alpha}{2} v = \sigma \frac{\alpha+2}{2} |v|^\alpha v .$$

Let now u be defined by $v(x) = au(bx)$ with $a = (\alpha/\sigma(\alpha+2))^{\frac{1}{\alpha}}$ and $b = (\alpha/2)^{\frac{1}{2}}$, so that u is a solution of (6.6.3) and

$$J(u) = J(v) = \sigma .$$

Since u satisfies equation (6.6.3), we deduce from Pohozaev's identity (see Lemma 8.1.2) that

$$\frac{1}{2}\|\nabla u\|_{L^2}^2 = -\frac{1}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2} ,$$

and that $2\|\nabla u\|_{L^2}^2 = N\|u\|_{L^2}^2$ (see formulae (8.1.21) and (8.1.22)), and so

$$J(u) = \frac{N}{N+2}\|u\|_{L^2}^{\frac{4}{N}} = \frac{2}{\alpha+2}\|u\|_{L^2}^{\alpha} . \tag{8.4.3}$$

Since R also satisfies equation (6.6.3), it satisfies the same identity. Since u minimizes J, we must have $J(R) \geq J(u)$, which implies that $\|u\|_{L^2} \leq \|R\|_{L^2}$. On the other hand, R being the ground state of (6.6.3), it is also the solution of (6.6.3) of minimal L^2-norm by (8.1.19) and (8.1.22), so that $\|R\|_{L^2} \leq \|u\|_{L^2}$. Therefore, $\|R\|_{L^2} = \|u\|_{L^2}$, and the result now follows from (8.4.3). □

CHAPTER 9

Further Results

In Sections 9.1 and 9.2 we present some results that follow easily from the techniques that we developed in the previous chapters. On the other hand, we describe in Sections 9.3 and 9.4 two results that do *not* fall into the scope of these methods. Finally, we briefly describe in Section 9.5 some further developments.

9.1. The Nonlinear Schrödinger Equation with a Magnetic Field

In this section we study the nonlinear Schrödinger equation in $\mathbb{R}^3$ in the presence of an external, constant magnetic field. Given $b \in \mathbb{R}$, $b \neq 0$, we consider the (vector-valued) potential Φ defined by

$$\Phi(x) = \frac{b}{2}(-x_2, x_1, 0) \quad \text{for } x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

which is the vector potential of the (constant) magnetic field $\vec{B} = \overrightarrow{\text{curl}}(\Phi)$, that is,

$$\vec{B} \equiv (0, 0, b).$$

We define the operator A on $L^2(\mathbb{R}^3)$ by

$$D(A) = \{u \in L^2(\mathbb{R}^3) : \nabla u + i\Phi u \in L^2(\mathbb{R}^3) \text{ and } \Delta u + 2i\Phi \cdot \nabla u - |\Phi|^2 u \in L^2(\mathbb{R}^3)\},$$

and

$$Au = \Delta u + 2i\Phi \cdot \nabla u - |\Phi|^2 u \quad \text{for } u \in D(A).$$

We consider the nonlinear Schrödinger equation

$$\begin{cases} iu_t + Au + g(u) = 0 \\ u(0) = \varphi, \end{cases} \tag{9.1.1}$$

and we refer to Avron, Herbst, and Simon [5, 6, 7], Combes, Schrader, and Seiler [93], Eboli and Marques [109], Kato [202], Reed and Simon [301], and B. Simon [313] for its physical relevance. We begin with the following observation.

LEMMA 9.1.1. *A is a self-adjoint, ≤ 0 operator on $L^2(\mathbb{R}^3)$.*

PROOF. Since $\mathcal{D}(\mathbb{R}^3) \subset D(A)$, $D(A)$ is dense in $L^2(\mathbb{R}^3)$. Furthermore, given $u, v \in D(A)$, $(Au, v)_{L^2} = -(\nabla u + i\Phi u, \nabla v + i\Phi v)_{L^2}$. Therefore, A is ≤ 0 and symmetric. It now remains to solve the equation $Au - \lambda u = f$ for every $f \in L^2(\mathbb{R}^3)$ and $\lambda > 0$. This follows easily by applying Lax-Milgram's lemma in the Hilbert space $H = \{u \in L^2(\mathbb{R}^3); \nabla u + i\Phi u \in L^2(\mathbb{R}^3)\}$, equipped with the scalar product $(u, v)_H = (\nabla u + i\Phi u, \nabla v + i\Phi v)_{L^2} + \lambda(u, v)_{L^2}$. □

We then may apply the results of Section 1.6. In particular, $D(A)$ is a Hilbert space when equipped with the norm

$$\|u\|_{D(A)}^2 = \|Au\|_{L^2}^2 + \|u\|_{L^2} \,,$$

and iA generates a group of isometries $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on the Hilbert space $(D(A))^\star$. The operators $(\mathcal{T}(t))_{t\in\mathbb{R}}$ restricted to any of the spaces $D(A)$, X_A, $L^2(\mathbb{R}^3)$, or $X_A^\star$ are a group of isometries, where X_A is defined by

$$X_A = \{u \in L^2(\mathbb{R}^3) : \nabla u + i\Phi u \in L^2(\mathbb{R}^3)\} \,,$$

and

$$\|u\|_{X_A}^2 = \|\nabla u + i\Phi u\|_{L^2}^2 + \|u\|_{L^2}^2 \,.$$

In addition, A can be extended to a self-adjoint, ≤ 0 operator on $(D(A))^\star$ (which we still denote by A), and A is bounded $X_A \to X_A^\star$ and $L^2(\mathbb{R}^3) \to (D(A))^\star$. Furthermore, we have the following result.

LEMMA 9.1.2. *The following properties hold:*

(i) $X_A \hookrightarrow L^p(\mathbb{R}^3)$ *for every* $2 \leq p \leq 6$.
(ii) $L^q(\mathbb{R}^3) \hookrightarrow X_A^\star$ *for every* $\frac{6}{5} \leq q \leq 2$.
(iii) $D(A) \hookrightarrow L^p(\mathbb{R}^3)$ *for every* $2 \leq p \leq \infty$.

PROOF. Let $u \in X_A$. We have

$$|\nabla(|u|)| = \left| \operatorname{Re}\left(\frac{\overline{u}}{|u|}(\nabla u + i\Phi u) \right) \right| \quad \text{a.e.}$$

on the set $\{x \in \mathbb{R}^3; u(x) \neq 0\}$. It follows that

$$|\nabla(|u|)| \leq |\nabla u + i\Phi u| \text{ a.e.} \tag{9.1.2}$$

Therefore, $\||u|\|_{H^1} \leq \|u\|_{X_A}$. Hence (i) is true. Note also that $\mathcal{D}(\mathbb{R}^3) \subset X_A$, from which we deduce that the embedding $X_A \hookrightarrow L^p(\mathbb{R}^3)$ is dense, and so (ii) follows from (i) by duality. Finally, let $u \in D(A)$ and set $f = Au \in L^2(\mathbb{R}^3)$. For every $j \in \{1,2,3\}$, let $v_j = \partial_j u + i\Phi_j u$. We have

$$Av_j - v_j = -(\partial_j - i\Phi_j)f - 2i(\nabla u + i\Phi u)\cdot(\partial_j\Phi - \nabla\Phi_j) - v_j \,. \tag{9.1.3}$$

Next, observe that $|\partial_j\Phi - \nabla\Phi_j| \leq b$. Furthermore, $\nabla + i\Phi$ is by definition a bounded operator $X_A \to L^2(\mathbb{R}^3)$, and so, by duality, $\nabla - i\Phi$ is bounded $L^2(\mathbb{R}^3) \to X_A^\star$. In particular, the right-hand side of (9.1.3) belongs to $X_A^\star$ and $\|Av_j - v_j\|_{X_A^\star} \leq C\|u\|_{D(A)}$. It follows easily that $v_j \in X_A$ and $\|v_j\|_{X_A} \leq C\|u\|_{D(A)}$. Letting successively $j = 1,2,3$ we obtain the inequality $\|\nabla u + i\Phi u\|_{X_A} \leq C\|u\|_{D(A)}$. Applying (i), we deduce that $\|\nabla u + i\Phi u\|_{L^6} \leq C\|u\|_{D(A)}$. Therefore, by (9.1.2), $\||\nabla(|u|)|\|_{L^6} \leq C\|u\|_{D(A)}$. Claim (iii) follows by Sobolev's embedding theorem. □

LEMMA 9.1.3. *If* $\varepsilon > 0$ *and* $1 \leq p < \infty$, *then* $(I - \varepsilon A)^{-1}$ *is continuous* $L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ *and* $\|(I-\varepsilon A)^{-1}\|_{\mathcal{L}(L^p,L^p)} \leq 1$.

PROOF. Let $\theta \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ be such that both θ and θ' are bounded, $\theta \geq 0$, $\theta' \geq 0$, and $\theta(0) = 0$. By applying the method of proof of Proposition 1.5.1, we need only show that

$$(Au, \theta(|u|^2)u)_{L^2} \leq 0 \quad \text{for all } u \in D(A) \,. \tag{9.1.4}$$

Consider $\rho \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \le \rho \le 1$ and $\rho(x) = 1$ for $|x| \le 1$, and set $\rho_m(x) = \rho(x/m)$ for $m \ge 1$. Let $u \in D(A)$. We have

$$(9.1.5) \qquad (Au, \theta(|u|^2)u)_{L^2} = \lim_{m\to\infty} (Au, \rho_m\theta(|u|^2)u)_{L^2} .$$

In addition, since $\rho_m\theta(|u|^2)u$ has compact support,

$$\begin{aligned}\big(Au, \rho_m\theta(|u|^2)u\big)_{L^2} = &- \operatorname{Re} \int_{\mathbb{R}^3} \nabla u \cdot \nabla\big(\rho_m\theta(|u|^2)\overline{u}\big) \\ &- 2\operatorname{Im} \int_{\mathbb{R}^3} \rho_m\theta(|u|^2)\overline{u}\Phi \cdot \nabla u - \int_{\mathbb{R}^3} \rho_m|\Phi|^2\theta(|u|^2)|u|^2 .\end{aligned}$$

It follows from the Cauchy-Schwarz inequality that

$$-2\operatorname{Im} \int_{\mathbb{R}^3} \rho_m\theta(|u|^2)\overline{u}\Phi \cdot \nabla u \le \int_{\mathbb{R}^3} \rho_m|\Phi|^2\theta(|u|^2)|u|^2 + \int_{\mathbb{R}^3} \rho_m\theta(|u|^2)|\nabla u|^2 ,$$

and so

$$\big(Au, \rho_m\theta(|u|^2)u\big)_{L^2} \le - \operatorname{Re} \int_{\mathbb{R}^3} \nabla u \cdot \nabla\big(\rho_m\theta(|u|^2)\overline{u}\big) + \int_{\mathbb{R}^3} \rho_m\theta(|u|^2)|\nabla u|^2 .$$

An elementary calculation shows that

$$\begin{aligned}&- \operatorname{Re}\big(\nabla u \cdot \nabla(\rho_m\theta(|u|^2)\overline{u})\big) + \rho_m\theta(|u|^2)|\nabla u|^2 \\ &\quad = -\rho_m\theta'(|u|^2)\big(|u|^2|\nabla u|^2 - \operatorname{Re}(\overline{u}^2\nabla u^2)\big) - \frac{1}{2}\nabla\rho_m \cdot \nabla\Theta(|u|^2) \\ &\quad \le -\frac{1}{2}\nabla\rho_m \cdot \nabla\Theta(|u|^2) \quad \text{a.e. where } \Theta(s) = \int_0^s \theta(\sigma)d\sigma ;\end{aligned}$$

therefore,

$$\big(Au, \rho_m\theta(|u|^2)u\big)_{L^2} \le \frac{1}{2}\int_{\mathbb{R}^3} \Theta(|u|^2)\Delta\rho_m \underset{m\to\infty}{\longrightarrow} 0 .$$

Applying (9.1.5), we obtain (9.1.4). □

Finally, we have the following estimate of $(\mathcal{T}(t))_{t\in\mathbb{R}}$.

LEMMA 9.1.4. *There exist $\delta > 0$ and $C < \infty$ such that $\mathcal{T}(t)$ is continuous $L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ for every $t \in (-\delta, \delta)$ and $t \neq 0$. Moreover,*

$$\|\mathcal{T}(t)u\|_{L^\infty} \le \frac{C}{|t|^{\frac{3}{2}}}\|u\|_{L^1}$$

for every $u \in L^1(\mathbb{R}^3)$ and $t \in (-\delta, \delta)$, $t \neq 0$.

PROOF. For every t such that $\sin(bt) \neq 0$, the following formula holds (see Avron, Herbst, and Simon [5]).

$$\mathcal{T}(t)u(x) = \frac{b}{4\pi(4\pi i t)^{\frac{1}{2}}\sin(bt)} \int_{\mathbb{R}^3} e^{-iF(x,y,t)}u(y)dy ,$$

where

$$F(x,y,t)=\frac{(x_3-y_3)^2}{4t}+\frac{b}{4}\left((x_1-y_1)^2+(x_2-y_2)^2\right)\operatorname{cotg}(bt)-\frac{b}{2}(x_1y_2-x_2y_1).$$

Therefore,

$$\|\mathcal{T}(t)\|_{\mathcal{L}(L^1,L^\infty)}\le\frac{|b|}{|t|^{\frac{1}{2}}|\sin(bt)|},$$

from which the result follows easily. ☐

Consider now g as in Example 3.2.11 with $N=3$; i.e.,

$$g(u)=Vu+f(\cdot,u(\cdot))+(W\star|u|^2)u$$

with $V\in L^p(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)$ with $p>3/2$, $W\in L^1(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)$ real-valued potentials, W even, and $f(x,u)$ is locally Lipschitz in u, uniformly in x, and satisfying $|f(x,u)-f(x,v)|\le C(1+|u|^\alpha+|v|^\alpha)|u-v|$ for some $0\le\alpha<4$. We set

$$G(u)=\int_{\mathbb{R}^3}\left\{\frac{1}{2}V(x)|u(x)|^2+F(x,u(x))+\frac{1}{4}(W\star|u|^2)(x)|u(x)|^2\right\}dx$$

with

$$F(x,z)=\int_0^{|z|}f(x,s)ds\quad\text{and}\quad E(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u+i\Phi u|^2\,dx-G(u).$$

We have the following result (see Cazenave and Esteban [62]; see also de Bouard [96] for related results for a more general equation).

THEOREM 9.1.5. *If g is as above, then the following properties hold.*

(i) *For every $\varphi\in X_A$, there exist $T_{\min}(\varphi),T_{\max}(\varphi)>0$, and a unique, maximal solution $u\in C((-T_{\min},T_{\max}),X_A)\cap C^1((-T_{\min},T_{\max}),X_A^\star)$ of problem* (9.1.1). *The solution u is maximal in the sense that if $T_{\max}<\infty$ (respectively, $T_{\min}<\infty$), then $\|u(t)\|_A\to\infty$ as $t\uparrow T_{\max}$ (respectively, as $t\downarrow -T_{\min}$).*

(ii) *There is conservation of charge and energy; that is,*

$$\|u(t)\|_{L^2}=\|\varphi\|_{L^2}\quad\textit{and}\quad E(u(t))=E(\varphi)\quad\textit{for all } t\in(-T_{\min},T_{\max}).$$

(iii) *There is continuous dependence of the solution on the initial value in the sense that both functions $T_{\min}(\varphi)$ and $T_{\max}(\varphi)$ are lower semicontinuous, and that if $\varphi_m\to\varphi$ in X_A and if $[-T_1,T_2]\subset(-T_{\min}(\varphi),T_{\max}(\varphi))$, then $u_m\to u$ in $C([-T_1,T_2],X_A)$, where u_m is the maximal solution of* (9.1.1) *with initial value φ_m.*

(iv) *If $\varphi\in D(A)$, then $u\in C((-T_{\min},T_{\max}),D(A))\cap C^1((-T_{\min},T_{\max}),L^2(\mathbb{R}^3))$.*

PROOF. It follows from Lemmas 9.1.1 to 9.1.4 that A and g satisfy the assumptions of Theorems 4.12.1 and 5.7.1. ☐

REMARK 9.1.6. By conservation of energy and Lemma 9.1.2(i), there exists $\delta>0$ such that if $\|\varphi\|_{X_A}\le\delta$, then the maximal solution u of (9.1.1) is global and $\sup\{\|u(t)\|_{X_A}:t\in\mathbb{R}\}<\infty$ (compare the proof of Corollary 6.1.2).

REMARK 9.1.7. In addition to the assumptions of Theorem 9.1.5, suppose that $W^+ \in L^q(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some $q > 3/2$, and that there exists $0 \le \delta < 4/3$ such that $F(x,u) \le C(1 + |u|^\delta)|u|^2$ for all $u \in \mathbb{C}$. It follows that for every $\varphi \in X_A$, the maximal solution u of (9.1.1) is global and $\sup\{\|u(t)\|_{X_A} : t \in \mathbb{R}\} < \infty$ (compare the proof of Corollary 6.1.2).

Concerning the existence of solutions of (9.1.1) for initial data in $L^2(\mathbb{R}^3)$, we have the following result (see Cazenave and Esteban [62]).

