LECTURE NOTES I: ON LOCAL AND GLOBAL THEORY FOR NONLINEAR SCHRÖDINGER EQUATION

NATAŠA PAVLOVIĆ AND NIKOLAOS TZIRAKIS

Abstract. The notes serve as an introduction to the analysis of dispersive partial differential equations. They are organized as follows:

- Part I focuses on basic theory for local and global analysis of the semilinear Schrödinger equation.
- Part II concentrates on basic local and global theory for the Korteveg de Vries equation.
- Part III gives a review of some recents results on a derivation of nonlinear dispersive equations from quantum many body systems.

Dislaimer. The notes are prepared as a study tool for participants of the MSRI summer school "Dispersive Partial Differential Equations", June 16-27, 2014. We tried to include many of the relevant references. However it is inevitable that we had to make sacrifices in the choice of the material that is included in the notes. As a consequence, there are many important works that we could not present in the notes.

1. What is a dispersive PDE

Informally speaking, a partial differential equation (PDE) is characterized as dispersive if, when no boundary conditions are imposed, its wave solutions spread out in space as they evolve in time. As an example consider the linear homogeneous Schrödinger equation on the real line

$$iu_t + u_{xx} = 0, \tag{1.1}$$

for a complex valued function $u = u(x,t)$ with $x \in \mathbb{R}$ and $t \in \mathbb{R}$. If we try to find a solution in the form of a simple wave

$$u(x,t) = Ae^{i(kx-\omega t)},$$

we see that it satisfies the equation if and only if

$$\omega = k^2. \tag{1.2}$$

The relation (1.2) is called the dispersive relation corresponding to the equation (1.1). It shows that the frequency is a real valued function of the wave number. If we denote the phase velocity by $v = \frac{\omega}{k}$, we can write the solution as $u(x,t) = Ae^{ik(x-v(k)t)}$ and notice that the wave travels with velocity k. Thus the wave

The work of N.P. is supported in part by NSF grant DMS-1101192.

The work of N.T. is supported in part by University of Illinois Research Board Grant RB-14054. Both authors are thankful to the MSRI staff for all help in organizing the workshop.

propagates in such a way that large wave numbers travel faster than smaller ones[1]. If we add nonlinear effects and study for example

$$iu_t + u_{xx} + |u|^{p-1}u = 0,$$

we will see that even the existence of solutions over small times requires delicate techniques.

Going back to the linear homogenous equation (1.1), let us now consider

$$u_0(x) = \int_{\mathbb{R}} \hat{u}_0(k)e^{ikx}dk.$$

For each fixed k the wave solution becomes

$$u(x,t) = \hat{u}_0(k)e^{ik(x-kt)} = \hat{u}_0(k)e^{ikx}e^{-ik^2t}.$$

Summing over k (integrating) we obtain the solution to our problem

$$u(x,t) = \int_{\mathbb{R}} \hat{u}_0(k)e^{ikx-ik^2t}dk.$$

Since $|\hat{u}(k,t)| = |\hat{u}_0(k)|$ we have that $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Thus the conservation of the L^2 norm (mass conservation or total probability) and the fact that high frequencies travel faster, leads to the conclusion that not only the solution will disperse into separate waves but that its amplitude will decay over time. This is not anymore the case for solutions over compact domains. The dispersion is limited and for the nonlinear dispersive problems we notice a migration from low to high frequencies. This fact is captured by zooming more closely in the Sobolev norm

$$\|u\|_{H^s} = \left(\int |\hat{u}(k)|^2(1+|k|)^{2s}dk\right)^{1/2}$$

and observing that it actually grows over time.

Another characterization of dispersive equations comes from the observation that the space-time Fourier transform (we usually denote by (ξ,τ) the dual variables of (x,t)) of their solutions are supported on hyper-surfaces that have non vanishing Gaussian curvature. For example taking the Fourier transform of the solution of the linear homogeneous Schrödinger equation

$$iu_t + \Delta u = 0,$$

for $x \in \mathbb{R}^n$ and $t \geq 0$, we obtain that $u(\xi,\tau)$ is supported[2] on $\tau = |\xi|^2$.

In dispersive equations there is usually a competition between dispersion that over time smooths out the initial data (in terms of extra regularity and/or in terms of extra integrability) and the nonlinearity that can cause concentration, blow-up or even ill-posedness in the Hadamard sense. We focus our attention on the following two dispersive equations:

---

[1] Trying a wave solution of the same form to the heat equation $u_t - u_{xx} = 0$, we obtain that the ω is complex valued and the wave solution decays exponential in time. On the other hand the transport equation $u_t - u_x = 0$ and the one dimensional wave equation $u_{tt} = u_{xx}$ have traveling waves with constant velocity.

[2] In this light the linear wave equation in dimension higher than two is dispersive as the solution is supported on the cone $\tau = |\xi|$.

- Nonlinear Schrödinger (NLS) equation given by
$$iu_t + \Delta u + f(u) = 0,$$
where $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$.

- The Korteweg-de Vries equation (KdV) given by
$$u_t + u_{xxx} + uu_x = 0,$$
where $u : M \times \mathbb{R} \to \mathbb{R}$ with $M \in \{\mathbb{R}, \mathbb{T}\}$

as two prime examples. However the methods that are reviewed in these notes apply equally well to other dispersive PDE. The competition mentioned above comes to light in a variety of ways. On one hand, we have the case of the NLS (4.2) of defocusing type with a polynomial nonlinearity of high enough power. In this case the global energy solution that we will obtain satisfy additional decay estimates that over time weaken the nonlinear effects. It is then possible to compare the dynamics of the NLS with the linear problem and show that as $t \to \infty$ the nonlinearity "disappears" and the solution approaches the free solution. On the other hand, we have the case of the KdV equation. There the dispersion and the nonlinearity are balanced in such away that solitary waves (global traveling wave solutions) exist for all times. These traveling waves are smooth solutions that prevent the equation from scattering even on the real line. Many different phenomena intertwine with dispersion but in these notes we can develop and partially answer only the most basic of questions. For more details the reader can consult [2, 4, 32, 44, 46].

To analyze further the properties of dispersive PDE and outline some recent developments we start with a concrete example.

2. The semi-linear Schrödinger equation.

Consider the semi-linear Schrödinger equation (NLS) in arbitrary dimensions
$$\begin{cases} iu_t + \Delta u + \lambda |u|^{p-1}u = 0, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad \lambda \pm 1, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}^n). \end{cases} \tag{2.1}$$
for any $1 < p < \infty$. Here $H^s(\mathbb{R}^n)$ denotes the s Sobolev space, which is a Banach space that contains all functions that along with their distributional s-derivatives belong to $L^2(\mathbb{R}^n)$. This norm is equivalent (through the basic properties of the Fourier transform) to
$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + |\xi|)^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty.$$
When $\lambda = -1$ the equation is called defocusing and when $\lambda = 1$ it is called focusing.

NLS is a basic dispersive model that appears in nonlinear optics and water wave theory, and it can be derived from quantum many body systems as we shall see in Part III of the notes.

Before we outline basic properties and questions of interest concerning solutions to (2.1), we review symmetries of the equation.

2.1. **Symmetries of the equation.** One of the questions that we shall consider is the following: for what values of $s \in \mathbb{R}$ one can expect reasonable solutions? The symmetries of the equation (2.1) can be very helpful in addressing this question.

(1) A symmetry that we shall often mention is the **scaling symmetry**, that can be formulated as follows. Let $\lambda > 0$. If u is a solution to (2.1) then

$$u^{\lambda}(x,t) = \lambda^{-\frac{2}{p-1}} u(\frac{x}{\lambda},\ \frac{t}{\lambda^2}), \quad u_0^{\lambda} = \lambda^{-\frac{2}{p-1}} u_0(\frac{x}{\lambda}),$$

is a solution to the same equation. If we compute $\|u_0^{\lambda}\|_{\dot{H}^s}$ we see that

$$\|u_0^{\lambda}\|_{\dot{H}^s} = \lambda^{s_c - s}\|u_0\|_{\dot{H}^s}$$

where $s_c = \frac{n}{2} - \frac{2}{p-1}$. It is then clear that as $\lambda \to \infty$:
   (a) If $s > s_c$ (**sub-critical case**) the norm of the initial data can be made small while at the same time the time interval is made longer. This is the best possible scenario for local well-posedness. Notice that u^{λ} lives on $[0, \lambda^2 T]$.
   (b) If $s = s_c$ (**critical case**) the norm of the initial data is invariant while the time interval gets longer. There is still hope in this case, but it turns out that to provide globally defined solutions one has to work very hard.
   (c) If $s < s_c$ (**super-critical case**) the norms grow as the time interval is made longer. Scaling works against us in this case; we cannot expect even locally defined strong solutions, at least in deterministic sense.

(2) Then we have the **Galilean Invariance**: If u is a solution to (2.1) then

$$e^{ix\cdot v} e^{-it|v|^2} u(x - 2vt,\ t)$$

is a solution to the same equation with data $e^{ix\cdot v}u_0(x)$.

(3) Other symmetries:
   (a) There is also **time reversal symmetry**. We can thus consider solutions in $[0, T]$ instead of $[-T, T]$.
   (b) **Spatial rotation symmetry** which leads to the property that if we start with radial initial data then we obtain a radially symmetric solution.
   (c) **Time translation invariance** that leads for smooth solutions to the conservation of energy

$$E(u)(t) = \frac{1}{2}\int |\nabla u(t)|^2 dx - \frac{\lambda}{p+1}\int |u(t)|^{p+1} dx = E(u_0). \tag{2.2}$$

   (d) **Phase rotation symmetry** $e^{i\theta}u$ that leads to mass conservation

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}. \tag{2.3}$$

   (e) **Space translation invariance** that leads to the conservation of the momentum

$$\vec{p}(t) = \Im \int_{\mathbb{R}^n} \bar{u}\nabla u dx = \vec{p}(0). \tag{2.4}$$

(4) In the case that $p = 1+\frac{4}{n}$, we also have the **pseudo-conformal symmetry** where if u is a solution to (2.1) then for $t \neq 0$

$$\frac{1}{|t|^{\frac{n}{2}}}\overline{u(\frac{x}{t},\frac{1}{t})}e^{\frac{i|x|^2}{4t}}$$

is also a solution. This leads to the pseudo-conformal conservation law

$$K(t) = \|(x+2it\nabla)u\|_{L^2}^2 - \frac{8t^2\lambda}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx = \|xu_0\|_{L^2}^2.$$

2.2. **Questions of interest and relevant notation.** We will study NLS and related equations via considering questions

- of local-in-time nature (local existence of solutions, uniqueness, regularity),
- of global-in-time nature (existence of solutions for large times, finite time blow-up, scattering).

The standard treatment of the subject is presented in the books of Cazenave [4] and Tao [46], among others. We will refer to these books, especially the first one, throughout the notes.

We start by listing some questions of interest:

1. Consider X a Banach space. Starting with initial data $u_0 \in H^s(\mathbb{R}^n)$, we say that the solution exists locally-in-time, if there exists $T > 0$ and a subset X of $C_t^0H_x^s([0,T]\times\mathbb{R}^n)$ such that there exists a unique solution to (1). Note that if $u(x,t)$ is a solution to (1) then $-u(-x,t)$ is also a solution. Thus we can extend any solution in $C_t^0H_x^s([0,T]\times\mathbb{R}^n)$ to a solution in $C_t^0H_x^s([-T,T]\times\mathbb{R}^n)$. We also demand that there is continuity with respect to the initial data in the appropriate topology.

2. If T can be taken to be arbitrarily large then we say that we have a global solution.

3. Assume $u_0 \in H^s(\mathbb{R}^n)$ and consider a local solution. If there is a $T^\star$ such that

$$\lim_{t\to T^\star}\|u(t)\|_{H^s} = \infty,$$

we say that the solution blows up in finite time. At this point, we can mention a statement of the so called "blow-up alternative" which is usually proved along with the local theory. More precisely, the blow-up alternative is a statement that characterizes the finite time of blow-up, which for example can be done along the following lines: if $(0,T^*)$ is the maximum interval of existence, then if $T^* < \infty$, we have $\lim_{t\to T^*}\|u(t)\|_{H^s} = \infty$. Analogous statements can be made for $(-T^*,0)$.

4. As a Corollary to the blow-up alternative one obtains globally defined solutions if there is an a priori bound of the H^s norms for all times. Such an a priori bound is of the form:

$$\sup_{t\in\mathbb{R}}\|u(t)\|_{H^s} < \infty,$$

and it usually comes from the conservation laws of the equation. For (2.1) this is usually the case for $s = 0,\ 1$. An *important* comment is in order. Our notion of global solutions in the point 2. described above does not require that $\|u(t)\|_{H^s}$ remains uniformly bounded in time. As we said unless $s = 0,\ 1$, it is not a triviality to obtain such a uniform bound. In case that we have quantum scattering, these uniform bounds are byproducts of the control we obtain on our solutions at infinity.

5. If $u_0 \in H^s(\mathbb{R}^n)$ and we have a well defined local solution, then for each $(0, T)$ we have that $u(t) \in H^s_x(\mathbb{R}^n)$. Persistence of regularity refers to the fact that if we consider $u_0 \in H^{s_1}(\mathbb{R}^n)$ with $s_1 > s$, then $u \in X \subset C^0_t H^{s_1}_x([0, T_1] \times \mathbb{R}^n)$, with $T_1 = T$. Notice that any H^{s_1} solution is in particular an H^s solution and thus $(0, T_1) \subset (0, T)$. Persistence of regularity affirms that $T_1 = T$ and thus u cannot blow-up in H^{s_1} before it blows-up in H^s both backward and forward in time.

6. Scattering is usually the most difficult problem of the ones mentioned above. Assume that we have a globally defined solution (which is true for arbitrary large data in the defocusing case). The problem then is divided into an easier (existence of the wave operator) and a harder (asymptotic completeness) problem. We will see shortly that the L^p norms of linear solutions decay in time. This time decay is suggestive that for large values of p the nonlinearity can become negligible as $t \to \pm\infty$. Thus we expect that u can be approximated by the solution of the linear equation. We have to add here that this theory is highly nontrivial for large data. For small data we can have global solutions and scattering even in the focusing problem.

7. A solution that will satisfy (at least locally) most of these properties will be called a **strong** solution. We will give a more precise definition later in the notes. This is a distinction that is useful as one can usually derive through compactness arguments weak solutions that are not unique. The equipment of the derived (strong) solutions with the aforementioned properties is of importance. For example the fact that local H^1 solutions satisfy the energy conservation law is a byproduct not only of the local-in-time existence but also of the regularity and the continuity with respect to the initial data properties.

8. To make the exposition easier we mainly consider H^s solutions where s is an integer. From a mathematical point of view one can investigate solutions that evolve from rougher and rougher initial data (and thus belong to larger classes of spaces).

3. Local Well-Posedness

When trying to establish existence of local (in time) solutions, an important step consists of constructing the aforementioned Banach space X. This process is delicate (the exception being the construction of smooth solutions that is done classically) and is built upon certain estimates that the linear solution satisfies. First we recall those estimates.

3.1. **Fundamental solution, Dispersive and Strichartz estimates.**

Recall (from an undergraduate or graduate PDE course) that we can obtain the solution to the linear problem by utilizing the Fourier transform. Then for smooth initial data (say in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$) the solution of the linear homogeneous equation is given as the convolution of the data with the tempered distribution

$$K_t(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{i\frac{i|x|^2}{4t}}.$$

Thus we can write the solution as:

$$u(x,t) = U(t)u_0(x) = e^{it\Delta}u_0(x) = K_t \star u_0(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy. \tag{3.1}$$

Another fact from our undergraduate (or graduate) machinery is Duhamel's principle:
Let I be any time interval and suppose that $u \in C_t^1\mathcal{S}(I \times \mathbb{R}^n)$ and that $F \in C_t^0\mathcal{S}(I \times \mathbb{R}^n)$. Then u solves

$$\begin{cases} iu_t + \Delta u = F, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,t_0) = u(t_0) \in \mathcal{S}(\mathbb{R}^n) \end{cases} \tag{3.2}$$

if and only if

$$u(x,t) = e^{i(t-t_0)\Delta}u(t_0) - i\int_0^t e^{i(t-s)\Delta}F(s)ds. \tag{3.3}$$

Definition 3.1. *Let I be a time interval which contains zero, $u_0 := u(x,0) \in H^s(\mathbb{R}^n)$ and*

$$F \in C(H^s(\mathbb{R}^n); H^{s-2}(\mathbb{R}^n)).$$

We say that

$$u \in C(I; H^s(\mathbb{R}^n)) \cap C^1(I; H^{s-2}(\mathbb{R}^n))$$

is a **strong solution** *of* (3.2) *on I, if it satisfies the equation for all $t \in I$ in the sense of H^{s-2} (thus as a distribution for low values of s) and $u(0) = u_0$.*

Remark 3.2. *By a little semigroup theory this definition of a strong solution is equivalent to saying that for all $t \in I$, u satisfies* (3.3)*.*

Now we state the basic dispersive estimate for solutions to the homogeneous equation (3.2), with $F = 0$. From the formula (3.1) we see that:

$$\|u\|_{L_x^\infty} \leq \frac{1}{(4|t|\pi)^{\frac{n}{2}}}\|u_0\|_{L^1}.$$

In addition the solution satisfies that $\hat{u}(\xi,t) = e^{-4\pi^2 it|\xi|^2}\hat{u}_0(\xi)$, which together with Plancherel's theorem implies that

$$\|u(t)\|_{L_x^2} = \|u_0\|_{L_x^2}.$$

Riesz-Thorin interpolation Lemma then implies that for any $p \geq 2$ and $t \neq 0$ we have that

$$\|u(t)\|_{L_x^p} \leq \frac{1}{(4|t|\pi)^{n(\frac{1}{2}-\frac{1}{p})}}\|u_0\|_{L^{p'}}, \tag{3.4}$$

where p' is the dual exponent of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

Fortunately, the basic dispersive estimates (3.4) can be extended by duality (using a $TT^\star$ argument) to obtain very useful Strichartz estimates, [4, 16, 27, 41]. In order to state Strichartz estimates, first, we recall the definition of an admissible pair of exponents.