THEOREM 9.1.8. *Let g be as in Theorem 9.1.5 and assume further that $\alpha < 4/3$ and that $W \in L^q(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some $q > 3/2$. Let*

$$r = \max\left\{\alpha + 2, \frac{2p}{p-1}, \frac{2q}{q-1}\right\},$$

and let (q, r) be the corresponding admissible pair. It follows that for every $\varphi \in L^2(\mathbb{R}^3)$, there exists a unique solution $u \in C(\mathbb{R}, L^2(\mathbb{R}^3)) \cap L^q_{\text{loc}}(\mathbb{R}, L^r(\mathbb{R}^3))$ with $u_t \in L^q_{\text{loc}}(\mathbb{R}, (D(A))^\star)$ of (9.1.1). In addition, $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in \mathbb{R}$, and $u \in L^\gamma_{\text{loc}}(\mathbb{R}, L^\rho(\mathbb{R}^3))$ for every admissible pair (γ, ρ). Furthermore, if $\varphi_m \to \varphi$ in $L^2(\mathbb{R}^3)$ and if u^m denotes the solution of (4.1.1) with initial value φ, then $u^m \to u$ in $u \in L^\gamma_{\text{loc}}(\mathbb{R}, L^\rho(\mathbb{R}^3))$ for every admissible pair (γ, ρ).

PROOF. One adapts easily the proofs of Theorem 4.6.4 and Corollary 4.6.5. □

REMARK 9.1.9. Under certain assumptions on g, one can adapt the methods of Section 6.5 and show that some solutions of (9.1.1) blow up in finite time (cf. Gonçalves Ribeiro [150]).

REMARK 9.1.10. For a certain class of nonlinearities, equation (9.1.1) has stationary states of the form $u(t,x) = e^{i\omega t}\varphi(x)$ (cf. Esteban and Lions [110]). One obtains stability results that are similar to those of Sections 8.2 and 8.3. For some nonlinearities, the ground states are stable (cf. Cazenave and Esteban [62]), and for other nonlinearities, the ground states are unstable (cf. Gonçalves Ribeiro [151]).

9.2. The Nonlinear Schrödinger Equation with a Quadratic Potential

We already studied the nonlinear Schrödinger equation in $\mathbb{R}^N$ with an external potential V, with $V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $p \ge 1$, $p > N/2$. Here we extend these results to the case of potentials U that are not localized, but have at most a quadratic growth at infinity, the model case being $U(x) = |x|^2$. More precisely, consider a real-valued potential $U \in C^\infty(\mathbb{R}^N)$ such that $U \ge 0$ and

$$D^\alpha U \in L^\infty(\mathbb{R}^N) \quad \text{for all } a \in \mathbb{N}^N$$

such that $|\alpha| \ge 2$. We define the operator A on $L^2(\mathbb{R}^N)$ by

$$\begin{cases} D(A) = \{u \in H^1(\mathbb{R}^N) : U|u|^2 \in L^1(\mathbb{R}^N) \text{ and } \Delta u - Uu \in L^2(\mathbb{R}^N)\}, \\ Au = \Delta u - Uu \quad \text{for } u \in D(A). \end{cases}$$

We consider the nonlinear Schrödinger equation

$$\begin{cases} iu_t + Au + g(u) = 0 \\ u(0) = \varphi. \end{cases} \tag{9.2.1}$$

We begin with the following observation.

LEMMA 9.2.1. *The operator A is self-adjoint and ≤ 0 operator on $L^2(\mathbb{R}^N)$.*

PROOF. Since $\mathcal{D}(\mathbb{R}^N) \subset D(A)$, $D(A)$ is dense in $L^2(\mathbb{R}^N)$. Furthermore, given $u, v \in D(A)$,

$$(Au, v)_{L^2} = -\operatorname{Re} \int_{\mathbb{R}^N} \nabla u \cdot \nabla \overline{v} + U u \overline{v}.$$

We deduce easily that A is ≤ 0 and symmetric. Therefore, it remains to solve the equation $Au - \lambda u = f$ for every $f \in L^2(\mathbb{R}^N)$ and $\lambda > 0$. This is done easily by applying Lax-Milgram's lemma in the Hilbert space $H = \{u \in H^1(\mathbb{R}^N) : U|u|^2 \in L^1(\mathbb{R}^N)\}$, equipped with the norm defined by

$$\|u\|_H^2 = \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^N} U|u|^2 + \lambda \|u\|_{L^2}^2 \quad \text{for all } u \in H.$$

□

We may then apply the results of Section 1.6. In particular, $D(A)$ is a Hilbert space when equipped with the norm

$$\|u\|_{D(A)}^2 = \|Au\|_{L^2}^2 + \|u\|_{L^2},$$

and iA generates a group of isometries $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on the Hilbert space $(D(A))^\star$. The group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ restricted to either of the spaces $D(A)$, X_A, $L^2(\mathbb{R}^N)$, $X_A^\star$ is a group of isometries, where X_A is defined by

$$X_A = \{u \in H^1(\mathbb{R}^N) : U|u|^2 \in L^1(\mathbb{R}^N)\}$$

and

$$\|u\|_{X_A}^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 + \int_{\mathbb{R}^N} U|u|^2.$$

In addition, A can be extended to a self-adjoint, ≤ 0 operator on $(D(A))^\star$ (which we still denote by A), and A is bounded $X_A \to X_A^\star$ and $L^2(\mathbb{R}^N) \to (D(A))^\star$. Furthermore, we have the following result.

LEMMA 9.2.2. *The following properties hold:*

(i) $X_A \hookrightarrow H^1(\mathbb{R}^N)$.
(ii) $H^{-1}(\mathbb{R}^N) \hookrightarrow X_A^\star$.
(iii) $D(A) \hookrightarrow L^p(\mathbb{R}^N)$ *for every* $2 \le p < \infty$ *such that* $\frac{1}{p} \ge \frac{1}{2} - \frac{2}{N}$.

PROOF. Claim (i) follows from the definition of X_A, and (ii) then follows by duality. We now prove (iii) for $N \ge 3$, the proof for $N = 1, 2$ being easily adapted. Let $u \in D(A)$ and let $f = Au$. Consider $p > 2$ and take the L^2-scalar product of the equation $\Delta u - Uu = f$ with $|u|^{p-2}u$. (In fact, a rigorous proof would require a regularization; see the proof of Lemma 9.2.3 below.) One obtains easily

$$\int_{\mathbb{R}^N} |u|^{p-2} |\nabla u|^2 \le \|f\|_{L^2} \|u\|_{L^{2(p-1)}}^{p-1}.$$

Since $p^2|u|^{p-2}|\nabla u|^2 = 4|\nabla(|u|^{p/2})|^2$, it follows from Sobolev's inequality that

$$\|u\|^p_{L^{\frac{Np}{N-2}}} \le C\|f\|_{L^2}\|u\|^{p-1}_{L^{2(p-1)}}\,.$$

Hence (iii) follows, by letting $p = \frac{2N-4}{N-4}$ if $N \ge 5$ and taking any $p < \infty$ if $N \le 4$, then applying Hölder's inequality. □

LEMMA 9.2.3. *If $\varepsilon > 0$ and $1 \le p < \infty$, then $(I - \varepsilon A)^{-1}$ is continuous $L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, and $\|(I - \varepsilon A)^{-1}\|_{\mathcal{L}(L^p,L^p)} \le 1$.*

PROOF. Let $\theta \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be such that both θ and θ' are bounded, $\theta \ge 0$, $\theta' \ge 0$, and $\theta(0) = 0$. By applying the method of proof of Proposition 1.5.1, we need only show that

$$(Au, \theta(|u|^2)u)_{L^2} \le 0 \quad \text{for all } u \in D(A)\,.$$

We have

$$(Au, \theta(|u|^2)u)_{L^2} = (\Delta u, \theta(|u|^2)u)_{L^2} - \int_{\mathbb{R}^N} U\theta(|u|^2)|u|^2 \le (\Delta u, \theta(|u|^2)u)_{L^2}\,,$$

and we already know that (see the proof of Proposition 1.5.1) $(\Delta u, \theta(|u|^2)u)_{L^2} \le 0$. The result follows. □

Finally, we have the following estimate of $(\mathcal{T}(t))_{t\in\mathbb{R}}$.

LEMMA 9.2.4. *There exist $\delta > 0$ and $C < \infty$ such that $\mathcal{T}(t)$ is continuous $L^1(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$ for every $t \in (-\delta, \delta)$, $t \ne 0$, and*

$$\|\mathcal{T}(t)u\|_{L^\infty} \le \frac{C}{|t|^{\frac{N}{2}}}\|u\|_{L^1} \tag{9.2.2}$$

for every $u \in L^1(\mathbb{R}^N)$ and $|t| \le \delta$, $t \ne 0$.

PROOF. This is a delicate result, based on a calculation of the kernel associated to $\mathcal{T}(t)$. See Oh [277], proposition 2.2. □

REMARK 9.2.5. Estimate (9.2.2) holds for $|t| \le \delta$. In fact, (9.2.2) does not in general hold for all $t \ne 0$. This can be seen in the special case $U(x) = \omega^2|x|^2/4$, where there is the following explicit formula (Mehler's formula; see Feynman and Hibbs [112])

$$\mathcal{T}(t)u(x) = \left(\frac{\omega}{4\pi i \sin(\omega t)}\right)^{\frac{N}{2}} \int_{\mathbb{R}^N} e^{(i\frac{\omega}{4\sin(\omega t)}((|x|^2+|y|^2)\cos(\omega t) - 2x\cdot y))}u(y)dy\,.$$

We see that $\|\mathcal{T}(t)u\|_{\mathcal{L}(L^1,L^\infty)} \le (\omega/4\pi i|\sin(\omega t)|)^{\frac{N}{2}}$ if $\sin(\omega t) \ne 0$ and that this estimate is optimal.

Consider now a real-valued potential $V : \mathbb{R}^N \to \mathbb{R}$ such that $V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $p \ge 1$, $p > N/2$.

Let

$$g(u) = Vu + f(\cdot, u(\cdot)) + (W \star |u|^2)u$$

as in Example 3.2.11, and set

$$G(u) = \int_{\mathbb{R}^N} \left\{ \frac{1}{2} V(x)|u(x)|^2 + F(x, u(x)) + \frac{1}{4} (W \star |u|^2)(x)|u(x)|^2 \right\} dx$$

and

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} U|u|^2 \, dx - G(u) .$$

We have the following result (see Oh [277, 278] for a similar result in the case where $g(u) = -\lambda |u|^\alpha u$).

THEOREM 9.2.6. *If g is as above, then the following properties hold:*

(i) *For every $\varphi \in X_A$, there exist $T_{\min}(\varphi), T_{\max}(\varphi) > 0$ and a unique, maximal solution $u \in C((-T_{\min}, T_{\max}), X_A) \cap C^1((-T_{\min}, T_{\max}), X_A^\star)$ of problem (9.2.1). The solution u is maximal in the sense that if $T_{\max} < \infty$ (respectively $T_{\min} < \infty$), then $\|u(t)\|_A \to \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow -T_{\min}$).*

(ii) *There is conservation of charge and energy, that is,*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2} \quad \text{and} \quad E(u(t)) = E(\varphi) \quad \text{for all } t \in (-T_{\min}, T_{\max}).$$

(iii) *There is continuous dependence of the solution on the initial value in the sense that both functions $T_{\min}(\varphi)$ and $T_{\max}(\varphi)$ are lower semicontinuous, and that if $\varphi_m \to \varphi$ in X_A and if $[-T_1, T_2] \subset (-T_{\min}(\varphi), T_{\max}(\varphi))$, then $u_m \to u$ in $C([-T_1, T_2], X_A)$, where u_m is the maximal solution of (9.2.1) with initial value φ_m.*

(iv) *If $\varphi \in D(A)$, then $u \in C((-T_{\min}, T_{\max}), D(A)) \cap C^1((-T_{\min}, T_{\max}), L^2(\mathbb{R}^N))$.*

PROOF. It follows from Lemmas 9.2.1 to 9.2.4 that A and g satisfy the assumptions of Theorems 4.12.1 and 5.7.1. □

REMARK 9.2.7. By conservation of energy and Lemma 9.2.2(i), there exists $\delta > 0$ such that if $\|\varphi\|_{X_A} \le \delta$, then the maximal solution u of (9.2.1) is global and $\sup\{\|u(t)\|_{X_A} : t \in \mathbb{R}\} < \infty$ (compare the proof of Corollary 6.1.5).

REMARK 9.2.8. In addition to the assumptions of Theorem 9.2.6, suppose that $W^+ \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $q \ge 1$, $q > N/2$, and that exists $0 \le \delta < 4/N$ such that $F(x, u) \le C(1 + |u|^\delta)|u|^2$ for all $u \in \mathbb{C}$. It follows that, for every $\varphi \in X_A$, the maximal solution u of (9.2.1) is global and $\sup\{\|u(t)\|_{X_A} : t \in \mathbb{R}\} < \infty$ (compare the proof of Corollary 6.1.2).

Concerning the existence of solutions of (9.2.1) for initial data in $L^2(\mathbb{R}^N)$, we have the following result.

THEOREM 9.2.9. *Let g be as in Theorem 9.2.6 and assume further that $\alpha < 4/N$ and that $W \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $q \ge 1$, $q > N/2$. Let*

$$r = \max\left\{ \alpha + 2, \frac{2p}{p-1}, \frac{2q}{q-1} \right\},$$

and let (q,r) be the corresponding admissible pair. It follows that for every $\varphi \in L^2(\mathbb{R}^N)$, there exists a unique solution $u \in C(\mathbb{R}, L^2(\mathbb{R}^N)) \cap L^q_{\mathrm{loc}}(\mathbb{R}, L^r(\mathbb{R}^N))$ with $u_t \in L^q_{\mathrm{loc}}(\mathbb{R}, (D(A))^\star)$ of (9.2.1). In addition, $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in \mathbb{R}$, and $u \in L^\gamma_{\mathrm{loc}}(\mathbb{R}, L^\rho(\mathbb{R}^N))$ for every admissible pair (γ, ρ). Furthermore, if $\varphi_m \to \varphi$ in $L^2(\mathbb{R}^N)$ and if u^m denotes the solution of (4.1.1) with initial value φ, then $u^m \to u$ in $u \in L^\gamma_{\mathrm{loc}}(\mathbb{R}, L^\rho(\mathbb{R}^N))$ for every admissible pair (γ, ρ).

PROOF. One adapts easily the proofs of Theorem 4.6.4 and Corollary 4.6.5. □

REMARK 9.2.10. Under certain assumptions on g, one can adapt the methods of Section 6.5 and show that some solutions of (9.2.1) blow up in finite time. More precisely, if we assume that $|x \cdot \nabla U| \le C(|x|^2 + U)$ and that g satisfies the assumptions of Proposition 6.5.1, one can show that (with the notation of Proposition 6.5.1)

$$\begin{aligned} f''(t) = 16E(\varphi) &+ \int\limits_{\mathbb{R}^N} (8(N+2)F(u) - 4N\,\mathrm{Re}(f(u)\overline{u}))dx \\ &+ 8 \int\limits_{\mathbb{R}^N} \Big(V + \frac{1}{2}x \cdot \nabla V\Big)|u|^2\,dx + 4 \int\limits_{\mathbb{R}^N} \Big(\Big(W + \frac{1}{2}x \cdot \nabla W\Big) \star |u|^2\Big)|u|^2\,dx \\ &- 8 \int\limits_{\mathbb{R}^N} \Big(U + \frac{1}{2}x \cdot \nabla U\Big)|u|^2\,dx\,. \end{aligned}$$

The proof of the above inequality is similar to that of Proposition 6.5.1. Assume further that g satisfies (6.5.24), (6.5.25), and (6.5.26), and that

$$U + \frac{1}{2}x \cdot \nabla U \ge 0\,.$$

If $\varphi \in X_A$ is such that $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$ and $E(\varphi) < 0$, then $T_{\max}(\varphi) < \infty$ and $T_{\min}(\varphi) < \infty$ (compare the proof of Theorem 6.5.4).

REMARK 9.2.11. In the model case $U(x) \equiv |x|^2$ and $g(u) = \lambda |u|^\alpha u$, where $\lambda > 0$ and $4/N \le \alpha < 4/(N-2)$ ($4/N \le \alpha < \infty$, if $N = 1$), it follows from Remark 9.2.10 that if $\varphi \in X_A$ is such that $E(\varphi) < 0$, then $T_{\max}(\varphi) < \infty$ and $T_{\min}(\varphi) < \infty$. For a more detailed study, see Carles [51, 52, 53], Fukuizumi [117], and Zhang [369, 370].