Definition 3.3. *Let $n \geq 1$. We call a pair (q, r) of exponents admissible if*

$$2 \leq q, r \leq \infty$$

are such that

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} \tag{3.5}$$

and $(q, r, n) \neq (2, \infty, 2)$.

Now we can state the Strichartz estimates:

Theorem 3.4. *Let $n \geq 1$. Then for any admissible exponents (q, r) and $(\tilde{q}, \tilde{r})$ we have the following estimates:*

- *The homogeneous estimate:*

$$\|e^{it\Delta}u_0\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2}, \tag{3.6}$$

- *The dual estimate:*

$$\|\int_{\mathbb{R}} e^{-it\Delta}F(\cdot, t)\, dt\|_{L_x^2(\mathbb{R}^n)} \leq \|F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}(\mathbb{R}\times\mathbb{R}^n)} \tag{3.7}$$

- *The non-homogeneous estimate:*

$$\|\int_0^t e^{i(t-s)\Delta}F(\cdot, s)\, ds\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^n)}, \tag{3.8}$$

where $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$.

Remark 3.5. *Actually the proof of Strichartz estimates implies more. In particular, the operator $e^{it\Delta}u_0(x)$ belongs to $C(\mathbb{R}, L_x^2)$ and $\int_0^t U(t-s)F(s)ds$ belongs to $C(\bar{I}, L_x^2)$ where $t \in I$ is any interval of $\mathbb{R}$.*

In the lectures and during TA sessions, we will discuss the non-endpoint case. The end-point case was proved in [27].

3.2. **Notion of local well-posedness.** We are now ready to give a precise definition of what we mean by local well-posedness of the initial value problem (IVP) (2.1).

Definition 3.6. *We say that the IVP (2.1) is locally well-posed (lwp) and admits a strong solution in $H^s(\mathbb{R}^n)$ if for any ball B in the space $H^s(\mathbb{R}^n)$, there exists a finite time T and a Banach space $X \subset L_t^\infty H_x^s([0, T] \times \mathbb{R}^n)$ such that for any initial data $u_0 \in B$ there exists a unique solution $u \in X \subset C_t^0 H_x^s([0, T] \times \mathbb{R}^n)$ to the integral equation*

$$u(x, t) = U(t)u_0 + i\lambda \int_0^t U(t-s)|u|^{p-1}u(s)ds.$$

Furthermore the map $u_0 \to u(t)$ is continuous as a map from $H^s(\mathbb{R}^n)$ into $C_t^0 H_x^s([0, T] \times \mathbb{R}^n)$. If uniqueness holds in the whole space $C_t^0 H_x^s([0, T] \times \mathbb{R}^n)$ then we say that the lwp is unconditional.

In what follows we assume that $p-1=2k$. This implies that the nonlinearity is sufficiently smooth to perform all the calculations in a straightforward way.

3.3. **Well-posedness for smooth solutions.** We start with the H^s well-posedness theory, with an integer $s>\frac{n}{2}$. For more general statements see [25].

Theorem 3.7. *Let $s>\frac{n}{2}$ be an integer. For every $u_0\in H^s(\mathbb{R}^n)$ there exists $T^*>0$ and a unique maximal solution $u\in C((0,T^*);H^s(\mathbb{R}^n))$ that satisfies* (2.1) *and in addition satisfies the following properties:*
i) If $T^*<\infty$ then $\|u(t)\|_{H^s}\to\infty$ as $t\to\infty$. Moreover $\limsup_{t\to T^*}\|u(t)\|_{L^\infty}=\infty$.
ii) u depends continuously on the initial data in the following sense. If $u_{n,0}\to u_0$ in H^s and if u_n is the corresponding maximal solution with initial data $u_{n,0}$, then $u_n\to u$ in $L^\infty((0,T);H^s(\mathbb{R}^n))$ for every interval $[0,T]\subset[0,T^*)$.
iii) In addition, the solution u satisfies conservation of energy (2.2) *and conservation of mass* (2.3)*.*

Remark 3.8. *A comment about uniqueness. Suppose that one proves existence and uniqueness in $C([-T,T];X_M)$ where X_M, $M=M(\|u_0\|_X)$, $T=T(M)$, is a fixed ball in the space X. One can then easily extend the uniqueness to the whole space X by shrinking time by a fixed amount. Indeed, shrinking time to T' we get existence and uniqueness in a larger ball $X_{M'}$. Now assume that there are two different solutions one staying in the ball X_M and one separating after hitting the boundary at some time $|t|<T'$. This is already a contradiction by the uniqueness in $X_{M'}$.*

3.3.1. *Preliminaries.* To prove Theorem 3.7 we need the following two lemmata:

Lemma 3.9. *Gronwall's inequality: Let $T>0$, $k\in L^1(0,T)$ with $k\geq 0$ a.e. and two constants $C_1,C_2\geq 0$. If $\psi\geq 0$, a.e in $L^1(0,T)$, such that $k\psi\in L^1(0,T)$ satisfies*

$$\psi(t)\leq C_1+C_2\int_0^t k(s)\psi(s)ds$$

for a.e. $t\in(0,T)$ then,

$$\psi(t)\leq C_1\exp\big(C_2\int_0^t k(s)ds\big).$$

Proof. For a proof, see e.g. Evans [15]. □

Lemma 3.10. *Let $g(u)=\pm|u|^{2k}u$ and consider and $s,l\geq 0$, integers with $l\leq s$ and $s>\frac{n}{2}$. Then*

$$\|g(u)\|_{H^s}\lesssim\|u\|_{H^s}^{2k+1}, \tag{3.9}$$

$$\|g(u)-g(v)\|_{L^2}\lesssim\big(\|u\|_{H^s}^{2k}+\|v\|_{H^s}^{2k}\big)\|u-v\|_{L^2}, \tag{3.10}$$

$$\|g^{(l)}(u)-g^{(l)}(v)\|_{L^\infty}\lesssim\big(\|u\|_{H^s}^{2k-l}+\|v\|_{H^s}^{2k-l}\big)\|u-v\|_{H^s}, \tag{3.11}$$

$$\|g(u)-g(v)\|_{H^s}\lesssim\big(\|u\|_{H^s}^{2k}+\|v\|_{H^s}^{2k}\big)\|u-v\|_{H^s}. \tag{3.12}$$

Proof. To prove (3.9) we use the algebra property of H^s for $s>\frac{n}{2}$ and the fact that $\|u\|_{H^s}=\|\bar{u}\|_{H^s}$.

To prove (3.10) and (3.11) note that since g is smooth we have that

$$|g(u)-g(v)| \lesssim \big(|u|^{2k}+|v|^{2k}\big)|u-v|,$$
$$|g^{(l)}(u)-g^{(l)}(v)| \lesssim \big(|u|^{2k-l}+|v|^{2k-l}\big)|u-v|.$$

Then

$$\|g(u)-g(v)\|_{L^2} \lesssim \big(\|u\|_{L^\infty}^{2k}+\|v\|_{L^\infty}^{2k}\big)\|u-v\|_{L^2} \lesssim \big(\|u\|_{H^s}^{2k}+\|v\|_{H^s}^{2k}\big)\|u-v\|_{L^2},$$
$$\|g^{(l)}(u)-g^{(l)}(v)\|_{L^\infty} \lesssim \big(\|u\|_{L^\infty}^{2k-l}+\|v\|_{L^\infty}^{2k-l}\big)\|u-v\|_{L^\infty} \lesssim \big(\|u\|_{H^s}^{2k-l}+\|v\|_{H^s}^{2k-l}\big)\|u-v\|_{L^2},$$

where we used the fact that H^s embeds in L^∞.

To prove (3.12) notice that the L^2 part of the left hand side follows from (3.10). For the derivative part consider a multi-index α with $|\alpha|=s$. Then $D^\alpha u$ is the sum (over $k\in\{1,2,...,s\}$) of terms of the form $g^{(k)}(u)\prod_{j=1}^k D^{\beta_j}u$ where $|\beta_j|\geq 1$ and $|\alpha|=|\beta_1|+...+|\beta_k|$. Now let $p_j=\frac{2s}{|\beta_j|}$ such that $\sum_{j=1}^k\frac{1}{p_j}=\frac{1}{2}$. We have by Hölder's inequality

$$\|g^{(k)}(u)\prod_{j=1}^k D^{\beta_j}u\|_{L^2} \lesssim \|g^{(k)}(u)\|_{L^\infty}\prod_{j=1}^k\|D^{\beta_j}u\|_{L^{p_j}}.$$

By complex interpolation (or Gagliardo-Nirenberg inequality) we obtain

$$\|D^{\beta_j}u\|_{L^{p_j}} \lesssim \|u\|_{H^s}^{\frac{|\beta_j|}{s}}\|u\|_{L^\infty}^{1-\frac{|\beta_j|}{s}}$$

and thus

$$\|g^{(k)}(u)\prod_{j=1}^k D^{\beta_j}u\|_{L^2} \lesssim \|g^{(k)}(u)\|_{L^\infty}\|u\|_{H^s}\|u\|_{L^\infty}^{k-1} \lesssim \|u\|_{H^s}^{2k+1}$$

where in the last inequality we used (3.11). Thus we obtain

$$\|D^\alpha u\|_{L^2} \lesssim \|u\|_{H^s}^{2k+1}. \tag{3.13}$$

Again notice that the term $D^\alpha(g(u)-g(v))$ is the sum of terms of the form

$$g^{(k)}(u)\prod_{j=1}^k D^{\beta_j}u - g^{(k)}(v)\prod_{j=1}^k D^{\beta_j}v = \big[g^{(k)}(u)-g^{(k)}(v)\big]\prod_{j=1}^k D^{\beta_j}u + g^{(k)}(v)\prod_{j=1}^k D^{\beta_j}w_j$$

where w_j's are equal to u or v except one that is equal to $u-v$. The second of the left hand side is estimated as in the proof of (3.13). For the first the same trick applies but now to estimate $\|g^{(k)}(u)-g^{(k)}(v)\|_{L^\infty}$ we use (3.12). □

3.3.2. *A proof of Theorem 3.7.* Now we present a proof of Theorem 3.7.