9.3. The Logarithmic Schrödinger Equation

Let $\Omega \subset \mathbb{R}^N$ be any open domain. We consider the following nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u + Vu + u\log(|u|^2) = 0 \\ u(0) = \varphi, \end{cases} \tag{9.3.1}$$

where V is some real-valued potential. The equation (9.3.1) arises in a model of nonlinear wave mechanics (see Bialinycki-Birula and Mycielski [31]). We cannot apply the results of Section 3.3 for solving the problem (9.3.1) because the function $z \mapsto z\log(|z|^2)$ is not Lipschitz continuous at $z = 0$, due to the singularity of the logarithm at the origin. Furthermore, it is not always clear in what space the nonlinearity makes sense. For example, if $\Omega = \mathbb{R}^N$ and $u \in H^1(\mathbb{R}^N)$, then $u\log(|u|^2)$ does not in general belong to any L^p for $p \le 2$, nor to $H^{-1}(\mathbb{R}^N)$ (this is again due to the singularity of the logarithm at the origin). However, we will solve the problem (9.3.1) by a compactness method, but before stating the precise existence result, we need to introduce some notation. Define

$$F(z) = |z|^2 \log(|z|^2) \quad \text{for every } z \in \mathbb{C}.$$

Furthermore, let the functions A, B, a, b be defined by

$$A(s) = \begin{cases} -s^2 \log(s^2) & \text{if } 0 \le s \le e^{-3} \\ 3s^2 + 4e^{-3}s - e^{-3} & \text{if } s \ge e^{-3}, \end{cases} \qquad B(s) = F(s) + A(s),$$

and

$$a(s) = \frac{A(s)}{s}, \qquad b(s) = \frac{B(s)}{s}.$$

Extend the functions a and b to the complex plane by setting

$$a(z) = \frac{z}{|z|} a(|z|), \quad b(z) = \frac{z}{|z|} b(|z|) \quad \text{for } z \in \mathbb{C}, z \neq 0.$$

It follows in particular that A is a convex C^1-function, which is C^2 and positive except at the origin. Let $A^\star$ be the convex conjugate function of A (see, e.g., Brezis [43]). The function $A^\star$ is also a convex C^1-function, which is positive except at the origin. Define the sets X and X' by

$$X = \{u \in L^1_{\text{loc}}(\Omega) : A(|u|) \in L^1(\Omega)\}, \quad X' = \{u \in L^1_{\text{loc}}(\Omega) : A^\star(|u|) \in L^1(\Omega)\}.$$

Finally, set

$$\|u\|_X = \inf\left\{k > 0 : \int_\Omega A\left(\frac{|u|}{k}\right) \le 1\right\} \quad \text{for } u \in X$$

and

$$\|u\|_{X'} = \inf\left\{k > 0 : \int_\Omega A^\star\left(\frac{|u|}{k}\right) \le 1\right\} \quad \text{for } u \in X'.$$

We have the following results (see Cazenave [58], lemmas 2.1 and 2.5, and Kranosel'skii and Rutickii [218]).

LEMMA 9.3.1. *The spaces X and X' are linear spaces. The inner product spaces $(X, \|\cdot\|_X)$ and $(X', \|\cdot\|_{X'})$ are reflexive Banach spaces and X' is the topological dual of X. Furthermore, the following properties hold:*

(i) *If $u^m \underset{m\to\infty}{\longrightarrow} u$ in X, then $A(|u^m|) \underset{m\to\infty}{\longrightarrow} A(|u|)$ in $L^1(\Omega)$.*

(ii) *If $u^m \underset{m\to\infty}{\longrightarrow} u$ a.e. and if*

$$\int_\Omega A(|u^m|) \underset{m\to\infty}{\longrightarrow} \int_\Omega A(|u|) < \infty,$$

then $u^m \underset{m\to\infty}{\longrightarrow} u$ in X.

LEMMA 9.3.2. *The operator $u \mapsto a(u)$ maps continuously $X \to X'$. The image under a of a bounded subset of X is a bounded subset of X'.*

Finally, consider the Banach space $W = H^1_0(\Omega) \cap X$ equipped with the usual norm. It follows from Proposition 1.1.3 that

$$W^\star = H^{-1}(\Omega) + X'.$$

Define

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega V|u|^2 - \frac{1}{2}\int_\Omega |u|^2 \log(|u|^2) \quad \text{for every } u \in W,$$

where the potential $V \in L^p(\Omega) + L^\infty(\Omega)$ for some $p \geq 1$, $p > N/2$. We have the following result.

LEMMA 9.3.3. *The operator $L : u \mapsto \Delta u + Vu + u\log(|u|^2)$ maps continuously $W \to W^\star$. The image under L of a bounded subset of W is a bounded subset of $W^\star$. The operator E is continuous $W \to \mathbb{R}$.*

PROOF. One easily verifies that for every $\varepsilon > 0$, there exists C_ε such that

$$|b(v) - b(u)| \leq C_\varepsilon(|u|^\varepsilon + |v|^\varepsilon)|v - u| \quad \text{for all } u, v \in \mathbb{C}\,. \tag{9.3.2}$$

Integrating inequality (9.3.2) on Ω, and applying Hölder's and Sobolev's inequalities, we obtain easily that $u \mapsto b(u)$ maps continuously $H^1_0(\Omega) \to H^{-1}(\Omega)$ and that the image under b of a bounded subset of $H^1_0(\Omega)$ is a bounded subset of $H^{-1}(\Omega)$. The same holds for Δ, and also for $u \mapsto Vu$ (by Hölder's inequality), and so the first part of the statement follows from Lemma 9.3.2. Finally,

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega V|u|^2 + \frac{1}{2}\int_\Omega A(|u|) - \frac{1}{2}\int_\Omega B(|u|)\,. \tag{9.3.3}$$

The first term in the right-hand side of (9.3.3) is continuous $H^1_0(\Omega) \to \mathbb{R}$, and it follows from Lemma 9.3.1(i) that the third term is continuous $X \to X'$. Furthermore,

$$|B(v) - B(u)| \leq C_\varepsilon(|u|^{1+\varepsilon} + |v|^{1+\varepsilon})|v - u|$$

by (9.3.2). Integrating the above inequality on Ω, and applying Hölder's and Sobolev's inequalities, we deduce that

$$\int_\Omega |B(v) - B(u)| \leq C\big(1 + \|u\|^2_{H^1} + \|u\|^2_{H^1}\big)\|v - u\|_{L^2} \quad \text{for all } u, v \in H^1_0(\Omega).$$

Therefore, the fourth term in the right-hand side of (9.3.3) is continuous $H^1_0(\Omega) \to \mathbb{R}$. Finally, if $V = V_1 + V_2$ with $V_1 \in L^p(\Omega)$ and $V_2 \in L^\infty(\Omega)$, then

$$\int_\Omega |V||u|^2 \leq \|V_1\|_{L^p}\|u\|^2_{L^{\frac{2p}{p-1}}} + \|V_2\|_{L^\infty}\|u\|^2_{L^2}\,. \tag{9.3.4}$$

Thus the second term in the right-hand side of (9.3.3) is continuous $H^1_0(\Omega) \to \mathbb{R}$, which completes the proof. □

Our main result of this section is the following (see Cazenave and Haraux [63]).

THEOREM 9.3.4. *Let V be a real-valued potential such that $V \in L^p(\Omega) + L^\infty(\Omega)$ for some $p \geq 1$, $p > N/2$. The following properties hold:*

(i) *For every $\varphi \in W$, there exists a unique, maximal solution $u \in C(\mathbb{R}, W) \cap C^1(\mathbb{R}, W^\star)$ of problem (9.3.1). Furthermore,* $\sup_{t\in\mathbb{R}} \|u(t)\|_W < \infty$.

(ii) *There is conservation of charge and energy; that is,*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2} \quad \textit{and} \quad E(u(t)) = E(\varphi) \quad \textit{for all } t \in \mathbb{R}.$$

(iii) *There is continuous dependence of the solution on the initial value in the sense that if $\varphi_m \to \varphi$ in W, then $u_m \to u$ in W uniformly on bounded intervals, where u_m is the maximal solution of (9.3.1) with initial value φ_m.*

For the proof of Theorem 9.3.4, we will use the following two lemmas.

LEMMA 9.3.5. *We have*

$$\big|\operatorname{Im}\big((v\log(|v|^2) - u\log(|u|^2))(\overline{v} - \overline{u})\big)\big| \le 4|v-u|^2 \quad \textit{for all } u, v \in \mathbb{C}.$$

PROOF. Note that

$$\operatorname{Im}\big((v\log(|v|^2) - u\log(|u|^2))(\overline{v} - \overline{u})\big) = 2(\log|v| - \log|u|)\operatorname{Im}(v\overline{u} - u\overline{v}).$$

Assuming, for example, $0 < |v| \le |u|$, we see that

$$|\log|v| - \log|u|| \le \frac{|v| - |u|}{|v|} \le \frac{|v-u|}{|v|}$$

and

$$|\operatorname{Im}(v\overline{u} - u\overline{v})| = |v(\overline{u} - \overline{v}) + \overline{v}(v-u)| \le 2|v||v-u|.$$

Hence the result follows. □

LEMMA 9.3.6. *Given $k \in \mathbb{N}$, set $\Omega_k = \Omega \cap \{x \in \Omega : |x| < k\}$. Let $(u^m)_{m\in\mathbb{N}} \subset L^\infty(\mathbb{R}, H_0^1(\Omega))$ be a bounded sequence. If $(u^m|_{\Omega_k})_{m\in\mathbb{N}}$ is a bounded sequence of $W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega_k))$ for every $k \in \mathbb{N}$, then there exists a subsequence, which we still denote by $(u^m)_{m\in\mathbb{N}}$, and there exists $u \in L^\infty(\mathbb{R}, H_0^1(\Omega))$ such that the following properties hold:*

(i) *$u|_{\Omega_k} \in W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega_k))$ for every $k \in \mathbb{N}$.*
(ii) *$u^m(t) \rightharpoonup u(t)$ in $H_0^1(\Omega)$ as $m \to \infty$ for every $t \in \mathbb{R}$.*
(iii) *For every $t \in \mathbb{R}$, there exists a subsequence m_j such that $u^{m_j}(t,x) \to u(t,x)$ as $k \to \infty$ for a.a. $x \in \Omega$.*
(iv) *$u^m(t,x) \to u(t,x)$ as $m \to \infty$ for a.a. $(t,x) \in \mathbb{R} \times \Omega$.*

PROOF. Fix $k \in \mathbb{N}$. $(u^m|_{\Omega_k})_{m\in\mathbb{N}}$ is a bounded sequence of $L^\infty((-k,k), H^1(\Omega_k)) \cap W^{1,\infty}((-k,k), H^{-1}(\Omega_k))$, so that (by Proposition 1.1.2) there exist a subsequence (which we still denote by $(u^m)_{m\in\mathbb{N}}$) and $u \in L^\infty((-k,k), H^1(\Omega_k))$ such that $u^m(t)|_{\Omega_k} \rightharpoonup u(t)$ in $H^1(\Omega_k)$. Letting $k \to \infty$ and considering a diagonal sequence, we see that there exist a subsequence (which we still denote by $(u^m)_{m\in\mathbb{N}}$) and $u \in L^\infty(\mathbb{R}, H^1(\Omega))$ such that $u^m(t)|_{\Omega_k} \rightharpoonup u(t)$ in $H^1(\Omega_k)$ for every $k \in \mathbb{N}$ and every $t \in \mathbb{R}$. This implies in particular that $u^m(t) \rightharpoonup u(t)$ in $H^1(\Omega)$. Therefore, $u \in L^\infty(\mathbb{R}, H_0^1(\Omega))$, and (ii) holds. In addition, since the embedding $H^1(\Omega_k) \hookrightarrow L^2(\Omega_k)$

is compact, we have $u^m(t)|_{\Omega_k} \to u(t)|_{\Omega_k}$ in $L^2(\Omega_k)$ for every $k \in \mathbb{N}$ and every $t \in \mathbb{R}$. Applying the dominated convergence theorem, we deduce that

$$\int_{-k}^{k} \int_{\Omega_k} |u^m - u|^2 \underset{m\to\infty}{\longrightarrow} 0 \quad \text{for every } k \in \mathbb{N}.$$

In particular, there exists a subsequence m_j for which $u^{m_j} \to u$ a.e. on $(-k,k)\times\Omega_k$ as $j \to \infty$. Letting $k \to \infty$ and considering a diagonal sequence, we see that (iv) holds. Furthermore, given $t \in \mathbb{R}$ and $k \in \mathbb{N}$, there exists a subsequence m_j for which $u^{m_j}(t) \to u(t)$ a.e. on Ω_k as $j \to \infty$. Letting $k \to \infty$ and considering a diagonal sequence, we see that (iii) holds. Finally, it follows from (i) and Remark 1.3.13(i) that $u|_{\Omega_k} \in W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega_k))$. Hence (i) is established. □

PROOF OF THEOREM 9.3.4. We apply a compactness method, and we proceed in four steps. Consider $\varphi \in W$.

STEP 1. Construction of approximate solutions. We have $V = V_1 + V_2$ with $V_1 \in L^p(\Omega)$ and $V_2 \in L^\infty(\Omega)$. Given $m \in \mathbb{N}$, define the potentials V_1^m and V_2^m by

$$V_j^m(x) = \begin{cases} V_j(x) & \text{if } |V_j(x)| \le m \\ 0 & \text{if } |V_j(x)| > m \end{cases} \quad \text{for } j = 1, 2.$$

Define the functions a_m and b_m by

$$a_m(z) = \begin{cases} a(z) & \text{if } |z| \ge \frac{1}{m} \\ mza(\frac{1}{m}) & \text{if } |z| \le \frac{1}{m}, \end{cases} \qquad b_m(z) = \begin{cases} b(z) & \text{if } |z| \le m \\ \frac{z}{m} b(m) & \text{if } |z| \ge m. \end{cases}$$

Finally, set

$$g_m(u) = V_1^m u + V_2^m u - a_m(u) + b_m(u) \quad \text{for } u \in H_0^1(\Omega).$$

Since $V_1^m, V_2^m \in L^\infty(\Omega)$ and both a_m and b_m are (globally) Lipschitz continuous $\mathbb{C} \to \mathbb{C}$, we see that g_m is Lipschitz continuous $L^2(\Omega) \to L^2(\Omega)$. It follows from Corollary 3.3.11 that there exists a unique solution $u^m \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, H^{-1}(\Omega))$ of the problem

$$\begin{cases} iu_t^m + \Delta u^m + g_m(u^m) = 0 \\ u^m(0) = \varphi. \end{cases} \tag{9.3.5}$$

In addition,

$$\|u^m(t)\|_{L^2} = \|\varphi\|_{L^2} \quad \text{and} \quad E_m(u^m(t)) = E_m(\varphi) \quad \text{for all } t \in \mathbb{R}, \tag{9.3.6}$$

where

$$E_m(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega V_1^m |u|^2 - \frac{1}{2}\int_\Omega V_2^m |u|^2 + \frac{1}{2}\int_\Omega \Phi_m(|u|) - \frac{1}{2}\int_\Omega \Psi_m(|u|),$$

and the functions Φ_m and Ψ_m are defined by

$$\Phi_m(z) = \frac{1}{2}\int_0^{|z|} a_m(s)ds \quad \text{and} \quad \Psi_m(z) = \frac{1}{2}\int_0^{|z|} b_m(s)ds \quad \text{for all } z \in \mathbb{C}.$$

STEP 2. Estimates of the approximate solutions. It follows from (9.3.6) that u^m is bounded in $L^\infty(\mathbb{R}, L^2(\Omega))$. Note that, by the dominated convergence theorem,

$$E_m(\varphi) \underset{m\to\infty}{\longrightarrow} E(\varphi)\,. \tag{9.3.7}$$

Applying (9.3.6) and (9.3.7), we deduce that (note that $\Phi_m \geq 0$ and compare inequality (9.3.4))

$$\|u^m(t)\|_{H^1} \leq C + C\|V_1^m\|_{L^p}\|u^m(t)\|_{H^1}^2 + \|\Psi_m(u^m(t))\|_{L^1}\,.$$

Note that $\|V_1^m\|_{L^p} \leq \|V_1\|_{L^p}$. Note also that we may assume that $\|V_1\|_{L^p}$ is arbitrarily small, by modifying V_2. In particular, we may assume that $C\|V_1^m\|_{L^p} \leq 1/4$. Finally, one easily verifies (see the proof of Lemma 9.3.3) that there exists C such that

$$\big\|\Psi_m(u^m(t))\big\|_{L^1} \leq \frac{1}{4}\|u^m(t)\|_{H^1}^2 + C\|u^m(t)\|_{L^2}^2\,,$$

therefore,

$$u^m \text{ is bounded in } L^\infty(\mathbb{R}, H_0^1(\Omega)). \tag{9.3.8}$$

Finally, it follows from elementary calculations that for every $\varepsilon > 0$, there exists C_ε such that

$$|g_m(u)| \leq |V_1^m||u| + |V_2^m||u| + C_\varepsilon(|u|^{1-\varepsilon} + |u|^{1+\varepsilon})\,.$$

We deduce easily from Hölder's and Sobolev's inequalities and (9.3.8) that, given $k \in \mathbb{N}$,

$$g_m(u^m) \text{ is bounded in } L^\infty(\mathbb{R}, L^{\frac{2p}{p-1}}(\Omega_k)), \tag{9.3.9}$$

where $\Omega_k = \Omega \cap \{x \in \Omega : |x| < k\}$. In particular, $(g_m(u^m))_{m\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}, H^{-1}(\Omega_k))$, and it follows from (9.3.5) that $(u^m|_{\Omega_k})_{m\in\mathbb{N}}$ is bounded in $W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega_k))$.