Existence and Uniqueness. We construct solutions by a fixed point argument.

Given $M,T>0$ to be chosen later, we set $I=(0,T)$ and consider the space

$$E=\{u\in L^\infty(I;H^s(\mathbb{R}^n)) : \|u\|_{L^\infty(I;H^s)}\leq M\},$$

equipped with the distance

$$d(u,v)=\|u-v\|_{L^\infty(I;L^2)}.$$

We note that (E,d) is a complete metric space.

Now based on the equation (2.1), with $\lambda = -1$, in the integral form, we introduce the mapping Φ as follows:

$$\Phi(u)(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}|u|^{2k}u(\tau)\,d\tau =: e^{it\Delta}u_0 + H(u)(t).$$

By Lemma 3.10, Minkowski's inequality and the fact that $e^{it\Delta}$ is an isometry in H^s we have that

$$\|\Phi(u)(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + T\|g(u)\|_{L^\infty(I;H^s)} \leq \|u_0\|_{H^s} + TC(M)M,$$

where we used the notation $g(u) = \pm|u|^{2k}u$ as in Lemma 3.10. Furthermore using Lemma 3.10 again we have

$$\|\Phi(u)(t) - \Phi(v)(t)\|_{L^2} \lesssim TC(M)\|u - v\|_{L^\infty(I;L^2)}. \tag{3.14}$$

Therefore we see that if $M = 2\|u_0\|_{H^s}$ and $TC(M) < \frac{1}{2}$, then Φ is a contraction of (E, d) and thus has a unique fixed point. Uniqueness in the full space follows by the remark above or alternatively by the remark and Gronwall's Lemma.

Blow-up alternative. Let $u_0 \in H^s$ and define

$$T^* = \sup\{T > 0 : \textit{there exists a solution on } [0,T]\}. \tag{3.15}$$

Now let $T^* < \infty$ and assume that there exists a sequence $t_j \to T^*$ such that $\|u(t_j)\|_{H^s} \leq M$. In particular for k such that t_k is close to T^* we have that $\|u(t_k)\|_{H^s} \leq M$. Now we solve our problem with initial data $u(t_k)$ and we extend our solution to the interval $[t_k,\ t_k + T(M)]$. But if we pick k such that

$$t_k + T(M) > T^*$$

we then contradict the definition of T^*. Thus $\lim_{t\to T^*}\|u(t)\|_{H^s} = \infty$ if $T^* < \infty$.

We now show that if $T^* < \infty$ then $\limsup_{t\to T^*}\|u(t)\|_{L^\infty} = \infty$. Indeed suppose that $\limsup_{t\to T^*}\|u(t)\|_{L^\infty} < \infty$. Since $u \in C([0,T^*); H^s)$ we have that

$$M = \sup_{0\leq t<T^*}\|u(t)\|_{L^\infty} < \infty$$

where we used the fact that H^s embeds in L^∞. By Duhamel's formula and Lemma 3.10 we have that

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + C(M)\int_0^t \|u(\tau)\|_{H^s}\,d\tau.$$

By Gronwall's lemma we have that $\|u(t)\|_{H^s} \leq \|u_0\|_{H^s}e^{T^*C(M)}$ for all $0 \leq t < T^*$. But this contradicts the blow-up of $\|u(t)\|_{H^s}$ at T^*.

Continuous dependence. Let $u_0 \in H^s$ and consider $u_{0,n} \subset H^s$ such that $u_{n,0} \to u_0$ in H^s as $n \to \infty$. Since for n sufficiently large we have that $\|u_{0,n}\|_{H^s} \leq 2\|u_0\|_{H^s}$ by the local theory there exists $T = T(\|u_0\|_{H^s})$ such that u and u_n are defined on $[0,T]$ for $n \geq N$ and

$$\|u\|_{L^\infty((0,T);H^s)} + \sup_{n\geq N}\|u_n\|_{L^\infty((0,T);H^s)} \leq 6\|u_0\|_{H^s}.$$

Now note that $u_n(t)-u(t)=e^{it\Delta}(u_{n,0}-u_0)+H(u_n)(t)-H(u)(t)$. If we use Lemma 3.10 we see that for all $t\in(0,T)$ and n sufficiently large, there exists C such that

$$\|u_n(t)-u(t)\|_{H^s}\le\|u_{n,0}-u_0\|_{H^s}+C\int_0^t\|u_n(\tau)-u(\tau)\|_{H^s}\,d\tau.$$

By Gronwall's lemma we see that $u_n\to u$ in H^s as $n\to\infty$. Iterating this property to cover any compact subset of $(0,T^*)$ we finish the proof.

As a final note we remark that if we solve the equation, starting from u_0 and $u(t_1)$ over the intervals $[0,t_1]$ and $[t_1,t_2]$ respectively, by continuous dependence, to prove that $C\big([0,T];H^s(\mathbb{R}^n)\big)$, it is enough to consider the difference $u(t_1)-u_0$ in the H^s norm. Since

$$u(t_1)-u_0=(e^{it_1\Delta}-1)u_0-i\int_0^{t_1}e^{i(t_1-\tau)}g(u)(\tau)\,d\tau,$$

using again Lemma 3.10 and the fact that $e^{it\Delta}u_0(x)\in C(\mathbb{R};H^s)$ we have

$$\|u(t_1)-u_0\|_{H^s}\lesssim\|(e^{it_1\Delta}-1)u_0\|_{H^s}+|t_1|\|u\|^{2k+1}_{L^\infty((0,t_1);H^s)}$$

which finishes the proof.

<u>Conservation laws:</u> Since we develop the H^1 theory below we implicitly have $s\ge 2$. We have at hand a solution that satisfies the equation in the classical sense for high enough s (in general in the H^{s-2} sense with $s\ge 2$ and thus in particular u satisfies the equation at least in the L^2 sense. All integrations below then can be justified in the Hilbert space L^2). To obtain the conservation of mass we can multiply the equation by $i\bar{u}$, integrate and then take the real part. To obtain the conservation of energy we multiply the equation by $\bar{u}_t$, take the real part and then integrate.

3.4. **Local well-posedness in the H^1 sub-critical case.**

For more details we refer to [4, 25, 26].

Theorem 3.11. *Let $1<p<1+\frac{4}{n-2}$, if $n\ge 3$ and $1<p<\infty$, if $n=1,2$. For every $u_0\in H^1(\mathbb{R}^n)$ there exists a unique strong H^1 solution of (2.1) defined on the maximal interval $(0,T^\star)$. Moreover*

$$u\in L^\gamma_{loc}((0,T^\star);\ W^{1,\rho}_x(\mathbb{R}^n))$$

for every admissible pair (γ,ρ). In addition

$$\lim_{t\to T^\star}\|u(t)\|_{H^1}=\infty$$

if $T^\star<\infty$, and u depends continuously on u_0 in the following sense: There exists $T>0$ depending on $\|u_0\|_{H^1}$ such that if $u_{0,n}\to u_0$ in H^1 and $u_n(t)$ is the corresponding solution of (2.1), then $u_n(t)$ is defined on $[0,T]$ for n sufficiently large and

$$u_n(t)\to u(t)\quad in\quad C([0,T];\ H^1)\tag{3.16}$$

for every compact interval $[0,T]$ of $(0,T^\star)$. Finally we have that

$$E(u)(t)=\frac{1}{2}\int|\nabla u(t)|^2dx-\frac{\lambda}{p+1}\int|u(t)|^{p+1}dx=E(u_0)$$

and

$$M(u)(t)=\|u(t)\|_{L^2}=\|u_0\|_{L^2}=M(u_0).$$

We note that $W^{1,\rho}$ is the Sobolev space of L^ρ functions with weak derivatives in L^ρ of order one.

Proof. First we establish:
Existence and Uniqueness. In order to define the space on which we shall apply the fixed point argument, we pick r to be $r := p+1$. Fix $M, T > 0$ to be chosen later and let q be such that the pair (q, r) is admissible.[3] Consider the set

$$E = \{u \in L_t^\infty H_x^1([0,T] \times \mathbb{R}^n) \cap L^q((0,T); W^{1,r}(\mathbb{R}^n)) : \tag{3.17}$$

$$\|u\|_{L_t^\infty((0,T);H^1)} \le M \text{ and } \|u\|_{L_t^q W_x^{1,r}} \le M\}. \tag{3.18}$$

equipped with the distance

$$d(u,v) = \|u-v\|_{L^q((0,T);L^r(\mathbb{R}^n))} + \|u-v\|_{L^\infty((0,T);L^2(\mathbb{R}^n))}.$$

It can be shown that (E, d) is a complete metric space.