STEP 3. Passage to the limit. By Step 2, $(u^m)_{m\in\mathbb{N}}$ satisfies the assumptions of Lemma 9.3.6. Let u be its limit. It follows from (9.3.5) that, for every $\psi \in \mathcal{D}(\Omega)$ and every $\phi \in \mathcal{D}(\mathbb{R})$,

$$\int_{\mathbb{R}} \langle iu_t^m + \Delta u^m + g_m(u^m), \psi\rangle_{\mathcal{D}',\mathcal{D}}\phi(t)dt = 0\,.$$

This means that

$$\int_{\mathbb{R}} (-\langle iu^m, \psi\rangle\phi'(t) + \langle u^m, \Delta\psi\rangle\phi(t))\,dt + \int_{\mathbb{R}}\int_{\Omega} g_m(u^m)\psi\phi\,dx\,dt = 0\,. \tag{9.3.10}$$

It follows easily from (9.3.8) and from property (ii) of Lemma 9.3.6 that

$$\begin{aligned} &\int_{\mathbb{R}} (-\langle iu^m, \psi\rangle\phi'(t) + \langle u^m, \Delta\psi\rangle\phi(t))\,dt \underset{m\to\infty}{\longrightarrow} \\ &\qquad\qquad \int_{\mathbb{R}} (-\langle iu, \psi\rangle\phi'(t) + \langle u, \Delta\psi\rangle\phi(t))\,dt\,. \end{aligned} \tag{9.3.11}$$

Furthermore, the function $h_m(t,x) = g_m(u^m)\psi(x)\phi(t)$ has compact support. We therefore deduce from (9.3.9) that h_m is bounded in $L^{\frac{2p}{p-1}}(\mathbb{R}\times\Omega)$. By property (iv) of Lemma 9.3.6, $h_m \to (Vu + u\log(|u|^2))\psi\phi$ a.e. on $\mathbb{R}\times\Omega$. Since h_m has compact support, it follows from Proposition 1.2.1 that $h_m \to (Vu+u\log(|u|^2))\psi\phi$ in $L^1(\mathbb{R}\times\Omega)$. Applying (9.3.10) and (9.3.11), we thus obtain

$$\int_{\mathbb{R}} (-\langle iu,\psi\rangle\phi'(t) + \langle u,\Delta\psi\rangle\phi(t))\,dt + \int_{\mathbb{R}}\int_{\Omega} (Vu + u\log|u|^2)\psi\phi\,dx\,dt = 0\,,$$

which implies that

$$\int_{\mathbb{R}} \langle iu_t + \Delta u + Vu + u\log|u|^2, \psi\rangle_{\mathcal{D}',\mathcal{D}}\phi(t)dt = 0\,.$$

It follows that, for all $t\in\mathbb{R}$,

$$iu_t + \Delta u + Vu + u\log(|u|^2) = 0 \quad \text{in } H^{-1}(\Omega_k) \text{ for every } k\in\mathbb{N}. \tag{9.3.12}$$

In addition, $u(0)=\varphi$ by property (ii) of Lemma 9.3.6. Finally, we deduce easily from (9.3.6), (9.3.7), and (9.3.8) that $\|\Phi_m(u^m(t))\|_{L^1}$ is bounded (see Step 2). Applying property (iii) of Lemma 9.3.6 and Fatou's lemma, we deduce that $u(t)\in X$ for all $t\in\mathbb{R}$ and that

$$\sup_{t\in\mathbb{R}} \|u(t)\|_X \le \infty\,.$$

Since X is reflexive, it follows that u is weakly continuous $\mathbb{R}\to X$. In particular, $u\in L^\infty(\mathbb{R},X)$ (see Remark 1.2.2(i)). Therefore, $u\in L^\infty(\mathbb{R},W)$, so that $u\in W^{1,\infty}(\mathbb{R},W^\star)$ by equation (9.3.12) and Lemma 9.3.3. In particular, equation (9.3.12) makes sense in $W^\star$ for all $t\in\mathbb{R}$. Therefore, we may take the $W-W^\star$ duality product of it with iu, and we obtain that

$$\langle u_t, u\rangle_{W^\star,W} = 0 \quad \text{for all } t\in\mathbb{R}\,,$$

which means that the function $t\mapsto \|u(t)\|_{L^2}^2$ is constant; hence there is conservation of charge. This implies that for every $t\in\mathbb{R}$, $\|u^m(t)\|_{L^2}\to\|u(t)\|_{L^2}$, and so $u^m(t)\to u(t)$ in $L^2(\Omega)$. Therefore, by boundedness of u^m in $H^1_0(\Omega)$ and Hölder's and Sobolev's inequalities, $u^m(t)\to u(t)$ in $L^q(\Omega)$ for every $2\le q<\frac{2N}{N-2}$ ($2\le q<\infty$ if $N=1,2$). We now may pass to the limit in (9.3.6). We apply the weak lower semicontinuity of the H^1 norm for the gradient term, we apply property (iii) of Lemma 9.3.6 and Fatou's lemma to the term Φ_m, and we apply Hölder's inequality to the other two terms. Taking (9.3.7) into account, we finally obtain

$$E(u(t)) \le E(\varphi) \quad \text{for all } t\in\mathbb{R}. \tag{9.3.13}$$

In conclusion, we have obtained the existence of a function $u\in L^\infty(\mathbb{R},W)\cap W^{1,\infty}(\mathbb{R},W^\star)$ that solves problem (9.3.1) and for which there is conservation of charge and the energy inequality (9.3.13).

STEP 4. Conclusion. Let us first prove uniqueness in the class $L^\infty(\mathbb{R},W)\cap W^{1,\infty}(\mathbb{R},W^\star)$. Let u and v be two solutions of (9.3.1) in that class. On taking the difference of the two equations and taking the $W-W^\star$ duality product with $i(v-u)$, we obtain that

$$\langle v_t - u_t, v-u\rangle_{W^\star,W} = -\operatorname{Im}\int_{\Omega} (v\log(|v|2) - u\log(|u|2))(\overline{v}-\overline{u})\,.$$

In view of Lemma 9.3.5, this implies that

$$\|v(t) - u(t)\|_{L^2}^2 \le 8 \int_0^t \|v(s) - u(s)\|_{L^2}^2 \, ds .$$

Uniqueness follows by Gronwall's lemma. Next, let u be a solution of (9.3.1). Considering the reverse equation and applying uniqueness (see Step 1 of the proof of Theorem 3.3.9), we deduce that u satisfies conservation of energy. Furthermore, by weak L^2 continuity and conservation of charge, $u \in C(\mathbb{R}, L^2(\Omega))$. Since u is bounded in $H_0^1(\Omega)$, it follows easily that the terms

$$\int_\Omega V|u|^2 \quad \text{and} \quad \int_\Omega B(|u|) \quad \text{are continuous } \mathbb{R} \to \mathbb{R}.$$

Thus, by conservation of energy,

$$\int_\Omega |\nabla u|^2 + \int_\Omega A(|u|) \quad \text{is continuous } \mathbb{R} \to \mathbb{R}. \tag{9.3.14}$$

Since both terms in (9.3.14) are lower semicontinuous $\mathbb{R} \to \mathbb{R}$ (the second one by Fatou's lemma), we deduce easily (see Cazenave and Haraux [63], lemma 2.4.4) that they are in fact continuous $\mathbb{R} \to \mathbb{R}$. In particular, $u \in C(\mathbb{R}, H_0^1(\Omega))$ and $u \in C(\mathbb{R}, X)$ (by Lemma 9.3.1(ii)). Therefore, $u \in C(\mathbb{R}, W)$, and by the equation and Lemma 9.3.3, $u \in C^1(\mathbb{R}, W^\star)$. Finally, one proves continuous dependence by a similar argument (compare Step 3 of the proof of Theorem 3.3.9). This completes the proof. □

REMARK 9.3.7. Strangely enough, one can apply the theory of maximal monotone operators to the equation (9.3.1). In particular, one can obtain stronger regularity if the initial value is smoother, and one can construct solutions of (9.3.1) for initial data in $L^2(\Omega)$ (see Cazenave and Haraux [63] and Haraux [157]). Note that one does not know whether the L^2 solutions are unique.

REMARK 9.3.8. At least in the case where $\Omega = \mathbb{R}^N$ and $V \equiv 0$, equation (9.3.1) has standing waves of the form $u(t,x) = e^{i\omega t}\varphi(x)$ for *every* $\omega \in \mathbb{R}$. The ground state, which is unique modulo space translations and rotations (cf. Section 8.1) is *explicitly* known. It is given by the formula

$$\varphi(x) = e^{\frac{N+\omega}{2}} e^{-\frac{|x|^2}{2}} ,$$

and it is stable in the sense of Section 8.2 (cf. Cazenave [58] and Cazenave and Lions [66]). Equation (9.3.1) has other interesting properties that are unusual with regard to Schrödinger equations when $\Omega = \mathbb{R}^N$. For example, it follows easily from conservation of energy that for every solution u of (9.3.1) (cf. Cazenave [58], proposition 4.3)

$$\inf_{t \in \mathbb{R}} \inf_{1 \le p \le \infty} \|u(t)\|_{L^p} > 0 .$$

Another interesting property is that every spherically symmetric (in space) solution of (9.3.1) has a relatively compact range in $L^2(\Omega)$ (cf. Cazenave [58], proposition 4.4).

9.4. Existence of Weak Solutions for Large Nonlinearities

Let $\Omega \subset \mathbb{R}^N$ be any open domain, and let $\eta > 0$ and $\alpha > 0$. Consider the following problem:

$$\begin{cases} iu_t + \Delta u - \eta |u|^\alpha u = 0, \\ u(0) = \varphi. \end{cases} \tag{9.4.1}$$

We already know that if $\alpha < 4/(N-2)$ ($\alpha < \infty$, if $N = 1, 2$), the problem (9.4.1) has a solution $u \in L^\infty(\mathbb{R}, H^1_0(\Omega)) \cap W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega))$ for every $\varphi \in H^1_0(\Omega)$ (see Section 3.4). In addition, if $\Omega = \mathbb{R}^N$, or if $N = 1$, or if $N = 2$ and $\alpha \le 2$, the solution is unique (see Corollary 4.3.3 and Remarks 3.5.4 and 3.6.4). However, those results do not apply when $\alpha \ge 4/(N-2)$. We present below a result of Strauss [324] (see also [321]) that applies for arbitrarily large α's. Before stating the result, we need some definitions. Let us denote by V the Banach space

$$V = H^1_0(\Omega) \cap L^{\alpha+2}(\Omega)$$

equipped with the usual norm (see Proposition 1.1.3). Since $\mathcal{D}(\Omega)$ is dense in both $H^1_0(\Omega)$ and $L^{\alpha+2}(\Omega)$,

$$V^\star = H^{-1}(\Omega) + L^{\frac{\alpha+2}{\alpha+1}}(\Omega),$$

where the Banach space $H^{-1}(\Omega) + L^{\frac{\alpha+2}{\alpha+1}}(\Omega)$ is equipped with its usual norm (see Proposition 1.1.3). Since Δ is continuous $H^1_0(\Omega) \to H^{-1}(\Omega)$ and $u \mapsto |u|^\alpha u$ is continuous $L^{\alpha+2}(\Omega) \to L^{\frac{\alpha+2}{\alpha+1}}(\Omega)$, it follows that the operator

$$\begin{cases} V \to V^\star \\ u \mapsto \Delta u - \eta |u|^\alpha u \end{cases}$$

is continuous. Therefore, if $u \in L^\infty(\mathbb{R}, V) \cap W^{1,\infty}(\mathbb{R}, V^\star)$, then equation (9.4.1) makes sense in $V^\star$. Finally, we define

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{\eta}{\alpha+2} \int_\Omega |u|^{\alpha+2} \quad \text{for all } u \in V.$$

We have the following result (see Strauss [324]).

THEOREM 9.4.1. *Let $\eta > 0$ and $\alpha > 0$. It follows that for every $\varphi \in V$, there exists a solution $u \in L^\infty(\mathbb{R}, V) \cap W^{1,\infty}(\mathbb{R}, V^\star)$ of equation* (9.4.1) *that satisfies*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2} \tag{9.4.2}$$

and

$$E(u(t)) \le E(\varphi) \tag{9.4.3}$$

for all $t \in \mathbb{R}$.

REMARK 9.4.2. Note that, in particular, $u \in C(\mathbb{R}, V^\star)$, and so u is weakly continuous $\mathbb{R} \to H^1_0(\Omega)$ and $\mathbb{R} \to L^{\alpha+2}(\Omega)$; in particular, $u(t) \in V$ for all $t \in \mathbb{R}$. Therefore, $u(0)$ makes sense (in V) and $E(u(t))$ is well defined for all $t \in \mathbb{R}$.

REMARK 9.4.3. Note that when $\alpha \le 4/(N-2)$ ($\alpha < \infty$, if $N = 1, 2$), then $H^1_0(\Omega) \hookrightarrow L^{\alpha+2}(\Omega)$, therefore, $V = H^1_0(\Omega)$.

REMARK 9.4.4. As observed before, when $\alpha < 4/(N-2)$ ($\alpha < \infty$, if $N = 1$), Theorem 9.4.1 follows from the results of Section 3.4.

Before proceeding to the proof of Theorem 9.4.1, we establish the following two lemmas.

LEMMA 9.4.5. *V and $V^\star$ are reflexive.*

PROOF. We need only show that V is reflexive. By Eberlein-Šmulian's theorem, we need to show that, given any bounded sequence $(u_m)_{m\in\mathbb{N}} \subset V$, there exist a subsequence m_k and $u \in V$ such that $u_{m_k} \rightharpoonup u$ in V as $k \to \infty$. Let $\rho = \alpha + 2$. We recall that if $u \in V$ and $\varphi \in V^\star$, then

$$\langle u, \varphi\rangle_{V,V^\star} = \langle u, \varphi_1\rangle_{H_0^1,H^{-1}} + \langle u, \varphi_2\rangle_{L^\rho,L^{\rho'}} ,$$

where $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in H^{-1}(\Omega)$ and $\varphi_2 \in L^\rho(\Omega)$ (see Bergh and Löfström [28], proof of Theorem 2.7.1). Note that there no ambiguity concerning the possible decompositions of φ, since if $\psi \in H^{-1}(\Omega) \cap L^{\rho'}(\Omega)$, then $\langle u, \psi\rangle_{H_0^1,H^{-1}} = \langle u, \psi\rangle_{L^\rho,L^{\rho'}}$.

If $(u_m)_{m\in\mathbb{N}}$ is bounded in V, then, in particular, $(u_m)_{m\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and in $L^{\alpha+2}(\Omega)$. Since both spaces are reflexive, there exists a subsequence m_k and there exist $u \in H_0^1(\Omega)$, $v \in L^{\alpha+2}(\Omega)$ such that $u_{m_k} \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_{m_k} \rightharpoonup v$ in $L^{\alpha+2}(\Omega)$. In particular, $u_{m_k} \to u$ and $u_{m_k} \to v$ in $\mathcal{D}'(\Omega)$; hence $u = v \in V$. It follows that for every $\varphi_1 \in H^{-1}(\Omega)$ and $\varphi_2 \in L^\rho(\Omega)$,

$$\langle u_{m_k}, \varphi_1\rangle_{H_0^1,H^{-1}} + \langle u_{m_k}, \varphi_2\rangle_{L^\rho,L^{\rho'}} \underset{k\to\infty}{\longrightarrow} \langle u, \varphi_1\rangle_{H_0^1,H^{-1}} + \langle u, \varphi_2\rangle_{L^\rho,L^{\rho'}} .$$

This implies that $u_{m_k} \rightharpoonup u$ in V. □

LEMMA 9.4.6. *Let $(u^m)_{m\in\mathbb{N}}$ be a bounded sequence in $L^\infty(\mathbb{R}, H_0^1(\Omega))$ and in $W^{1,\infty}(\mathbb{R}, V^\star)$. It follows that there exists a subsequence, which we still denote by $(u^m)_{m\in\mathbb{N}}$, and there exists $u \in L^\infty(\mathbb{R}, H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}, V^\star)$ such that the following properties hold:*

(i) *$u^m(t) \rightharpoonup u(t)$ in $H_0^1(\Omega)$ as $m \to \infty$ for every $t \in \mathbb{R}$.*

(ii) *For every $t \in \mathbb{R}$, there exists a subsequence m_k such that $u^{m_k}(t,x) \to u(t,x)$ as $k \to \infty$ for a.a. $x \in \Omega$.*

(iii) *$u^m(t,x) \to u(t,x)$ as $m \to \infty$ for a.a. $(t,x) \in \mathbb{R} \times \Omega$.*

PROOF. Let $k \in \mathbb{N}$ and let $\Omega_k = \Omega \cap \{x \in \Omega; |x| < k\}$ for $k \in \mathbb{N}$. Consider an integer $q > N/2$. It follows from Sobolev's embedding theorem that $H_0^q(\Omega_k) \hookrightarrow L^{\alpha+2}(\Omega_k)$, from which we obtain by duality $L^{\frac{\alpha+2}{\alpha+1}}(\Omega_k) \hookrightarrow H^{-q}(\Omega_k)$. Therefore, $u^m|_{\Omega_k}$ is bounded in $L^\infty((-k,k), H^1(\Omega_k)) \cap W^{1,\infty}((-k,k), H^{-q}(\Omega_k))$. Therefore (by Proposition 1.1.2), there exist a subsequence (which we still denote by $(u^m)_{m\in\mathbb{N}}$) and $u \in L^\infty((-k,k), H^1(\Omega_k))$ such that $u^m(t)|_{\Omega_k} \rightharpoonup u(t)$ in $H^1(\Omega_k)$. Letting $k \to \infty$ and considering a diagonal sequence, we see that there exist a subsequence (which we still denote by $(u^m)_{m\in\mathbb{N}}$) and $u \in L^\infty(\mathbb{R}, H^1(\Omega))$ such that $u^m(t)|_{\Omega_k} \rightharpoonup u(t)$ in $H^1(\Omega_k)$ for every $k \in \mathbb{N}$ and every $t \in \mathbb{R}$. This implies in particular that $u^m(t) \rightharpoonup u(t)$ in $H^1(\Omega)$. Therefore, $u \in L^\infty(\mathbb{R}, H_0^1(\Omega))$, and (i) holds. In addition, since the embedding $H^1(\Omega_k) \hookrightarrow L^2(\Omega_k)$ is compact, we have $u^m(t)|_{\Omega_k} \to u(t)|_{\Omega_k}$

in $L^2(\Omega_k)$ for every $k \in \mathbb{N}$ and every $t \in \mathbb{R}$. Applying the dominated convergence theorem, we deduce that

$$\int_{-k}^{k} \int_{\Omega_k} |u^m - u|^2 \underset{m\to\infty}{\longrightarrow} 0 \quad \text{for every } k \in \mathbb{N}.$$

In particular, there exists a subsequence m_j for which $u^{m_j} \to u$ a.e. on $(-k,k)\times\Omega_k$ as $j \to \infty$. Letting $k \to \infty$ and considering a diagonal sequence, we see that (iii) holds. Furthermore, given $t \in \mathbb{R}$ and $k \in \mathbb{N}$, there exists a subsequence m_j for which $u^{m_j}(t) \to u(t)$ a.e. on Ω_k as $j \to \infty$. Letting $k \to \infty$ and considering a diagonal sequence, we obtain (ii). Finally, it follows from (i) and Lemma 9.4.5 that $u^m(t) \rightharpoonup u(t)$ in V (hence in $V^\star$) for all $t \in \mathbb{R}$. By Theorem 1.2.4 and Remark 1.3.13(i), $u \in L^\infty(\mathbb{R}, V) \cap W^{1,\infty}(\mathbb{R}, V^\star)$. This completes the proof. □

PROOF OF THEOREM 9.4.2. We construct the solution u by a compactness method, and we proceed in three steps.