We write the solution map via Duhamel's formula as follows:

$$\Phi(u)(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}|u|^{p-1}u(\tau)\,d\tau. \tag{3.19}$$

Now we provide a few estimates that we shall use in order to justify that the mapping Φ is a contraction on (E, d). Notice that for $r = p+1$ we have

$$\||u|^{p-1}u\|_{L_x^{r'}} \lesssim \|u\|_{L_x^r}^p$$

and thus by Hölder

$$\||u|^{p-1}u\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty L_x^r}^{p-1}\|u\|_{L_t^q L_x^r}. \tag{3.20}$$

However by Sobolev embedding we have that

$$\|u\|_{L_x^r} \lesssim \|u\|_{H^1},$$

which together with (3.20) implies that:

$$\||u|^{p-1}u\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{p-1}\|u\|_{L_t^q L_x^r}. \tag{3.21}$$

Similarly, since the nonlinearity is smooth,

$$\|\nabla(|u|^{p-1}u)\|_{L_t^q L_x^{r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{p-1}\|\nabla u\|_{L_t^q L_x^r}. \tag{3.22}$$

Now we combine (3.21) and (3.22) to obtain for $u \in E$:

$$\||u|^{p-1}u\|_{L_t^q W_x^{1,r'}} \lesssim \|u\|_{L_t^\infty H_x^1}^{p-1}\|u\|_{L_t^q W_x^{1,r}} \tag{3.23}$$

Furthermore, applying Hölder's inequality in time, followed by an application of (3.23) gives:

$$\begin{aligned}\||u|^{p-1}u\|_{L_t^{q'} W_x^{1,r'}} &\lesssim T^{\frac{q-q'}{q'q}}\||u|^{p-1}u\|_{L_t^q W_x^{1,r'}} \\ &\lesssim T^{\frac{q-q'}{q'q}}\|u\|_{L_t^\infty H_x^1}^{p-1}\|u\|_{L_t^q W_x^{1,r}}. \end{aligned}\tag{3.24}$$

[3]Since the admissibility condition reads $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$, and $r = p+1$, we have that $q = \frac{4(p+1)}{n(p-1)}$.

Now we are ready to show that Φ is a contraction on (E,d). Using Duhamel's formula (3.19) and Strichartz estimates we obtain:

$$\begin{aligned}\|\Phi(u)(t)\|_{L^q_t W^{1,r}_x} &\lesssim \|e^{it\Delta}u_0\|_{L^q_t W^{1,r}_x} + \||u|^{p-1}u\|_{L^{q'}_t W^{1,r'}_x} \\ &\lesssim \|u_0\|_{H^1} + T^{\frac{q-q'}{q'q}} \|u\|^{p-1}_{L^\infty_t H^1_x} \|u\|_{L^q_t W^{1,r}_x}, \qquad (3.25)\end{aligned}$$

where to obtain (3.25) we used (3.24). Also by Duhamel's formula (3.19), Strichartz estimates and (3.24) we have:

$$\|\Phi(u)(t)\|_{L^\infty_t H^1_x} \lesssim \|u_0\|_{H^1} + T^{\frac{q-q'}{q'q}} \|u\|^{p-1}_{L^\infty_t H^1_x} \|u\|_{L^q_t W^{1,r}_x}. \qquad (3.26)$$

Hence (3.25) and (3.26) imply:

$$\|\Phi(u)(t)\|_{L^q_t W^{1,r}_x} + \|\Phi(u)(t)\|_{L^\infty_t H^1_x} \leq C\|u_0\|_{H^1} + CT^{\frac{q-q'}{q'q}} M^{p-1} \|u\|_{L^q_t W^{1,r}_x}. \qquad (3.27)$$

Now we set $M = 2C\|u_0\|_{H^1}$ and then choose T small enough such that

$$CT^{\frac{q-q'}{q'q}} M^{p-1} \leq \frac{1}{2}.$$

We note that such choice of T is indeed possible thanks to the fact that for $p < 1 + \frac{4}{n-2}$ we have that $q > 2$ and thus $q > q'$. For such $T \sim T(\|u_0\|_{H^1})$ we have that $\|\Phi(u)(t)\|_E \leq M$ whenever $u \in E$ and thus $\Phi : E \to E$. In a similar way, one can obtain the following estimate on the difference:

$$\|\Phi(u)(t) - \Phi(v)(t)\|_{L^q_t W^{1,r}_x} + \|\Phi(u)(t) - \Phi(v)(t)\|_{L^\infty_t L^2_x}$$

provides a unique solution $u \in E$. Notice that by the above estimates and the Strichartz estimates we have that $u \in C^0_t((0,T); H^1(\mathbb{R}^n))$.

To extend uniqueness in the full space we assume that we have another solution v and consider an interval $[0,\delta]$ with $\delta < T$. Then as before

$$\|u(t)-v(t)\|_{L^q_\delta W^{1,r}_x} + \|u(t)-v(t)\|_{L^\infty_\delta H^1_x} \leq C\delta^\alpha (\|u\|^{p-1}_{L^\infty_T H^1_x} + \|v\|^{p-1}_{L^\infty_T H^1_x}) \|u-v\|_{L^q_\delta W^{1,r}_x}$$

But if we set

$$K = \max(\|u\|_{L^\infty_T H^1_x} + \|v\|_{L^\infty_T H^1_x}) < \infty$$

then for δ small enough we obtain

$$\|u(t)-v(t)\|_{L^q_\delta W^{1,r}_x} + \|u(t)-v(t)\|_{L^\infty_\delta H^1_x} \leq \frac{1}{2}(\|u(t)-v(t)\|_{L^q_\delta W^{1,r}_x} + \|u(t)-v(t)\|_{L^\infty_\delta H^1_x})$$

which forces $u = v$ on $[0,\delta]$. To cover the whole $[0,T]$ we iterate the previous argument $\frac{T}{\delta}$ times.

Membership in the Strichartz space. The fact that

$$u \in L^\gamma_{loc}((0,T^*);\ W^{1,\rho}_x(\mathbb{R}^n))$$

for every admissible pair (γ,ρ), follows from the Strichartz estimates on any compact interval inside $(0,T^*)$.

Blow-up alternative. The proof is the same as in the smooth case.

Continuous dependence can be obtained via establishing estimates on

$$\|u_n(t) - u(t)\|_{L^q_t W^{1,r}_x} + \|u_n(t) - u(t)\|_{L^\infty_t H^1_x}.$$

We skip details and refer the interested reader to [4].

Conservation laws. The proof of the conservation of mass is similar to the smooth case but now we use the pairing $(u_t, u)_{H^1-H^{-1}}$. A proof of conservation of energy is more involved since we need more derivatives to make sense of the energy functional. Details can be found in e.g. [4]. □

Remark 3.12. *We pause to give a couple of comments:*

(1) *Notice that when $\lambda = -1$ (defocusing case), the mass and energy conservation provide a global a priori bound*
$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} \le C_{M(u_0),E(u_0)}.$$
By the blow-up alternative we then have that $T^* = \infty$ and the problem is globally well-posed (gwp).

(2) *Let $I = [0, T]$. An inspection of the proof reveals that we can run the lwp argument in the space $\mathcal{S}^1(I \times \mathbb{R}^n)$ with the norm*
$$\|u\|_{\mathcal{S}^1(I\times\mathbb{R}^n)} = \|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} + \|\nabla u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)}$$
where
$$\|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} = \sup_{(q,r)-admissible} \|u\|_{L^q_{t\in I}L^r_x}.$$

3.5. **Well-posedness for the L^2 sub-critical problem.** We now state the lwp and gwp theory for the L^2 sub-critical problem. The reader can consult e.g. [49] for details.

Theorem 3.13. *Consider $1 < p < 1 + \frac{4}{n}$, $n \ge 1$ and an admissible pair (q, r) with $p + 1 < q$. Then for every $u_0 \in L^2(\mathbb{R}^n)$ there exists a unique strong solution of*
$$\begin{cases} iu_t + \Delta u + \lambda|u|^{p-1}u = 0, \\ u(x, 0) = u_0(x) \end{cases} \tag{3.28}$$
defined on the maximal interval $(0, T^\star)$ such that
$$u \in C^0_t((0, T^\star); L^2(\mathbb{R}^n)) \cap L^q_{loc}((0, T^\star); L^r(\mathbb{R}^n)).$$
Moreover
$$u \in L^\gamma_{loc}((0, T^\star); L^\rho(\mathbb{R}^n))$$
for every admissible pair (γ, ρ). In addition
$$\lim_{t\to T^\star} \|u(t)\|_{L^2} = \infty$$
if $T^\star < \infty$ and u depends continuously on u_0 in the following sense: There exists $T > 0$ depending on $\|u_0\|_{L^2}$ such that if $u_{0,n} \to u_0$ in L^2 and $u_n(t)$ is the corresponding solution of (3.28), then $u_n(t)$ is defined on $[0, T]$ for n sufficiently large and
$$u_n(t) \to u(t) \quad in \quad L^\gamma_{loc}([0, T];\ L^\rho(\mathbb{R}^n)) \tag{3.29}$$
for every admissible pair (γ, ρ) and every compact interval $[0, T]$ of $(0, T^\star)$. Finally we have that
$$M(u)(t) = \|u(t)\|_{L^2} = \|u_0\|_{L^2} = M(u_0) \quad and\ thus\ T^\star = \infty. \tag{3.30}$$

Remark 3.14. *We give a couple of comments:*

(1) *Notice that global well-posedness follows immediately.*
(2) *The equation makes sense in H^{-2}.*

Finally we state the L^2-critical lwp theory when $p = 1 + \frac{4}{n}$, [5]. We should mention that a similar theory holds for the H^1 critical problem ($p = 1 + \frac{4}{n-2}$), [5]. For dimensions $n = 1, 2$ the problem is always energy sub-critical.