STEP 1. Construction of a sequence of approximate solutions. Given an integer $m \geq 1$, let

$$f_m(z) = \begin{cases} -\eta|z|^\alpha z & \text{if } |z| \leq m \\ -\eta m^\alpha z & \text{if } |z| \geq m. \end{cases}$$

In particular, f_m is globally Lipschitz continuous $\mathbb{C} \to \mathbb{C}$. Let

$$G_m(z) = \int_0^{|z|} f_m(s)ds\,.$$

Given $u \in H_0^1(\Omega)$, let

$$g_m(u)(x) = f_m(u(x)) \quad \text{for a.a. } x \in \Omega$$

and

$$E_m(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega G_m(u)\,.$$

Applying Corollary 3.3.11, we see that there exists a unique solution $u^m \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, H^{-1}(\Omega))$ of

$$\begin{cases} iu_t^m + \Delta u^m + g_m(u^m) = 0 \\ u^m(0) = \varphi. \end{cases} \tag{9.4.4}$$

Furthermore,

$$\|u^m(t)\|_{L^2} = \|\varphi\|_{L^2} \tag{9.4.5}$$

and

$$E_m(u^m(t)) = E_m(\varphi) \quad \text{for every } t \in \mathbb{R}. \tag{9.4.6}$$

STEP 2. Estimates of u^m. Since $G_m \geq 0$, it follows from (9.4.5) and (9.4.6) that

$$u^m \text{ is bounded in } L^\infty(\mathbb{R}, H_0^1(\Omega)) \tag{9.4.7}$$

and

$$G_m(u^m) \text{ is bounded in } L^\infty(\mathbb{R}, L^1(\Omega))\,. \tag{9.4.8}$$

On the other hand, one easily verifies that

$$|g_m(z)|^{\frac{\alpha+2}{\alpha+1}} \le (\alpha+2)G_m(z) \quad \text{for all } z \in \mathbb{C} \text{ and all } m \in \mathbb{N}.$$

Applying (9.4.8), we deduce that

$$g_m(u^m) \text{ is bounded in } L^\infty(\mathbb{R}, L^{\frac{\alpha+2}{\alpha+1}}(\Omega)). \tag{9.4.9}$$

Therefore, it follows from (9.4.4) that

$$u_t^m \text{ is bounded in } L^\infty(\mathbb{R}, V^\star). \tag{9.4.10}$$

STEP 3. Conclusion. It follows from (9.4.7) and (9.4.10) that we may apply Lemma 9.4.6 to the sequence u^m. Let u be the limit of u^m. By Lemma 9.4.6(i) and (ii), the weak lower semicontinuity of the H^1 norm and Fatou's lemma, we deduce that $u(t) \in L^{\alpha+2}(\Omega)$ for every $t \in \mathbb{R}$ and that (9.4.3) holds. In particular, $u \in L^\infty(\mathbb{R}, L^{\alpha+2}(\Omega))$, and so $u \in L^\infty(\mathbb{R}, V)$. Furthermore, it follows from property (i) that $u(0) = \varphi$. Finally, we deduce from the equation (9.4.4) that for every $\phi \in \mathcal{D}(\mathbb{R})$ and every $\psi \in \mathcal{D}(\Omega)$,

$$\int_{\mathbb{R}} \langle iu_t^m + \Delta u^m + g_m(u^m), \psi \rangle_{\mathcal{D}', \mathcal{D}} \phi(t) dt = 0.$$

This means that

$$\int_{\mathbb{R}} (-\langle iu^m, \psi\rangle \phi'(t) + \langle u^m, \Delta\psi\rangle \phi(t)) dt + \int_{\mathbb{R}} \int_{\Omega} g_m(u^m)\psi\phi \, dx\, dt = 0. \tag{9.4.11}$$

It follows easily from (9.4.7) and from property (i) of Lemma 9.4.6 that

$$\int_{\mathbb{R}} (-\langle iu^m, \psi\rangle \phi'(t) + \langle u^m, \Delta\psi\rangle \phi(t)) dt \underset{m\to\infty}{\longrightarrow} \int_{\mathbb{R}} (-\langle iu, \psi\rangle \phi'(t) + \langle u, \Delta\psi\rangle \phi(t)) dt. \tag{9.4.12}$$

Furthermore, the function $h_m(t,x) = g_m(u^m)\psi(x)\phi(t)$ has compact support. Therefore, it follows from (9.4.9) that h_m is bounded in $L^{\frac{\alpha+2}{\alpha+1}}(\mathbb{R} \times \Omega)$. By property (iii) of Lemma 9.4.6, $h_m \to -\eta|u|^\alpha u\psi\phi$ a.e. on $\mathbb{R} \times \Omega$. Since h_m has compact support, we deduce from Proposition 1.2.1 that $h_m \to -\eta|u|^\alpha u\psi\phi$ in $L^1(\mathbb{R} \times \Omega)$. Applying (9.4.11) and (9.4.12), we thus obtain

$$\int_{\mathbb{R}} (-\langle iu, \psi\rangle \phi'(t) + \langle u, \Delta\psi\rangle \phi(t)) dt - \eta \int_{\mathbb{R}} \int_{\Omega} |u|^\alpha \psi\phi \, dx\, dt = 0,$$

which implies that

$$\int_{\mathbb{R}} \langle iu_t + \Delta u - \eta|u|^\alpha u, \psi\rangle_{\mathcal{D}', \mathcal{D}} \phi(t) dt = 0.$$

Since $u \in L^\infty(\mathbb{R}, V)$, we obtain easily that $u_t \in L^\infty(\mathbb{R}, V^\star)$ and that u satisfies (9.4.1). It remains to establish conservation of charge. This follows easily by taking the $V - V^\star$ duality product of the equation with $iu_t \in V^\star$. This completes the proof. □

REMARK 9.4.7. In the case where $\alpha > 4/(N-2)$, it is not known whether the solution given by Theorem 9.4.1 is unique or not, even when $\Omega = \mathbb{R}^N$. We do not know either whether the energy is conserved.

REMARK 9.4.8. Remember that Theorem 3.3.5 applies to the case $\eta < 0$ and $\alpha < 4/(N-2)$. On the contrary, in the case $\alpha \geq 4/(N-2)$, the method of proof of Theorem 9.4.1 does not apply when $\eta < 0$. We do not know whether it is possible to construct (local) solutions of (9.4.1) in this case.

9.5. Comments

The conservation laws that we used in these notes are conservation of charge and energy, and the pseudoconformal conservation laws. They are related to the invariance of the equation for some groups of transformations. On this subject, consult Ginibre and Velo [139], Olver [285]. When $N = 1$ and $g(u) = \lambda|u|^2u$, there are infinitely many conservation laws (cf. Zakharov and Shabat [367]), while in general, there do not seem to be other useful conservation laws (cf. Serre [310]). In relation with the invariance properties of nonlinear Schrödinger equations, one can construct families of explicit solutions for some nonlinearities (cf. Fushchich and Serov [121, 122, 123]). Unfortunately, these solutions do not in general belong to the energy space.

Nonlinearities of different types than those studied here were also considered. See Baillon, Cazenave, and Figueira [9], Cazenave [57], Stubbe and Vazquez [328, 329], Adami, and Teta [2] and Adami, Dell'Antonio, Figari, and Teta [1], and Colin [85, 86].

Quasilinear Schrödinger equations require in general completely different methods for proving the existence of solutions, making an essential use of the smoothing properties of the Schrödinger group. See, for example, Biagioni and Linares [29], Chang, Shatah, and Uhlenbeck [77], Chihara [79], Colliander et al. [88, 91], Hayashi [166], Hayashi and Hirata [170], Hayashi and Kaikina [171], Hayashi, Kaikina, and Naumkin [173], Hayashi and Kato [176], Hayashi and Naumkin [180], Hayashi and Ozawa [190, 191], Katayama and Tsutsumi [201], Kenig, Ponce, and Vega [212, 215], Klainerman and Ponce [217], Ozawa and Tsutsumi [292], Takaoka [331, 333], and Y. Tsutsumi [346]. See also Lange [222] for a suggestive numerical study.

Systems of Schrödinger equations or coupled systems with other equations (Klein-Gordon, for example) are also of a great interest. See, for example, Cipolatti and Zumpichiatti [84], and Colin and Weinstein [87] (systems of Schrödinger equations); Castella [54] (Schrödinger-Poisson system); Baillon and Chadam [10], Bachelot [8], and Ozawa and Tsutsumi [290] (Schrödinger-Klein-Gordon system); Ginibre and Velo [145], Guo, Nakamitsu, and Strauss [156], Nakamitsu and Tsutsumi [254], and Y. Tsutsumi [345, 347] (Maxwell-Schrödinger system); Schochet and Weinstein [307], Lee [224], Ozawa and Tsutsumi [289, 291], Glangetas and Merle [146, 147], Kenig, Ponce, and Vega [213], Merle [247], Ginibre, Tsutsumi, and Velo [131], Bourgain [36], Bourgain and Colliander [40], Colliander and Staffilani [92], Masselin [241], Takaoka [332], and Tzvetkov [349] (Zakharov system); and Ghidaglia and Saut [124], Cipolatti [82, 83], Ozawa [288], Hayashi [167], Hayashi and Hirata [168, 169], and Ohta [279, 280, 281] (Davey-Stewartson system).

Stochastic nonlinear Schrödinger equations (i.e., with a probabilistic noise) were also considered. They display interesting phenomena, in particular concerning blowup. See de Bouard and Debussche [98, 99, 100, 101], and de Bouard, Debussche, and Di Menza [102].

Bibliography

[1] Adami, R., Dell'Antonio, G., Figari, R., and Teta, A. The Cauchy problem for the Schrödinger equation in dimension three with concentrated nonlinearity. Preprint, 2001.

[2] Adami, R., and Teta, A. A class of nonlinear Schrödinger equations with concentrated nonlinearity. *J. Funct. Anal.* 180: 148–175, 2001.

[3] Adams, R. A. *Sobolev spaces.* Academic, New York, 1975.

[4] Agmon, S., Douglis, A., and Nirenberg, L. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. *Comm. Pure Appl. Math.* 17: 35–92, 1964.

[5] Avron, J., Herbst, I., and Simon, B. Schrödinger operators with magnetic fields. I. General interactions. *Duke Math. J.* 45: 847–883, 1978.

[6] ______. Separation of center of mass in homogeneous magnetic fields. *Ann. Physics* 114: 431–451, 1978.

[7] ______. Schrödinger operators with magnetic fields. III. Atoms in homogeneous magnetic field. *Comm. Math. Phys.* 79: 529–572, 1981.

[8] Bachelot, A. Problème de Cauchy pour des systèmes hyperboliques semi-linéaires. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1: 453–478, 1984.

[9] Baillon, J.-B., Cazenave T., and Figueira, M. Equation de Schrödinger avec non-linéarité intégrale. *C. R. Acad. Sci. Paris Sér. I Math.* 284: 939–942, 1977.

[10] Baillon, J.-B., and Chadam, J. M. The Cauchy problem for the coupled Schrödinger-Klein-Gordon equations. *Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Jane.)*, 37–44. North Holland Mathematics Studies, 30. North Holland, Amsterdam–New York, 1978.

[11] Balabane, M. Décroissance des solutions d'équations de type de Schrödinger. *C. R. Acad. Sci. Paris Sér. I Math.* 292: 511–513, 1981.

[12] ______. On a regularizing effect of Schrödinger type groups. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 6: 1–14, 1989.

[13] Balabane, M., and Emammirad, H. A. Smooth distribution group and Schrödinger equation in $L^p(\mathbb{R}^N)$. *J. Math. Anal. Appl.* 70: 61–71, 1979.

[14] ______. L^p estimates for Schrödinger evolution equations. *Trans. Amer. Math. Soc.* 292: 357–373, 1985.

[15] Ball, J. M. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford Ser. (2)* 28: 473–486, 1977.

[16] ______. Finite time blow-up in nonlinear problems. *Nonlinear evolution equations (Proc. Sympos., Univ. Wisconsin, Madison, Wis., 1977)*, 189–205. M. G. Crandall, ed. Publications of the Mathematics Research Center, University of Wisconsin, 40. Academic, New York–London, 1978.

[17] Barab, J. E. Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation. *J. Math. Phys.* 25: 3270–3273, 1984.

[18] Bégout, P. Convergence to scattering states in the nonlinear Schrödinger equation. *Commun. Contemp. Math.* 3(3): 403–418, 2001.

[19] ______. Necessary conditions and sufficient conditions for global existence in the nonlinear Schrödinger equation. *Adv. Math. Sci. Appl.* 12(2): 817–827, 2002.

[20] ______. Maximum decay rate for the nonlinear Schrödinger equation. *NoDEA Nonlinear Differential Equations Appl.*, to appear.

[21] Ben-Artzi, M., and Devinatz, A. Local smoothing and convergence properties of Schrödinger type equations. *J. Funct. Anal.* 101: 231–254, 1991.

[22] Ben-Artzi, M., and Klainerman, S. Decay and regularity for the Schrödinger equation. *J. Anal. Math.* 58: 25–37, 1992.

[23] Berestycki, H., and Cazenave, T. Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires. *C. R. Acad. Sci. Paris Sér. I Math.* 293: 489–492, 1981.

[24] Berestycki, H., Gallouët, T., and Kavian, O. Équations de champs scalaires euclidiens non linéaires dans le plan. *C. R. Acad. Sci. Paris Sér. I Math.* 297: 307–310, 1983.

[25] Berestycki, H., and Lions, P.-L. Nonlinear scalar field equations. *Arch. Ration. Mech. Anal.* 82: 313–375, 1983.

[26] Berestycki, H., Lions, P.-L., and Peletier, L. A. An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$. *Indiana Univ. Math. J.* 30: 141–157, 1981.

[27] Bergé, L. Wave collapse in physics: principles and applications to light and plasma waves. *Phys. Rep.* 303: 259–370, 1998.

[28] Bergh, J., and Löfström, J. *Interpolation spaces.* Springer, New York, 1976.

[29] Biagioni, H. A., and Linares, F. Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations. *Trans. Amer. Math. Soc.* 353: 3649–3659, 2001.

[30] Białynicki-Birula, I., and Mycielski, J. Wave equations with logarithmic nonlinearities. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 23: 461–466, 1975.

[31] ______. Nonlinear wave mechanics. *Ann. Physics* 100: 62–93, 1976.

[32] Birnir, B., Kenig, C., Ponce, G., Svanstedt, N., and Vega, L. On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations. *J. London Math. Soc. (2)* 53: 551–559, 1996.

[33] Blanchard, P., Stubbe, J., and Vázquez, L. On the stability of solitary waves for classical scalar fields. *Ann. Inst. H. Poincaré Phys. Théor.* 47: 309–336, 1987.

[34] Bourgain, J. Exponential sums and nonlinear Schrödinger equations. *Geom. Funct. Anal.* 3: 157–178, 1993.

[35] ______. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. *Geom. Funct. Anal.* 3(2): 107–156, 1993.

[36] ______. On the Cauchy and invariant measure problem for the periodic Zakharov system. *Duke Math. J.* 76: 175–202, 1994.

[37] ______. Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* 166: 1–26, 1994.

[38] ______. *Global solutions of nonlinear Schrödinger equations.* American Mathematical Society Colloquium Publications, 46. American Mathematical Society, Providence, R.I., 1999.

[39] ______. Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case. *J. Amer. Math. Soc.* 12(1): no. 1, 145–171, 1999.