Theorem 3.15. *Consider $p = 1 + \frac{4}{n}$, $n \geq 1$. Then for every $u_0 \in L^2(\mathbb{R}^n)$ there exists a unique strong solution of*

$$\begin{cases} iu_t + \Delta u + \lambda |u|^{\frac{4}{n}} u = 0, \\ \quad u(x, 0) = u_0(x) \end{cases} \tag{3.31}$$

defined on the maximal interval $(0, T^\star)$ such that

$$u \in C_t^0((0, T^\star); L^2(\mathbb{R}^n)) \cap L_{loc}^{p+1}((0, T^\star); L^{p+1}(\mathbb{R}^n)).$$

Moreover

$$u \in L_{loc}^{\gamma}((0, T^\star); L^{\rho}(\mathbb{R}^n))$$

for every admissible pair (γ, ρ). In addition if $T^\star < \infty$

$$\lim_{t \to T^\star} \|u(t)\|_{L^q_{loc}((0,T^\star);L^r(\mathbb{R}^n))} = \infty$$

for every admissible pair (q, r) with $r \geq p + 1$. u also depends continuously on u_0 in the following sense: If $u_{0,n} \to u_0$ in L^2 and $u_n(t)$ is the corresponding solution of (3.31), then $u_n(t)$ is defined on $[0, T]$ for n sufficiently large and

$$u_n(t) \to u(t) \quad \text{in} \quad L^q([0, T]); \; L^r(\mathbb{R}^n)) \tag{3.32}$$

for every admissible pair (q, r) and every compact interval $[0, T]$ of $(0, T^\star)$. Finally we have that

$$M(u)(t) = \|u(t)\|_{L^2} = \|u_0\|_{L^2} = M(u_0) \quad \text{for all } t \in (0, T^\star). \tag{3.33}$$

Remark 3.16. *Again, we give a few comments:*

(1) *Notice that the blow-up alternative in this case is not in terms of the L^2 norm, which is the conserved quantity of the problem. This is because the problem is critical and the time of local well-posedness depends not only on the norm but also on the profile of the initial data. On the other hand if we have a global Strichartz bound on the solution global well-posedness is guaranteed by the Theorem. We will see later that this global Strichartz bound is sufficient for proving scattering also.*
(2) *It is easy to see that if $\|u_0\|_{L^2} < \mu$, for μ small enough, then by the Strichartz estimates*

$$\|e^{it\Delta} u_0\|_{L_t^{p+1} L_x^{p+1}(\mathbb{R} \times \mathbb{R}^n)} < C\mu < \eta.$$

Thus for sufficiently small initial data $T^\star = \infty$ and after only one iteration we have global well-posedness for the focusing or defocusing problem. In addition we have that $u \in L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^n))$ for every admissible pair (q, r) and thus we also have scattering for small data. But this is not true for large data as the following example shows.

Consider $\lambda > 0$. We know that there exists nontrivial solutions of the form

$$u(x,t) = e^{i\omega t}\phi(x)$$

where ϕ is a smooth nonzero solution of

$$-\Delta\phi + \omega\phi = |\phi|^{p-1}\phi$$

with $\omega > 0$. But

$$\|\phi\|_{L^r_x(\mathbb{R}^n)} \leq M$$

for every $r \geq 2$ and thus $u \notin L^q_t(\mathbb{R}; L^r_x(\mathbb{R}^n))$ for any $q < \infty$.

Although some recent results have appeared for super-critical equations, the theory has been completed only for the defocusing critical problem and those developments are recent. More precisely, global energy solutions for the 3d defocusing energy-critical problem with radially symmetric initial data was obtained in [3]. The radially symmetric assumption was removed in [10]. For $n \geq 4$ the problem was solved in [42, 50]. The defocusing mass-critical problem is now solved in all dimensions in a series of papers, [11, 12, 13].

To obtain global-in-time solutions for the focusing problems, as we have seen, one needs to assume a bound on the norm of the data. For the energy-critical focusing problem one can consult the work [28], where a powerful program that helped settle many critical problems, has been introduced; for higher dimensions see e.g. [29]. Results concerning the mass-critical focusing problem are obtained in [14] in all dimensions.

4. Morawetz type inequalities

To study in more details the local or global solutions of the above problems we have to revisit the symmetries of the equation. We first write down the local conservation laws or the conservation laws in differentiable form. The differential form of the conservation law is more flexible and powerful as it can be localized to any given region of space-time by integrating against a suitable cut-off function or contracting against a suitable vector fields. One then does not obtain a conserved quantity but rather a monotone quantity. Thus from a single conservation law one can generate a variety of useful estimates. We can also use these formulas to study the blow-up and concentration problems for the focusing NLS and the scattering problem for the defocusing NLS.

The question of scattering or in general the question of dispersion of the nonlinear solution is tied to weather there is some sort of decay in a certain norm, such as the L^p norm for $p > 2$. In particular knowing the exact rate of decay of various L^p norms for the linear solutions, it would be ideal to obtain estimates that establish similar rates of decay for the nonlinear problem. The decay of the linear solutions can immediately establish weak quantum scattering in the energy space but to estimate the linear and the nonlinear dynamics in the energy norm we usually looking for the L^p norm of the nonlinear solution to go to zero as $t \to \infty$.

Strichartz type estimates assure us that certain L^p norms going to zero but only for the linear part of the solution. For the nonlinear part we need to obtain general decay estimates on solutions of defocusing equations. The mass and energy conservation laws establish the boundedness of the L^2 and the H^1 norms but are insufficient to provide a decay for higher powers of Lebesgue norms. In these notes we provide a summary of recent results that demonstrate a straightforward method to obtain such estimates by taking advantage of the momentum conservation law

$$\Im \int_{\mathbb{R}^n} \overline{u} \nabla u dx = \Im \int_{\mathbb{R}^n} \overline{u_0} \nabla u_0 dx. \tag{4.1}$$

Thus we want to establish a priori estimates for the solutions to the power type nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda |u|^{p-1} u, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}^n) \end{cases} \tag{4.2}$$

for any $p > 1$ and $\lambda \in \mathbb{R}$. Equation (4.2) is the Euler-Lagrange equation for the Lagrangian density

$$L(u) = -\frac{1}{2}\Delta(|u|^2) + \lambda \frac{p-1}{p+1}|u|^{p+1}.$$

Space translation invariance leads to momentum conservation

$$\vec{p}(t) = \Im \int_{\mathbb{R}^n} \bar{u} \nabla u dx, \tag{4.3}$$

a quantity that has no definite sign. It turns out that one can also use this conservation law in the defocusing case and prove monotonicity formulas that are very useful in studying the global-in-time properties of the solutions at $t = \infty$. For most of these classical results the reader can consult [4], [46].

The study of the problem at infinity is an attempt to describe and classify the asymptotic behavior-in-time for the global solutions. To handle this issue, one tries to compare the given nonlinear dynamics with suitably chosen simpler asymptotic dynamics. For the semilinear problem (4.2), the first obvious candidate for the simplified asymptotic behavior is the free dynamics generated by the group $S(t) = e^{-it\Delta}$. The comparison between the two dynamics gives rise to the questions of the existence of wave operators and of the asymptotic completeness of the solutions. More precisely, we have:
i) Let $v_+(t) = S(t)u_+$ be the solution of the free equation. Does there exist a solution u of equation (4.2) which behaves asymptotically as v_+ as $t \to \infty$, typically in the sense that $\|u(t) - v_+\|_{H^1} \to 0, \text{ as } t \to \infty$. If this is true, then one can define the map $\Omega_+ : u_+ \to u(0)$. The map is called the wave operator and the problem of existence of u for given u_+ is referred to as the problem of the *existence of the wave operator*. The analogous problem arises as $t \to -\infty$.
ii) Conversely, given a solution u of (4.2), does there exist an asymptotic state u_+ such that $v_+(t) = S(t)u_+$ behaves asymptotically as $u(t)$, in the above sense. If that is the case for any u with initial data in X for some $u_+ \in X$, one says that *asymptotic completeness* holds in X.

In effect the existence of wave operators asks for the construction of global solutions that behave asymptotically as the solution of the free Schrödinger equation while the asymptotic completeness requires all solutions to behave asymptotically in this manner. It is thus not accidental that asymptotic completeness is a much harder problem than the existence of the wave operators (except in the case of small data theory which follows from the iterative methods of the local well-posedness theory).