[40] Bourgain, J., and Colliander, J. On wellposedness of the Zakharov system. *Internat. Math. Res. Notices* 1996(11): 515–546.

[41] Bourgain, J., and Wang, W. Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity. Dedicated to Ennio De Giorgi. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 25(1-2): 197–215, 1997.

[42] Brezis, H. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Mathematics Studies, 5. Notas de Matemática, 50. North-Holland, Amsterdam–London, 1972. American Elsevier, New York, 1973.

[43] ______. *Analyse fonctionnelle.* Théorie et applications. Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1983.

[44] Brezis, H., and Cazenave, T. *Nonlinear evolution equations.* In preparation.

[45] Brezis, H., and Gallouët, T. Nonlinear Schrödinger evolution equations. *Nonlinear Anal.* 4: 677–681, 1980.

[46] Burq, N., Gérard, P., and Tzvetkov, N. An instability property of the nonlinear Schrödinger equation on S^d. *Math. Res. Lett.* 9: 323–335, 2002.

[47] ______. On nonlinear Schrödinger equations in exterior domains. Preprint, 2003.

[48] ______. An example of singular dynamics for the nonlinear Schrödinger equation on bounded domains. *Proc. Conf. Hyperbolic PDEs and related topics (Cortona, September 2002).* F. Colombini and T. Nishitani, eds. To appear.

[49] ______. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math.*, to appear.

[50] ______. Two singular dynamics of the nonlinear Schrödinger equation on a plane domain. *Geom. Funct. Anal.*, to appear.

[51] Carles, R. Semiclassical Schrödinger equations with harmonic potential and nonlinear perturbation. Preprint, 2001.

[52] ______. Critical nonlinear Schrödinger equations with and without harmonic potential. *Math. Models Methods Appl. Sci.* 12: 1513–1523, 2002.

[53] ______. Remarks on nonlinear Schrödinger equations with harmonic potential. *Ann. Henri Poincaré* 3: 757–772, 2002.

[54] Castella, F. L^2-solutions to the Schrödinger-Poisson system: existence, uniqueness, time behaviour and smoothing effect. *Math. Models Methods Appl. Sci.* 7: 1051–1083, 1997.

[55] Cazenave, T. *An introduction to nonlinear Schrödinger equations.* Textos de Métodos Matemáticos, 22. Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 1989.

[56] ______. *Blow up and scattering in the nonlinear Schrödinger equation.* Textos de Métodos Matemáticos, 30. Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 1994.

[57] ______. Equations de Schrödinger non linéaires en dimension deux. *Proc. Roy. Soc. Edinburgh Sect. A* 84: 327–346, 1979.

[58] ______. Stable solutions of the logarithmic Schrödinger equation. *Nonlinear Anal.* 7: 1127–1140, 1983.

[59] ______. Stability and instability of stationary states in nonlinear Schrödinger equations. *Contributions to nonlinear partial differential equations (Madrid, 1981)*, 123–129. Research Notes in Mathematics, 89. Pitman, Boston, 1983.

[60] ______. Some remarks on the asymptotic behaviour of solutions to nonlinear Schrödinger equations. *Trends and developments in the eighties (Bielefeld, 1982/1983)*, 116–129. World Scientific, Singapore, 1985.

[61] Cazenave, T., Dias, J.-P., and Figueira, M. A remark on the decay of the solutions of some Schrödinger equations. *Rend. Sem. Mat. Univ. Politec. Torino* 40: 129–137, 1982.

[62] Cazenave, T., and Esteban, M. J. On the stability of stationary states for nonlinear Schrödinger equations with an external magnetic field. *Mat. Apl. Comput.* 7: 155–168, 1988.

[63] Cazenave, T., and Haraux, A. Équations d'évolution avec non linéarité logarithmique. *Ann. Fac. Sci. Toulouse Math. (5)* 2(1): 21–51, 1980.

[64] ______. *Introduction aux problèmes d'évolution semi-linéaires.* Mathématiques et Applications, 1. Ellipses, Paris, 1990.

[65] ______. *An introduction to semilinear evolution equations.* Oxford Lecture Series in Mathematics and Its Applications, 13. Clarendon, Oxford University Press, New York, 1998.

[66] Cazenave, T., and Lions, P.-L. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.* 85(4): 549–561, 1982.

[67] Cazenave, T., Vega, L., and Vilela, M.-C. A note on nonlinear Schrödinger equations in weak L^p spaces. *Commun. Contemp. Math.* 3: 153–162, 2001.

[68] Cazenave, T., and Weissler, F. B. The Cauchy problem for the nonlinear Schrödinger equation in H^1. *Manuscripta Math.* 61: 477–494, 1988.

[69] ______. Some remarks on the nonlinear Schrödinger equation in the critical case. *Nonlinear semigroups, partial differential equations and attractors (Washington, DC, 1987)*, 18–29. Lecture Notes in Mathematics, 1394. Springer, Berlin, 1989.

[70] ______. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Anal.* 14: 807–836, 1990.

[71] ______. The structure of solutions to the pseudo-conformally invariant nonlinear Schrödinger equation. *Proc. Roy. Soc. Edinburgh Sect. A* 117(3-4): 251–273, 1991.

[72] ______. Rapidly decaying solutions of the nonlinear Schrödinger equation. *Comm. Math. Phys.* 147: 75–100, 1992.

[73] ______. Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations. *Math. Z.* 228: 83–120, 1998.

[74] ______. More self-similar solutions of the nonlinear Schrödinger equation. *NoDEA Nonlinear Differential Equations Appl.* 5: 355–365, 1998.

[75] ______. Scattering theory and self-similar solutions for the nonlinear Schrödinger equation. *SIAM J. Math. Anal.* 31: 625–650, 2000.

[76] Chadam, J. M., and Glassey, R. T. Global existence of solutions to the Cauchy problem for time-dependent Hartree equations. *J. Math. Phys.* 16: 1122–1130, 1975.

[77] Chang, N.-H., Shatah, J., and Uhlenbeck, K. Schrödinger maps. *Comm. Pure Appl. Math.* 53: 590–602, 2000.

[78] Chen, Y.-M. Global existence to the solution of nonlinear Schrödinger equation in exterior domains. *Acta. Math. Appl. Sinica Engl. Ser.* 2: 191–212, 1985.

[79] Chihara, H. Gain of regularity for semilinear Schrödinger equations. *Math. Ann.* 315: 529–567, 1999.

[80] Christ, M., Colliander, J., and Tao, T. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. Preprint, 2002.

[81] Cid, C., and Felmer, P. Orbital stability of standing waves for the nonlinear Schrödinger equation with potential. *Rev. Math. Phys.* 13: 1529–1546, 2001.

[82] Cipolatti, R. On the existence of standing waves for a Davey-Stewartson system. *Comm. Partial Differential Equations* 17: 967–988, 1992.

[83] ______. On the instability of ground states for a Davey-Stewartson system. *Ann. Inst. H. Poincaré Phys. Théor.* 58: 85–104, 1993.

[84] Cipolatti, R., and Zumpichiatti, W. Orbitally stable standing waves for a system of coupled nonlinear Schrödinger equations. *Nonlinear Anal.* 42: 445–461, 2000.

[85] Colin, T. On the standing wave solutions to a nonlocal, nonlinear Schrödinger equation occurring in plasma physics. *Phys. D* 64: 215–236, 1993.

[86] ______. On the Cauchy problem for a nonlocal nonlinear Schrödinger equation occurring in plasma physics. *Differential Integral Equations* 6: 1431–1450, 1993.

[87] Colin, T., and Weinstein, M. I. On the ground states of vector nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* 65: 57–79, 1996.

[88] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., and Tao, T. Global well-posedness for Schrödinger equations with derivative. *SIAM J. Math. Anal.* 33: 649–669, 2001.

[89] ______. Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation. *Math. Res. Lett.* 9: 1–24, 2002.

[90] ______. Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$. Preprint, 2003.

[91] ______. A refined global well-posedness result for Schrödinger equations with derivative. *SIAM J. Math. Anal.*, to appear.

[92] Colliander, J., and Staffilani, G. Regularity bounds on Zakharov system evolutions. *Electron. J. Differential Equations* 2002(11): 1–11.

[93] Combes, J. M., Schrader, R., and Seiler, R. Classical bounds and limits for energy distributions of Hamilton operators in electromagnetic fields. *Ann. Phys.* 111: 1–18, 1978.

[94] Constantin, P., and Saut, J.-C. Local smoothing properties of dispersive equations. *J. Amer. Math. Soc.* 1: 413–439, 1988.

[95] ______. Local smoothing properties of Schrödinger equations. *Indiana Univ. Math. J.* 38: 791–810, 1989.

[96] de Bouard A. Nonlinear Schroedinger equations with magnetic fields. *Differential Integral Equations* 4(1): 73–88, 1991.

[97] ______. Analytic solutions to nonelliptic nonlinear Schrödinger equations. *J. Differential Equations* 104: 196–213, 1993.

[98] de Bouard, A., and Debussche, A. A stochastic nonlinear Schrödinger equation with multiplicative noise. *Comm. Math. Phys.* 205: 161–181, 1999.

[99] ______. Finite-time blow-up in the additive supercritical stochastic nonlinear Schrödinger equation: the real noise case. *The legacy of the inverse scattering transform in applied mathematics (South Hadley, MA, 2001)*, 183–194. Contemporary Mathematics, 301. American Mathematical Society, Providence, R.I., 2002.

[100] ______. On the effect of a noise on the solutions of the focusing supercritical nonlinear Schrödinger equation. *Probab. Theory Related Fields* 123: 76–96, 2002.

[101] ______. The stochastic nonlinear Schrödinger equation in H^1. *Stochastic Anal. Appl.* 21(1): 97–126, 2003.

[102] de Bouard, A., Debussche, A., and Di Menza, L. Theoretical and numerical aspects of stochastic nonlinear Schrödinger equations. Monte Carlo and probabilistic methods for partial differential equations (Monte Carlo, 2000). *Monte Carlo Methods Appl.* 7(1-2): 55–63, 2001.

[103] Dias, J.-P. Time decay for the solutions of some nonlinear Schrödinger equations. *Trends and developments in the eighties (Bielefeld, 1982/1983)*, 130–138. World Scientific, Singapore, 1985.

[104] Dias, J.-P., and Figueira, M. Conservation laws and time decay for the solutions to some nonlinear Schrödinger-Hartree equations. *J. Math. Anal. Appl.* 84: 486–508, 1981.

[105] Diestel, I., and Uhl, J. J., Jr. *Vector measures.* Mathematical Surveys, 15. American Mathematical Society, Providence, R.I., 1977.

[106] Dinculeanu, N. *Vector measures*, Pergamon Press, New York, 1967.

[107] Dong, G.-C., and Li, S. J. On the initial value problem for a nonlinear Schrödinger equation. *J. Differential Equations* 42: 353–365, 1981.

[108] Dunford, C., and Schwartz, J. T. *Linear operators.* I. General Theory. Pure and Applied Mathematics, 7. Interscience, New York–London, 1958.

[109] Eboli, O. J. P., and Marques, G. C. Solitons as Newtonian particles. *Phys. Rev. B* 28: 689–696, 1983.

[110] Esteban, M. J., and Lions, P.-L. Stationary solutions of nonlinear Schrödinger equations with an external magnetic field. *Partial differential equations and the calculus of variations*, vol. 1, 401–449. Progress in Nonlinear Differential Equations and Their Applications, 1. Birkhäuser, Boston, 1989.

[111] Esteban, M. J., and Strauss, W. A. Stability and instability results for solitary waves of some nonlinear Schrödinger equations in exterior domains. *Comm. Partial Differential Equations* 19: 177–197, 1994.

[112] Feynman, R. P., and Hibbs, A. R. *Quantum mechanics and path integrals.* McGraw-Hill, New York–Toronto–London, 1965.

[113] Friedman, A., and McLeod, B. Blow-up of positive solutions of semilinear heat equations. *Indiana Univ. Math. J.* 34: 425–447, 1985.

[114] Frisch, H., Sulem, C., and Sulem, P.-L. Tracing complex singularities with spectral methods. *J. Comp. Phys.* 50: 138–161, 1983.

[115] Fujiwara, D. A construction of the fundamental solution for the Schrödinger equation. *J. Anal. Math.* 35: 41–96, 1979.

[116] ______. Remarks on convergence of the Feynman path integrals. *Duke Math. J.* 47(3): 559–600, 1980.

[117] Fukuizumi, R. Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential. *Discrete Contin. Dynam. Systems* 7: 525–544, 2000.

[118] Fukuizumi, R., and Ohta, M. Instability of standing waves for nonlinear Schrödinger equations with potentials. Preprint, 2001.
[119] Furioli, G. On the existence of self-similar solutions of the nonlinear Schrödinger equation with power nonlinearity between 1 and 2. *Differential Integral Equations* 14: 1259–1266, 2001.
[120] Furioli, G., and Terraneo, E. Besov spaces and unconditional well-posedness for the nonlinear Schrödinger equation. *Commun. Contemp. Math.*, in press.
[121] Fushchich, B. W. I., and Serov, N. I. On some exact solutions of the three-dimensional nonlinear Schrödinger equation. *J. Phys. A* 20(15): L929–L933, 1987.
[122] ______. On some exact solutions of a system of nonlinear differential equations for spinor and vector fields. *J. Phys. A* 20: 4173–4190, 1987.
[123] ______. On the reduction and some new exact solutions of the nonlinear Dirac and Dirac-Klein-Gordon equations. *J. Phys. A* 21: L5–L9, 1988.
[124] Ghidaglia, J-M., and Saut, J-C. On the initial value problem for the Davey-Stewartson systems. *Nonlinearity* 3: 475–506, 1990.
[125] Gidas, B., Ni, W. M., and Nirenberg, L. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$. *Mathematical analysis and applications*, Part A, 369–402. Advances in Mathematics Supplementary Studies, 7. Academic, New York–London, 1981.
[126] Giga, Y., and Kohn, R. V. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.* 42: 845–885, 1989.
[127] Gilbarg, D., and Trudinger, N. S. *Elliptic partial differential equations of second order.* 2nd ed. Grundlehren der Mathematischen Wissenschaften, 224. Springer, Berlin, 1983.
[128] Ginibre, J. *Introduction aux équations de Schrödinger non linéaires.* Paris Onze Edition, L161. Université Paris-Sud, Paris, 1998.
[129] Ginibre, J., and Ozawa, T. Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$. *Comm. Math. Phys.* 151(3): 619–645, 1993.
[130] Ginibre, J., Ozawa, T., and Velo, G. On the existence of the wave operators for a class of nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* 60: 211–239, 1994.
[131] Ginibre, J., Tsutsumi, Y., and Velo, G. On the Cauchy problem for the Zakharov system. *J. Funct. Anal.* 151: 384–436, 1997.
[132] Ginibre, J., and Velo, G. On a class of nonlinear Schrödinger equations, special theories in dimensions 1, 2 and 3. *Ann. Inst. Henri Poincaré* 28: 287–316, 1978.
[133] ______. On a class of nonlinear Schrödinger equations. *J. Funct. Anal.* 32: 1–71, 1979.
[134] ______. On a class of nonlinear Schrödinger equations with non local interaction. *Math. Z.* 170: 109–136, 1980.
[135] ______. On the global Cauchy problem for some nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1: 309–323, 1984.
[136] ______. The global Cauchy problem for the nonlinear Schrödinger equation revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 2: 309–327, 1985.
[137] ______. Scattering theory in the energy space for a class of nonlinear Schrödinger equations. *J. Math. Pures Appl.* 64: 363–401, 1985.

[138] ______. Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* 43: 399–442, 1985.

[139] ______. Sur une équation de Schrödinger non linéaire avec interaction non locale. *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980)*, 155–199, 391–392. Research Notes in Mathematics, 60. Pitman, Boston–London, 1982.

[140] ______. The global Cauchy problem for the nonlinear Klein-Gordon equation. *Math. Z.* 189: 487–505, 1985.

[141] ______. Long range scattering and modified wave operators for some Hartree type equations. I. *Rev. Math. Phys.* 12(3): 361–429, 2000.

[142] ______. Long range scattering and modified wave operators for some Hartree type equations. II. *Ann. Henri Poincaré* 1(4): 753–800, 2000.

[143] ______. Scattering theory in the energy space for a class of Hartree equations. *Nonlinear wave equations (Providence, RI, 1998)*, 29–60. Contemporary Mathematics, 263. American Mathematical Society, Providence, R.I., 2000.

[144] ______. Long range scattering and modified wave operators for some Hartree type equations. III. *J. Differential Equations* 175: 415–501, 2001.

[145] ______. Long range scattering and modified wave operators for the Maxwell-Schrödinger system. I. The case of vanishing asymptotic magnetic field. Preprint, 2002.

[146] Glangetas, L., and Merle, F. Existence of self-similar blow-up solutions for Zakharov equation in dimension two. I. *Comm. Math. Phys.* 160(1): 173–215, 1994.

[147] ______. Concentration properties of blow-up solutions and instability results for Zakharov equation in dimension two. II. *Comm. Math. Phys.* 160(2): 349–389, 1994.

[148] Glassey, R. T. On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. *J. Math. Phys.* 18: 1794–1797, 1977.

[149] Gogny, D., and Lions, P.-L. Hartree-Fock theory in nuclear physics. *RAIRO Modél. Math. Anal. Numér.* 20(4): 571–637, 1986.