Asymptotic completeness for large data not only require a repulsive nonlinearity but also some decay for the nonlinear solutions. A key example of these ideas is contained in the following generalized virial inequality, [31]:

$$\int_{\mathbb{R}^n\times\mathbb{R}}(-\Delta\Delta a(x))|u(x,t)|^2dxdt+\lambda\int_{\mathbb{R}^n\times\mathbb{R}}(\Delta a(x))|u(x,t)|^{p+1}dxdt\leq C \tag{4.4}$$

where $a(x)$ is a convex function, u is a solution to (4.2), and C a constant that depends only on the energy and mass bounds.

An inequality of this form, which we will call a one-particle inequality, was first derived in the context of the Klein-Gordon equation by Morawetz in [34], and then extended to the NLS equation in [31]. Most of these estimates are referred in the literature as Morawetz type estimates. The inequality was applied to prove asymptotic completeness first for the nonlinear Klein-Gordon and then for the NLS equation in [41], and then in [31] for slightly more regular solutions in space dimension $n\geq 3$. The case of general finite energy solutions for $n\geq 3$ was treated in [18] for the NLS and in [17] for the Hartree equation. The treatment was then improved to the more difficult case of low dimensions in [35, 36].

The bilinear a priori estimates that we outline here give stronger bounds on the solutions and in addition simplify the proofs of the results in the papers cited above. For a detailed summary of the method see [19]. In the original paper by Morawetz, the weight function that was used was $a(x)=|x|$. This choice has the advantage that the distribution $-\Delta\Delta(\frac{1}{|x|})$ is positive for $n\geq 3$. More precisely it is easy to compute that $\Delta a(x)=\frac{n-1}{|x|}$ and that

$$-\Delta\Delta a(x)=\begin{cases} 8\pi\delta(x), & \text{if} \quad n=3\\ \frac{(n-1)(n-3)}{|x|^2}, & \text{if} \quad n\geq 4.\end{cases}$$

In particular, the computation in (4.4) gives the following estimate for $n=3$ and λ positive

$$\int_{\mathbb{R}}|u(t,0)|^2dt+\int_{\mathbb{R}^3\times\mathbb{R}}\frac{|u(x,t)|^{p+1}}{|x|}dxdt\leq C. \tag{4.5}$$

Similar estimates are true in higher dimensions. The second, nonlinear term, or certain local versions of it, have played central role in the scattering theory for the nonlinear Schrödinger equation, [3], [18], [22], [31]. The fact that in 3d, the bi-harmonic operator acting on the weight $a(x)$ produces the $\delta-$measure can be exploited further. In [9], a quadratic Morawetz inequality was proved by correlating two nonlinear densities $\rho_1(x)=|u(x)|^2$ and $\rho_2(y)=|u(y)|^2$ and define as $a(x,y)$ the distance between x and y in 3d. The authors obtained an a priori estimate of the form $\int_{\mathbb{R}^3\times\mathbb{R}}|u(x,t)|^4dx\leq C$ for solutions that stay in the energy space. A frequency localized version of this estimate has been successfully implemented to remove the radial assumption of Bourgain, [3], and prove global well-posedness and scattering for the energy-critical (quintic) equation in 3d, [10]. For $n\geq 4$ new quadratic Morawetz estimates were given in [47]. Finally in [6] and in [40] these estimates were extended to all dimensions.

We should mention that taking as the weight function the distance between two points in $\mathbb{R}^n$ is not the only approach, see [7] for a recent example. Nowadays it is well understood that the bilinear Morawetz inequalities provide a unified approach for proving energy scattering for energy sub-critical solutions of the NLS when $p > 1 + \frac{4}{n}$ (L^2 super-critical nonlinearities). This last statement has been rigorously formalized only recently due to the work of the aforementioned authors, and a general exposition has been published in [19]. Sub-energy solution scattering in the same range of powers has been initiated in [9]. For the L^2-critical problem, scattering is a very hard problem, but the problem has now been resolved in a series of new papers by B. Dodson, [11, 12, 13]. For mass sub-critical solutions, scattering even in the energy space is a very hard problem, and is probably false. Nevertheless, two particle Morawetz estimates have been used for the problem of the existence (but not uniqueness) of the wave operator for mass subcritical problems, [24]. We have already mentioned their implementation to the hard problem of energy critical solutions in [3], [22], and [10]. Recent preprints have used these inequalities for the mass critical problem, [11], and the energy super-critical problem, [30]. For a frequency localized one particle Morawetz inequality and its application to the scattering problem for the mass-critical equation with radial data see [48].

We start with the equation

$$iu_t + \Delta u = \lambda |u|^{p-1} u \tag{4.6}$$

with $p \geq 1$ and $\lambda \in \mathbb{R}$. We use Einstein's summation convention throughout. According to this convention, when an index variable appears twice in a single term, once in an upper (superscript) and once in a lower (subscript) position, it implies that we are summing over all of its possible values. We will also write $\nabla_j u$ for $\frac{\partial u}{\partial x_j}$. For a function $a(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ we define $\nabla_{x,j}\ a(x,y) = \frac{\partial a(x,y)}{\partial x_j}$ and similarly for $\nabla_{x,k}\ a(x,y)$.

We define the mass density ρ and the momentum vector $\vec{p}$, by the relations

$$\rho = |u|^2, \qquad\qquad p_k = \Im(\bar{u} \nabla_k u).$$

It is well known, [4], that smooth solutions to the semilinear Schrödinger equation satisfy mass and momentum conservation. The local conservation of mass reads

$$\partial_t \rho + 2 div \vec{p} = \partial_t \rho + 2 \nabla_j p^j = 0 \tag{4.7}$$

and the local momentum conservation is

$$\partial_t p^j + \nabla^k \Big(\delta^j_k \big(-\frac{1}{2} \Delta \rho + \lambda \frac{p-1}{p+1} |u|^{p+1} \big) + \sigma^j_k \Big) = 0 \tag{4.8}$$

where the symmetric tensor σ_{jk} is given by

$$\sigma_{jk} = 2\Re(\nabla_j u \nabla_k \overline{u}).$$

Notice that the term $\lambda \frac{p-1}{p+1} |u|^{p+1}$ is the only nonlinear term that appears in the expression. One can express the local conservation laws purely in terms of the mass density ρ and the momentum $\vec{p}$ if we write

$$\lambda\frac{p-1}{p+1}|u|^{p+1} = 2^{\frac{p+1}{2}}\lambda\frac{p-1}{p+1}\rho^{\frac{p+1}{2}}$$

and

$$\sigma_{jk} = 2\Re(\nabla_j u\nabla_k\overline{u}) = \frac{1}{\rho}(2p_jp_k + \frac{1}{2}\nabla_j\rho\nabla_k\rho),$$

but we will not use this formulation in these notes.

4.1. **One particle Morawetz inequalities.** We are ready to state the main theorem of this section:

Theorem 4.1. [6, 9, 40, 47] *Consider $u \in C_t(\mathbb{R}; C_0^\infty(\mathbb{R}^n))$ a smooth and compactly supported solution to (4.6) with $u(x,0) = u(x) \in C_0^\infty(\mathbb{R}^n)$. Then for $n \geq 2$ we have that*

$$C\|D^{-\frac{n-3}{2}}(|u|^2)\|^2_{L_t^2L_x^2} + (n-1)\lambda\frac{p-1}{p+1}\int_{\mathbb{R}_t}\int_{\mathbb{R}_x^n\times\mathbb{R}_y^n}\frac{|u(y,t)|^2|u(x,t)|^{p+1}}{|x-y|}dxdydt$$
$$\leq \|u_0\|^2_{L^2}\sup_{t\in\mathbb{R}}|M_y(t)|,$$

where

$$M_y(t) = \int_{\mathbb{R}^n}\frac{x-y}{|x-y|}\cdot\Im\big(\overline{u}(x)\nabla u(x)\big)dx,$$

D^α is defined on the Fourier side as $\widehat{D^\alpha f}(\xi) = |\xi|^\alpha\widehat{u}(\xi)$ for any $\alpha \in \mathbb{R}$ and C is a positive constant that depends only on n, [45]*. For $n = 1$ the estimate is*

$$\|\partial_x(|u|^2)\|^2_{L_t^2L_x^2} + \lambda\frac{p-1}{p+1}\|u\|^{p+3}_{L_t^{p+3}L_x^{p+3}} \leq \frac{1}{2}\|u_0\|^3_{L^2}\sup_{t\in\mathbb{R}}\|\partial_x u\|_{L^2}.$$

Remarks on Theorem 4.1.

1. By the Cauchy-Schwarz inequality it follows that for any $n \geq 2$,

$$\sup_{0,t}|M_y(t)| \lesssim \|u_0\|_{L^2}\sup_{t\in\mathbb{R}}\|\nabla u(t)\|_{L^2}.$$

A variant of Hardy's inequality gives

$$\sup_{0,t}|M_y(t)| \lesssim \sup_{t\in\mathbb{R}}\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}},$$

For details, see [19].