[150] Gonçalves Ribeiro, J. M. Finite time blow-up for some nonlinear Schrödinger equations with an external magnetic field. *Nonlinear Anal.* 16(11): 941–948, 1991.

[151] ______. Instability of symmetric stationary states for some nonlinear Schrödinger equations with an external magnetic field. *Ann. Inst. H. Poincaré Phys. Théor.* 54(4): 403–433, 1991.

[152] ______. Arbitrary growth and oscillation versus finite time blow up in nonlinear Schrödinger equations. *Portugal. Math.* 49(2): 191–204, 1992.

[153] Grillakis, M. Linearized instability for nonlinear Schrödinger and Klein-Gordon equations. *Comm. Pure Appl. Math.* 41: 747–774, 1988.

[154] Grillakis, M., Shatah, J., and Strauss, W. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.* 74: 160–197, 1987.

[155] ______. Stability theory of solitary waves in the presence of symmetry. II. *J. Funct. Anal.* 94: 308–348, 1990.

[156] Guo, Y., Nakamitsu, K., and Strauss, W. A. Global finite-energy solutions of the Maxwell-Schrödinger system. *Comm. Math. Phys.* 170: 181–196, 1995.

[157] Haraux, A. *Nonlinear evolution equations—global behavior of solutions.* Lecture Notes in Mathematics, 841. Springer, Berlin–New York, 1981.

[158] ______. Linear semi-groups in Banach spaces. *Semigroups, theory and applications, Vol. II (Trieste, 1984)*, 93–135. Pitman Research Notes in Mathematics Series, 152. Longman, Harlow, 1986.

[159] Hayashi, N. Asymptotic behavior of solutions to time-dependent Hartree equations. *Arch. Ration. Mech. Anal.* 100: 191–206, 1988.

[160] ______. Time decay of solutions to the Schrödinger equation in exterior domains. *Ann. Inst. H. Poincaré Phys. Théor.* 50: 71–93, 1989.

[161] ______. Global existence of small analytic solutions to nonlinear Schrödinger equations. *Duke Math. J.* 60: 717–727, 1990.

[162] ______. Smoothing effect for nonlinear Schrödinger equations in exterior domains. *J. Funct. Anal.* 89: 444–458, 1990.

[163] ______. Smoothing effect of small analytic solutions to nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* 57: 385–394, 1992.

[164] ______. Global existence of small radially symmetric solutions to quadratic nonlinear wave equations in an exterior domain. *Manuscripta Math.* 81: 15–39, 1993.

[165] ______. Global existence of small radially symmetric solutions to quadratic nonlinear evolution equations in an exterior domain. *Math. Z.* 215: 281–319, 1994.

[166] ______. Global and almost global solutions to quadratic nonlinear Schrödinger equations with small initial data. *Dynam. Contin. Discrete Impuls. Systems* 2(1): 109–129, 1996.

[167] ______. Local existence in time of small solutions to the Davey-Stewartson system. *Ann. Inst. H. Poincaré Phys. Théor.* 65: 313–366, 1996.

[168] Hayashi, N., and Hirata, H. Global existence and asymptotic behavior in time of small solutions to the elliptic-hyperbolic Davey-Stewartson system. 4th Symposium on Mathematical Analysis and Its Applications (Aran-delovac, 1997). *Mat. Vesnik* 49(3-4): 273–278, 1997.

[169] ______. Local existence in time of small solutions to the elliptic-hyperbolic Davey-Stewartson system in the usual Sobolev space. *Proc. Roy. Soc. Edinburgh Sect. A* 40: 563–581, 1997.

[170] ______. Global existence of small solutions to nonlinear Schrödinger equations. *Nonlinear Anal.* 31(5-6): 671–685, 1998.

[171] Hayashi, N., and Kaikina, E. I. Local existence of solutions to the Cauchy problem for nonlinear Schrödinger equations. *SUT J. Math.* 34(2): 111–137, 1998.

[172] Hayashi, N., Kaikina, E. I., and Naumkin, P. E. On the scattering theory for the cubic nonlinear Schrödinger and Hartree type equations in one space dimension. *Hokkaido Math. J.* 27: 651–667, 1998.

[173] ______. Large time behavior of solutions to the generalized derivative nonlinear Schrödinger equation. *Discrete Contin. Dynam. Systems* 5: 93–106, 1999.

[174] Hayashi, N., and Kato, K. Regularity in time of solutions to nonlinear Schrödinger equations. *J. Funct. Anal.* 128: 253–277, 1995.

[175] ______. Analyticity in time and smoothing effect of solutions to nonlinear Schrödinger equations. *Comm. Math. Phys.* 184: 273–300, 1997.

[176] ______. Global existence of small solutions to quadratic nonlinear Schrödinger equations. *Comm. Partial Differential Equations* 22: 773–798, 1997.

[177] Hayashi, N., Nakamitsu, K., and Tsutsumi, M. On solutions of the initial value problem for the nonlinear Schrödinger equations in one space dimension. *Math. Z.* 192: 637–650, 1986.

[178] ______. On solutions of the initial value problem for the nonlinear Schrödinger equations. *J. Funct. Anal.* 71: 218–245, 1987.

[179] ______. Nonlinear Schrödinger equations in weighted Sobolev spaces. *Funkcial. Ekvac.* 31: 363–381, 1988.

[180] Hayashi, N., and Naumkin, P. I. Asymptotic behavior in time of solutions to the derivative nonlinear Schrödinger equation. *Ann. Inst. H. Poincaré Phys. Théor.* 68: 159–177, 1998.

[181] ______. Asymptotic for large time of solutions to the nonlinear Schrödinger and Hartree equations. *Amer. J. Math.* 120: 369–389, 1998.

[182] ______. Remarks on scattering theory and large time asymptotics of solutions to Hartree type equations with a long range potential. *SUT J. Math.* 34: 13–24, 1998.

[183] ______. On the quadratic nonlinear Schrödinger equation in three space dimensions. *Internat. Math. Res. Notices* 2000(3): 115–132.

[184] Hayashi, N., Naumkin, P. I., and Ozawa, T. Scattering theory for the Hartree equation. *SIAM J. Math. Anal.* 29: 1256–1267, 1998.

[185] Hayashi, N., and Ozawa, T. Time decay of solutions to the Cauchy problem for time-dependent Schrödinger-Hartree equations. *Comm. Math. Phys.* 110: 467–478, 1987.

[186] ______. Scattering theory in the weighted $L^2(\mathbb{R}^n)$ spaces for some Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* 48(1): 17–37, 1988.

[187] ______. Lower bounds for order of decay or of growth in time for solutions to linear and nonlinear Schrödinger equations. *Publ. Res. Inst. Math. Sci.* 25(6): 847–859, 1989.

[188] ______. Smoothing effect for some Schrödinger equations. *J. Funct. Anal.* 85: 307–348, 1989.

[189] ______. Time decay for some Schrödinger equations. *J. Funct. Anal.* 85: 307–348, 1989.

[190] ______. On the derivative nonlinear Schrödinger equation. *Phys. D* 55: 14–36, 1992.

[191] ______. Remarks on nonlinear Schrödinger equations in one space dimension. *Differential Integral Equations* 7: 453–461, 1994.

[192] Hayashi, N., and Saitoh, S. Analyticity and global existence of small solutions to some nonlinear Schrödinger equations. *Comm. Math. Phys.* 129(1): 27–41, 1990.

[193] ______. Analyticity and smoothing effect for the Schrödinger equation. *Ann. Inst. H. Poincaré Phys. Théor.* 52(2): 163–173, 1990.

[194] Hayashi, N., and Tsutsumi, Y. Scattering theory for Hartree type equations. *Ann. Inst. H. Poincaré Phys. Théor.* 46: 187–213, 1987.

[195] Herrero, M. A., and Velázquez, J. J. L. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 10: 131–189, 1993.

[196] ______. Explosion de solutions d'équations paraboliques semi-linéaires supercritiques. *C. R. Acad. Sci. Paris Sér. I Math.* 319: 141–145, 1994.

[197] Jensen, A., and Kato, T. Spectral properties of Schrödinger operators and time decay of wave functions. *Duke Math. J.* 46: 583–611, 1979.

[198] Jones, C. K. R. T. An instability mechanism for radially symmetric standing waves of a nonlinear Schrödinger equation. *J. Differential Equations* 71: 34–62, 1988.

[199] Jones, C., and Küpper, T. On the infinitely many solutions of a semilinear elliptic equation. *SIAM J. Math. Anal.* 17: 803–836, 1986.

[200] Journé, J.-L., Soffer, A., and Sogge, C. Decay estimates for Schrödinger equations. *Comm. Pure Appl. Math.* 44: 572–604, 1991.

[201] Katayama, S., and Tsutsumi, Y. Global existence of solutions for nonlinear Schrödinger equations in one space dimension. *Comm. Partial Differential Equations* 19: 1971–1997, 1994.

[202] Kato, T. *Perturbation theory for linear operators.* Die Grundlehren der mathematischen Wissenschaften, 132. Springer, New York 1966.

[203] ______. On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* 46: 113–129, 1987.

[204] ______. Nonlinear Schrödinger equations. *Schrödinger operators (Sønderborg, 1988)*, 218–263. Lecture Notes in Physics, 345. Springer, Berlin, 1989.

[205] ______. An $L^{q,r}$-theory for nonlinear Schrödinger equations. *Spectral and scattering theory and applications*, 223–238. Advanced Studies in Pure Mathematics, 23. Mathematical Society of Japan, Tokyo, 1994.

[206] ______. On nonlinear Schrödinger equations. II. H^s-solutions and unconditional well-posedness. *J. Anal. Math.* 67: 281–306, 1995.

[207] Kato, T., and Yajima, K. Some examples of smooth operators and the associated smoothing effect. *Rev. Math. Phys.* 1: 481–496, 1989.

[208] Kavian, O. A remark on the blowing-up of solutions to the Cauchy problem for nonlinear Schrödinger equations. *Trans. Amer. Math. Soc.* 299: 193–205, 1987.

[209] Kavian, O., and Weissler, F. B. Self-similar solutions of the pseudo-conformally invariant nonlinear Schrödinger equation. *Michigan Math. J.* 41: 151–173, 1994.

[210] Keel, M., and Tao, T. Endpoint Strichartz inequalities. *Amer. J. Math.* 120: 955–980, 1998.

[211] Kenig, C., Ponce, G., and Vega, L. Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.* 40: 33–69, 1991.

[212] ______. Small solutions to nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 10: 255–288, 1993.

[213] ______. On the Zakharov and Zakharov-Schulman systems. *J. Funct. Anal.* 127: 204–234, 1995.

[214] ______. Quadratic forms for the 1-D semilinear Schrödinger equation. *Trans. Amer. Math. Soc.* 348: 3323–3353, 1996.

[215] ______. Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations. *Invent. Math.* 134: 489–545, 1998.

[216] ______. On the ill-posedness of some canonical dispersive equations. *Duke Math. J.* 106: 617–633, 2001.

[217] Klainerman, S., and Ponce, G. Global, small amplitude solutions to non-linear evolution equations. *Comm. Pure Appl. Math.* 36: 133–141, 1983.

[218] Kranosel′skiĭ, M. A., and Rutickiĭ, Y. B. *Convex functions and Orlicz spaces.* Noordhoff, Groningen, 1961.

[219] Kwong, M. K. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. *Arch. Ration. Mech. Anal.* 105(3): 243–266, 1989.

[220] Lange, H. On some nonlinear Schrödinger equations of Hartree type. *Methods and techniques of mathematical physics (Oberwolfach, 1980)*, 99–111. Methoden und Verfahren der Mathematischen Physik, 21. Lang, Frankfurt, 1981.

[221] ______. On nonlinear Schrödinger equations in the theory of quantum mechanics dissipative systems. *Nonlinear Anal.* 9: 1115–1133, 1985.

[222] ______. Asymptotical behaviour of solutions to quasilinear Schrödinger equations. *Differential equations (Xanthi, 1987)*, 411–421. Lecture Notes in Pure and Applied Mathematics, 118. Dekker, New York, 1989.

[223] Lebowitz, J. L., Rose, H. A., and Speer, E. R. Statistical mechanics of the nonlinear Schrödinger equation. *J. Statist. Phys.* 50(3-4): 657–687, 1988.

[224] Lee, J.-H. Global solvability of the derivative nonlinear Schrödinger equation. *Trans. Amer. Math. Soc.* 314(1): 107–118, 1989.

[225] Lemarié-Rieusset, P.-G. *Recent developments in the Navier-Stokes problem.* Chapman and Hall/CRC Research Notes in Mathematics, 431. Chapman and Hall/CRC, Boca Raton, Fla., 2002.

[226] LeMesurier, B., Papanicolaou, G., Sulem, C., and Sulem, P.-L. The focusing singularity of the nonlinear Schrödinger equation. *Directions in partial differential equations (Madison, WI, 1985)*, 159–201. Publications of the Mathematics Research Center, University of Wisconsin, 54. Academic, Boston, 1987.

[227] ______. Focusing and multi-focusing solutions of the nonlinear Schrödinger equation. *Phys. D* 31(1): 78–102, 1988.

[228] ______. Local structure of the self-focusing singularity of the nonlinear Schrödinger equation. *Phys. D* 32(2): 210–266, 1988.

[229] Lieb, E. H., and Simon, B. The Hartree-Fock theory for Coulomb systems. *Comm. Math. Phys.* 53(3): 185–194, 1977.

[230] Lin, J. E., and Strauss, W. A. Decay and scattering of solutions of a nonlinear Schrödinger equation. *J. Funct. Anal.* 30: 245–263, 1978.

[231] Lions, J.-L. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod; Gauthiers-Villars, Paris, 1969.

[232] Lions, J.-L., and Magenes, E. *Problèmes aux limites non homogènes et applications.* Dunod; Gauthiers-Villars, Paris, 1968.

[233] Lions, P.-L. The Choquard equation and related questions. *Nonlinear Anal.* 4: 1063–1073, 1980.

[234] ______. Some remarks on Hartree equation. *Nonlinear Anal.* 5: 1245–1256, 1981.

[235] ______. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1: 109–145, 1984.

[236] ______. The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1: 223–283, 1984.

[237] Marcus, M., and Mizel, V. J. Nemitsky operators on Sobolev spaces. *Arch. Ration. Mech. Anal.* 51: 347–370, 1973.

[238] ______. Complete characterization of functions which act, via superposition, on Sobolev spaces. *Trans. Amer. Math. Soc.* 251: 187–218, 1979.

[239] ______. Every superposition operator mapping one Sobolev space into another is continuous. *J. Funct. Anal.* 33: 217–229, 1979.

[240] Martel, Y. Blow-up for the nonlinear Schrödinger equation in nonisotropic spaces. *Nonlinear Anal.* 28(12): 1903–1908, 1997.

[241] Masselin, V. A result on the blow-up rate for the Zakharov system in dimension 3. *SIAM J. Math. Anal.* 33: 440–447, 2001.

[242] McLaughlin, D., Papanicolaou, G. C., Sulem, C., and Sulem, P.-L. The focusing singularity of the cubic Schrödinger equation. *Phys. Rev. A* 34(2): 1200–1210, 1986.

[243] Merle, F. Limit of the solution of the nonlinear Schrödinger equation at the blow-up time. *J. Funct. Anal.* 84: 201–214, 1989.

[244] ______. Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity. *Comm. Math. Phys.* 129: 223–240, 1990.

[245] ______. On uniqueness and continuation properties after blowup time of self-similar solutions of nonlinear Schrödinger equations with critical exponent and critical mass. *Comm. Pure Appl. Math.* 45: 203–254, 1992.

[246] ______. Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. *Duke Math. J.* 69: 427–454, 1993.

[247] ______. Lower bounds for the blowup rate of solutions of the Zakharov equation in dimension two. *Comm. Pure Appl. Math.* 49: 765–794, 1996.

[248] ______. Personal communication.

[249] Merle, F., and Raphael, P. Blow up dynamic and upper bound on the blow up rate for critical nonlinear Schrödinger equation. Preprint, 2003.

[250] ______. Sharp upper bound on the blow up rate for critical nonlinear Schrödinger equation. *Geom. Funct. Anal.*, to appear.

[251] Merle, F., and Tsutsumi, Y. L^2-concentration of blow-up solutions for the nonlinear Schrödinger equation with the critical power nonlinearity. *J. Differential Equations* 84: 205–214, 1990.

[252] Montgomery-Smith, S. J. Time decay for the bounded mean oscillation of solutions of the Schrödinger and wave equations. *Duke Math. J.* 91(2): 393–408, 1998.

[253] Murata, M. Rate of decay of local energy and spectral properties of elliptic operators. *Japan J. Math.* 6: 77–127, 1980.

[254] Nakamitsu, K., and Tsutsumi, M. The Cauchy problem for the coupled Maxwell-Schrödinger equations. *J. Math. Phys.* 27: 211–216, 1986.

[255] Nakamura M., and Ozawa, T. Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces. *Rev. Math. Phys.* 9: 397–410, 1997.

[256] ______. Nonlinear Schrödinger equations in the Sobolev space of critical order. *J. Funct. Anal.* 155: 363–380, 1998.

[257] ______. Small solutions to nonlinear Schrödinger equations in the Sobolev spaces. *J. Anal. Math.* 81: 305–329, 2000.

[258] Nakanishi, K. Energy scattering for Hartree equations. *Math. Res. Lett.* 6(1): 107–118, 1999.

[259] ______. Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2. *J. Funct. Anal.* 169: 201–225, 1999.

[260] ______. Asymptotically-free solutions for the short-range nonlinear Schrödinger equation. *SIAM J. Math. Anal.* 32(6): 1265–1271, 2001.

[261] ______. Modified wave operators for the Hartree equation with data, image and convergence in the same space. II. *Ann. Henri Poincaré* 3(3): 503–535, 2002.

[262] ______. Modified wave operators for the Hartree equation with data, image and convergence in the same space. *Commun. Pure Appl. Anal.* 1(2): 237–252, 2002.

[263] Nakanishi, K., and Ozawa, T. Remarks on scattering for nonlinear Schrödinger equations. *NoDEA Nonlinear Differential Equations Appl.* 9(1): 45–68, 2002.

[264] Nawa, H. "Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity. II. *Kodai Math. J.* 13(3): 333–348, 1990.

[265] ______. Formation of singularities on solutions of the nonlinear Schrödinger equation. *Proc. Japan Acad. Ser. A Math. Sci.* 67: 29–34, 1991.

[266] ______. "Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity. *Funkcial. Ekvac.* 35(1): 1–18, 1992.

[267] ______. Asymptotic profiles of blow-up solutions of the nonlinear Schrödinger equation with critical power nonlinearity. *J. Math. Soc. Japan* 46(4): 557–586, 1994.

[268] ______. Limiting profiles of blow up solutions of the nonlinear Schrödinger equation with critical power nonlinearity. *Proc. Japan Acad. Ser. A Math. Sci.* 73: 171–175, 1997.

[269] ______. Two points blow up in solutions of the nonlinear Schrödinger equation with quartic potential on $\mathbb{R}$. *J. Statist. Phys.* 91: 439–458, 1998.

[270] ______. Asymptotic and limiting profiles of blowup solutions of the nonlinear Schrödinger equation with critical power. *Comm. Pure Appl. Math.* 52: 193–270, 1999.

[271] Nawa, H., and Ozawa, T. Nonlinear scattering with nonlocal interaction. *Comm. Math. Phys.* 146: 259–275, 1992.

[272] Nirenberg, L. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 13: 115–162, 1959.

[273] Ogawa, T. A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations. *Nonlinear Anal.* 14(9): 765–769, 1990.

[274] Ogawa, T., and Ozawa, T. Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem. *J. Math. Anal. Appl.* 155(2): 531–540, 1991.

[275] Ogawa, T., and Tsutsumi, Y. Blow-up of H^1 solutions for the nonlinear Schrödinger equation. *J. Differential Equations* 92: 317–330, 1991.

[276] ______. Blow-up of H^1 solutions for the one dimensional nonlinear Schrödinger equation with critical power nonlinearity. *Proc. Amer. Math. Soc.* 111: 487–496, 1991.

[277] Oh, Y.-G. Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$. *Comm. Partial Differential Equations* 13: 1499–1519, 1988.

[278] ______. Lyapunov stability of semiclassical bound states of nonlinear Schrödinger equations with potentials. *Comm. Math. Phys.* 121: 11–33, 1989.

[279] Ohta, M. Stability of standing waves for the generalized Davey-Stewartson system. *J. Dynam. Differential Equations* 6(2): 325–334, 1994.

[280] ______. Instability of standing waves for the generalized Davey-Stewartson system. *Ann. Inst. H. Poincaré Phys. Théor.* 62(1): 69–80, 1995.

[281] ______. Blow-up solutions and strong instability of standing waves for the generalized Davey-Stewartson system in $\mathbb{R}^2$. *Ann. Inst. H. Poincaré Phys. Théor.* 63(1): 111–117, 1995.

[282] ______. Stability and instability of standing waves for one-dimensional nonlinear Schrödinger equations with double power nonlinearity. *Kodai Math. J.* 18(1): 68–74, 1995.

[283] ______. Stability of solitary waves for coupled nonlinear Schrödinger equations. *Nonlinear Anal.* 26: 933–939, 1996.

[284] ______. Stability of solitary waves for the coupled Klein-Gordon-Schrödinger equations. *Nonlinear Anal.* 27: 455–461, 1996.

[285] Olver, P. J. *Applications of Lie groups to differential equations.* Graduate Texts in Mathematics, 107. Springer, New York, 1986.

[286] Oru, F. Rôle des oscillations dans quelques problèmes d'analyse non-linéaire. Doctoral dissertation, École Normale Supérieure de Cachan, 1998.

[287] Ozawa, T. Long range scattering for the nonlinear Schrödinger equations in one space dimension. *Comm. Math. Phys.* 139(3): 479–493, 1991.

[288] ______. Exact blowup solutions to the Cauchy problem for the Davey-Stewartson systems. *Proc. Roy. Soc. London Ser. A* 436(1897): 345–349, 1992.

[289] Ozawa, T., and Tsutsumi, Y. Existence and smoothing effect of solutions for the Zakharov equations. *Publ. Res. Inst. Math. Sci.* 28(3): 329–361, 1992.

[290] ______. Asymptotic behavior of solutions for the coupled Klein-Gordon-Schrödinger equations. *Spectral and scattering theory and applications*, 295–305. Advanced Studies in Pure Mathematics, 23. Mathematical Society of Japan, Tokyo, 1994.

[291] ______. Global existence and asymptotic behavior of solutions for the Zakharov equations in three space dimensions. *Adv. Math. Sci. Appl.* 3: 301–334, 1993/94.

[292] ______. Space-time estimates for null gauge forms and nonlinear Schrödinger equations. *Differential Integral Equations* 11(2): 201–222, 1998.

[293] Patera, A., Sulem, C., and Sulem, P.-L. Numerical simulation of singular solutions to the two-dimensional cubic Schrödinger equation. *Comm. Pure Appl. Math.* 37(6): 755–778, 1984.

[294] Pazy, A. *Semi-groups of linear operators and applications to partial differential equations.* Springer, Berlin, 1983.

[295] Pecher, H. Solutions of semilinear Schrödinger equations in H^s. *Ann. Inst. H. Poincaré Phys. Théor.* 67: 259–296, 1997.

[296] Pecher, H., and Von Wahl, W. Time dependent nonlinear Schrödinger equations. *Manuscripta Math.* 27: 125–157, 1979.

[297] Perelman, G. On the formation of singularities in solutions of the critical nonlinear Schrödinger equation. *Ann. Henri Poincaré* 2(4): 605–673, 2001.

[298] Planchon, F. On the Cauchy problem in Besov spaces for a nonlinear Schrödinger equation. *Commun. Contemp. Math.* 2: 243–254, 2000.

[299] ______. Dispersive estimates and the 2D cubic NLS equation. *J. Anal. Math.* 86: 319–334, 2002.

[300] Rauch, J. Local decay of scattering solutions to Schrödinger equations. *Comm. Math. Phys.* 61: 149–168, 1978.

[301] Reed, M., and Simon, B. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness.* Academic, New York–London, 1975.

[302] Ribaud, F., and Youssfi, A. Regular and self-similar solutions of nonlinear Schrödinger equations. *J. Math. Pures Appl.* 77: 1065–1079, 1998.

[303] Rose, H. A., and Weinstein, M. I. On the bound states of the nonlinear Schrödinger equation with a linear potential. *Phys. D* 30: 207–218, 1988.

[304] Rudin, W. *Functional analysis.* 2nd ed. McGraw-Hill, New York, 1991.

[305] ______. *Real and complex analysis.* McGraw-Hill, New York–Toronto–London, 1966.

[306] Ruiz, A., and Vega, L. On local regularity of Schrödinger equations. *Internat. Math. Res. Notices* 1993(1): 13–27.

[307] Schochet, H., and Weinstein, M. I. The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence. *Comm. Math. Phys.* 106: 569–580, 1986.

[308] Schonbek, T. Decay of solutions of Schroedinger equations. *Duke Math. J.* 46(1): 203–213, 1979.

[309] Segal, I. E. Non-linear semi-groups. *Ann. of Math. (2)* 78: 339–364, 1963.

[310] Serre, D. A propos des invariants de diverses équations d'évolution. *Nonlinear partial differential equations and their applications*, 326–336. Collège de France Seminar. Vol. III. H. Brezis and J.-L. Lions, eds. Research Notes in Mathematics, 70, Pitman, Boston–London, 1982.

[311] Shatah, J., and Strauss, W. A. Instability of nonlinear bound states. *Comm. Math. Phys.* 100: 173–190, 1985.

[312] Shatah, J., and Struwe, M. *Geometric wave equations.* Courant Lecture Notes in Mathematics, 2. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, R.I., 1998.

[313] Simon, B. *Functional integration and quantum physics.* Academic, New York, 1979.

[314] Simon, J. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)* 146: 65–96, 1987.

[315] Sjölin, P. Regularity of solutions to Schrödinger equations. *Duke Math. J.* 55: 699–715, 1987.

[316] Soffer, A., and Weinstein, M. I. Multichannel nonlinear scattering theory for nonintegrable equations. *Comm. Math. Phys.* 133: 119–146, 1990.

[317] ______. Multichannel nonlinear scattering theory for nonintegrable equations. II. The case of anisotropic potentials and data. *J. Differential Equations* 98: 376–390, 1992.

[318] Staffilani, G. Quadratic forms for a 2-D semilinear Schrödinger equation. *Duke Math. J.* 86: 79–107, 1997.

[319] Stein, E. M. *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton, N.J., 1970.

[320] Strauss, W. A. On the continuity of functions with values in various Banach spaces. *Pacific J. Math.* 19: 543–551, 1966.

[321] ______. On weak solutions of semi-linear hyperbolic equations. *An. Acad. Brasil. Ci.* 42: 645–651, 1970.

[322] ______. Nonlinear scattering theory. *Scattering theory in mathematical physics*, 53–78. J. A. Lavita and J.-P. Marchand, eds. Reidel, Dordrecht, Holland–Boston, 1974.

[323] ______. Existence of solitary waves in higher dimensions. *Comm. Math. Phys.* 55(2): 149–162, 1977.

[324] ______. *Nonlinear invariant wave equations. Invariant wave equations (Proc. "Ettore Majorana" Internat. School of Math. Phys., Erice, 1977)*, 197–249. Lecture Notes in Physics, 73. Springer, Berlin–New York, 1978.

[325] ______. Nonlinear scattering theory at low energy. *J. Funct. Anal.* 41: 110–133, 1981.

[326] ______. *Nonlinear wave equations.* CBMS Regional Conference Series in Mathematics, 73. American Mathematical Society, Providence, R.I., 1989.

[327] Strichartz, M. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.* 44: 705–714, 1977.

[328] Stubbe, J., and Vázquez, L. Nonlinear Schrödinger equations with magnetic field effect: existence and stability of solitary waves. *Integrable systems and applications (Île d'Oléron, 1988)*, 336–342. Lecture Notes in Physics, 342. Springer, Berlin, 1989.

[329] ______. A nonlinear Schrödinger equation with magnetic field effect: solitary waves with internal rotation. *Portugal. Math.* 46: 493–499, 1989.

[330] Sulem, C., and Sulem, P.-L. *The nonlinear Schrödinger equation. Self-focusing and wave collapse.* Applied Mathematical Sciences, 139. Springer, New York, 1999.

[331] Takaoka, H. Well-posedness for the one dimensional nonlinear Schrödinger equation with the derivative nonlinearity. *Adv. Differential Equations* 4: 561–580, 1999.

[332] ______. Well-posedness for the Zakharov system with the periodic boundary condition. *Differential Integral Equations* 12: 789–810, 1999.

[333] ______. Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces. *Electron. J. Differential Equations* 2001(42): 1–23.

[334] Tao, T. Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation. *Comm. Partial Differential Equations* 25: 1471–1485, 2000.

[335] ______. Multilinear weighted convolution of L^2 functions, and applications to nonlinear dispersive equations. *Amer. J. Math.* 123: 839–908, 2001.

[336] Tourigny, Y., and Sanz-Serna, J. M. The numerical study of blowup with application to a nonlinear Schrödinger equation. *J. Comp. Phys.* 102: 407–416, 1992.

[337] Triebel, H. *Interpolation theory, function spaces, differential operators.* North Holland Mathematical Library, 18. North Holland, Amsterdam–New York, 1978.

[338] ______. *Theory of function spaces.* Monographs in Mathematics, 78. Birkhäuser, Basel, 1983.

[339] Tsutsumi, M. On global solutions to the initial-boundary value problem for the nonlinear Schrödinger equation in exterior domains. *Comm. Partial Differential Equations* 16(6-7): 885–907, 1991.

[340] Tsutsumi, Y. Global solutions of the nonlinear Schrödinger equation in exterior domains. *Comm. Partial Differential Equations* 8: 1337–1374, 1984.

[341] ______. Scattering problem for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* 43: 321–347, 1985.

[342] ______. Global strong solutions for nonlinear Schrödinger equations. *Nonlinear Anal.* 11: 1143–1154, 1987.

[343] ______. L^2-solutions for nonlinear Schrödinger equations and nonlinear groups. *Funkcial. Ekvac.* 30: 115–125, 1987.

[344] ______. Rate of L^2-concentration of blow-up solutions for the nonlinear Schrödinger equation with the critical nonlinearity. *Nonlinear Anal.* 15: 719–724, 1990.

[345] ______. Global existence and asymptotic behavior of solutions for the Maxwell-Schrödinger equations in three space dimensions. *Comm. Math. Phys.* 151: 543–576, 1993.

[346] ______. The null gauge condition and the one-dimensional nonlinear Schrödinger equation with cubic nonlinearity. *Indiana Univ. Math. J.* 43(1): 241–254, 1994.

[347] ______. Global existence and uniqueness of energy solutions for the Maxwell-Schrödinger equations in one space dimension. *Hokkaido Math. J.* 24: 617–639, 1995.

[348] Tsutsumi, Y., and Yajima, K. The asymptotic behavior of nonlinear Schrödinger equations. *Bull. Amer. Math. Soc.* 11: 186–188, 1984.

[349] Tzvetkov, N. Low regularity solutions for a generalized Zakharov system. *Differential Integral Equations* 13: 423–440, 2000.

[350] Vargas, A., and Vega, L. Global wellposedness for 1-D nonlinear Schrödinger equation for data with an infinite L^2 norm. *J. Math. Pures Appl.* 80: 1029–1044, 2001.

[351] Vega, L. Schrödinger equations: pointwise convergence to the initial data. *Proc. Amer. Math. Soc.* 102: 874–878, 1988.

[352] Velázquez, J. J. L. Classification of singularities for blowing up solutions in higher dimensions. *Trans. Amer. Math. Soc.* 338: 441–464, 1993.

[353] Vilela, M. C. Las estimaciones de Strichartz bilineales en el contexto de la ecuación de Schrödinger. Doctoral dissertation, Universidad del País Vasco, Bilbao, 2003.

[354] Vladimirov, M. V. On the solvability of mixed problem for a nonlinear equation of Schrödinger type. *Sov. Math. Dokl.* 29: 281–284, 1984.

[355] Weinstein, A. A symbol class for Schrödinger equations on $\mathbb{R}^n$. *Amer. J. Math.* 107: 1–23, 1985.

[356] Weinstein, M. I. Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.* 87: 567–576, 1983.

[357] ______. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.* 16: 567–576, 1985.

[358] ______. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.* 29: 51–68, 1986.

[359] ______. On the structure and formation of singularities of solutions to nonlinear dispersive evolution equations. *Comm. Partial Differential Equations* 11: 545–565, 1986.

[360] ______. The nonlinear Schrödinger equation–Singularity formation, stability and dispersion. *The connection between infinite-dimensional and finite-dimensional systems (Boulder, CO, 1987)*, 213–232. Contemporary Mathematics, 99. American Mathematical Society, Providence, R.I., 1989.

[361] Weissler, F. B. An L^∞ blowup estimate for a nonlinear heat equation. *Comm. Pure Appl. Math.* 38: 291–296, 1985.

[362] ______. L^p-energy and blow-up for a semilinear heat equation. *Nonlinear functional analysis and its applications, Part 2 (Berkeley, Calif., 1983)*, 545–551. Proceedings of Symposia in Pure Mathematics, 45. American Mathematical Society, Providence, R.I., 1986.

[363] ______. Asymptotically self-similar solutions of the two-power nonlinear Schrödinger equation. *Adv. Differential Equations* 6: 419–440, 2001.

[364] Yajima, K. Existence of solutions for Schrödinger evolution equations. *Comm. Math. Phys.* 110: 415–426, 1987.

[365] Yao, J.-Q. Comportement à l'infini des solutions d'une équation de Schrödinger non linéaire dans un domaine extérieur. *C. R. Acad. Sci. Paris Sér. I Math.* 294: 163–166, 1982.

[366] Yosida, K. *Functional analysis.* Springer, New York, 1965.

[367] Zakharov, V. E., and Shabat, A. B. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP* 34(1): 62–69, 1972; translated from *Ž. Èksper. Teoret. Fiz.* 61(1) 118–134, 1971.

[368] Zelditch, S. Reconstruction of singularities for solutions of Schrödinger's equation. *Comm. Math. Phys.* 90(1): 1–26, 1983.

[369] Zhang, J. Sharp criteria for blowup and global existence in nonlinear Schrödinger equations under harmonic potential. Preprint, 1999.

[370] ______. Stability of standing waves for nonlinear Schrödinger equations with unbounded potentials. *Z. Angew. Math. Phys.* 51: 498–503, 2000.