2. Concerning our main theorem, we note that both the integrated functions in the second term on the left hand side of the inequalities are positive. Thus when $\lambda > 0$, which corresponds to the defocusing case, and for H^1 data say, we obtain for $n \geq 2$:

$$\|D^{-\frac{n-3}{2}}(|u|^2)\|_{L_t^2L_x^2} \lesssim \|u_0\|^{\frac{3}{2}}_{L^2}\sup_{t\in\mathbb{R}}\|\nabla u(t)\|^{\frac{1}{2}}_{L^2} \lesssim M(u_0)^{\frac{3}{2}}E(u_0)^{\frac{1}{2}},$$

and for $n = 1$

$$\|\partial_x(|u|^2)\|^2_{L_t^2L_x^2} \lesssim \|u_0\|^{\frac{3}{2}}_{L^2}\sup_{t\in\mathbb{R}}\|\partial_x u(t)\|^{\frac{1}{2}}_{L^2} \lesssim M(u_0)^{\frac{3}{2}}E(u_0)^{\frac{1}{2}}.$$

These are easy consequences of the conservation laws of mass (2.3) and energy (2.2). They provide the global a priori estimates that are used in quantum scattering in the energy space, [19].

3. Analogous estimates hold for the case of the Hartree equation $iu_t + \Delta u = \lambda(|x|^{-\gamma} \star |u|^2)u$ when $0 < \gamma < n$, $n \geq 2$. For the details, see [24]. We should point out that for $0 < \gamma \leq 1$ scattering fails for the Hartree equation, [23], and thus the estimates given in [24] for $n \geq 2$ cover all the interesting cases.

4. Take $\lambda > 0$. The expression

$$\|D^{-\frac{n-3}{2}}(|u|^2)\|_{L_t^2 L_x^2},$$

for $n = 3$, provides an estimate for the $L_t^4 L_x^4$ norm of the solution. For $n = 2$ by Sobolev embedding one has that

$$\|u\|^2_{L_t^4 L_x^8} = \||u|^2\|_{L_t^2 L_x^4} \lesssim \|D^{\frac{1}{2}}(|u|^2)\|_{L_t^2 L_x^2} \lesssim C_{M(u_0),E(u_0)}.$$

For $n \geq 4$ the power of the D operator is negative but some harmonic analysis and interpolation with the trivial inequality

$$\|D^{\frac{1}{2}} u\|_{L_t^\infty L_x^2} \lesssim \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}$$

provides an estimate in a Strichartz norm. For the details see [47].

5. In the defocusing case all the estimates above give a priori information for the $\dot{H}^{\frac{1}{4}}$-critical Strichartz norm. We remind the reader that the $\dot{H}^s$-critical Strichartz norm is $\|u\|_{L_t^q L_x^r}$ where the pair (q, r) satisfies $\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - s$. In principle the correlation of k particles will provide a priori information for the $\dot{H}^{\frac{1}{2k}}$ critical Strichartz norm. In 1d an estimate that provides a bound on the $\dot{H}^{\frac{1}{8}}$ critical Strichartz norm has been given in [8].

6. To make our presentation easier we considered smooth solutions of the NLS equation. To obtain the estimates in Theorem 4.1 for arbitrary H^1 functions we have to regularize the solutions and then take a limit. The process is described in [19].

7. A more general bilinear estimate can be proved if one correlates two different solutions (thus considering different density functions ρ_1 and ρ_2). Unfortunately, one can obtain useful estimates only for $n \geq 3$. The proof is based on the fact that $-\Delta^2|x|$ is a positive distribution only for $n \geq 3$. For details the reader can check [9]. Our proof shows that the diagonal case when $\rho_1 = \rho_2 = |u|^2$ provides useful monotonicity formulas in all dimensions.

Proof. We define the Morawetz action centered at zero by

$$M_0(t) = \int_{\mathbb{R}} \nabla a(x) \cdot \vec{p}(x)\, dx, \tag{4.9}$$

where the weight function $a(x) : \mathbb{R}^n \to \mathbb{R}$ is for the moment arbitrary. The minimal requirements on $a(x)$ call for the matrix of the second partial derivatives $\partial_j \partial_k a(x)$

to be positive definite. Throughout our paper we will take $a(x) = |x|$, but many estimates can be given with different weight functions, see for example [7] and [29]. If we differentiate the Morawetz action with respect to time we obtain:

$$\partial_t M_0(t) = \int_{\mathbb{R}^n} \nabla a(x) \cdot \partial_t \vec{p}(x)\ dx = \int_{\mathbb{R}^n} \nabla_j a(x) \partial_t p^j(x)\ dx$$

$$= \int_{\mathbb{R}^n} \big(\nabla_j \nabla^k a(x)\big) \delta_k^j \big(-\frac{1}{2}\Delta\rho + \lambda \frac{p-1}{p+1}|u|^{p+1}\big)dx + 2\int_{\mathbb{R}^n} \big(\nabla_j \nabla^k a(x)\big) \Re\big(\nabla^j \overline{u} \nabla_k u\big)dx,$$

where we use equation (4.8). We rewrite and name the equation as follows

$$\partial_t M_0(t) = \int_{\mathbb{R}^n} \Delta a(x) \big(-\frac{1}{2}\Delta\rho + \lambda \frac{p-1}{p+1}|u|^{p+1}\big)dx + 2\int_{\mathbb{R}^n} \big(\nabla_j \nabla^k a(x)\big) \Re\big(\nabla^j \overline{u} \nabla_k u\big)dx. \tag{4.10}$$

Notice that for $a(x) = |x|$ the matrix $\nabla_j \nabla_k a(x)$ is positive definite and the same is true if we translate the weight function by any point $y \in \mathbb{R}^n$ and consider $\nabla_{x,j} \nabla^{x,k} a(x-y)$ for example. That is for any vector function on $\mathbb{R}^n$, $\{v_j(x)\}_{j=1}^n$, with values on $\mathbb{R}$ or $\mathbb{C}$ we have that

$$\int_{\mathbb{R}^n} \big(\nabla_j \nabla^k a(x)\big) v^j(x) v_k(x) dx \geq 0.$$

To see this, observe that for $n \geq 2$ we have $\nabla_j a = \frac{x_j}{|x|}$ and $\nabla_j \nabla_k a = \frac{1}{|x|}\big(\delta_{kj} - \frac{x_j x_k}{|x|^2}\big)$. Summing over $j = k$ we obtain $\Delta a(x) = \frac{n-1}{|x|}$. Then

$$\nabla_j \nabla^k a(x) v^j(x) v_k(x) = \frac{1}{|x|}\big(\delta_j^k - \frac{x_j x^k}{|x|^2}\big) v^j(x) v_k(x) = \frac{1}{|x|}\Big(|\vec{v}(x)|^2 - \big(\frac{x \cdot \vec{v}(x)}{|x|}\big)^2\Big) \geq 0$$

by the Cauchy-Schwarz inequality. Notice that it does not matter if the vector function is real or complex valued for this inequality to be true. In dimension one (4.10) simplifies to

$$\partial_t M_0(t) = \int_{\mathbb{R}} a_{xx}(x) \big(-\frac{1}{2}\Delta\rho + \lambda \frac{p-1}{p+1}|u|^{p+1} + 2|u_x|^2\big)dx. \tag{4.11}$$

In this case for $a(x) = |x|$, we have that $a_{xx}(x) = 2\delta(x)$. Since the identity (4.10) does not change if we translate the weight function by $y \in \mathbb{R}^n$ we can define the Morawetz action with center at $y \in \mathbb{R}^n$ by

$$M_y(t) = \int_{\mathbb{R}^n} \nabla a(x-y) \cdot \vec{p}(x)\ dx.$$

We can then obtain like before

$$\partial_t M_y(t) = \int_{\mathbb{R}^n} \Delta_x a(x-y) \big(-\frac{1}{2}\Delta\rho + \lambda \frac{p-1}{p+1}|u|^{p+1}\big)dx \tag{4.12}$$

$$+2\int_{\mathbb{R}^n} \big(\nabla_{x,j} \nabla^{x,k} a(x-y)\big) \Re\big(\nabla^{x,j} \overline{u} \nabla_{x,k} u\big)dx. \tag{4.13}$$

Recall that

$$\partial_t M_0 = \int_{\mathbb{R}^n} \Delta a(x) \left(\frac{\lambda(p-1)}{p+1}|u|^{p+1} - \frac{1}{2}\Delta\rho\right) dx + \int_{\mathbb{R}^n} (\partial_j \partial^k a(x)) \sigma_k^j dx$$

for a general weight function $a(x)$.
If we pick $a(x) = |x|^2$, then $\Delta a(x) = 2n$ and $\partial_j \partial_k a(x) = 2\delta_{kj}$. Therefore

$$\begin{aligned}
\partial_t M &= \frac{2n\lambda(p-1)}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx + 2 \int_{\mathbb{R}^n} |\nabla u|^2 dx \\
&= 8 \left(\frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{\lambda}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx \right) - \frac{2\lambda}{p+1} \left(4 - n(p-1)\right) \int_{\mathbb{R}^n} |u|^{p+1} dx \\
&= 8E(u(t)) - \frac{2\lambda}{p+1} \left(4 - n(p-1)\right) \int_{\mathbb{R}^n} |u|^{p+1} dx. \qquad (4.14)
\end{aligned}$$

Thus if we define the quantity

$$V(t) = \int_{\mathbb{R}^n} a(x)\rho(x) dx,$$

with $a(x) = |x|^2$, we have that

$$\partial_t V(t) = \int_{\mathbb{R}^n} a(x) \partial_t \rho(x) dx = -2 \int_{\mathbb{R}^n} a(x) \ \nabla \cdot \vec{p} \ dx = 2M(t) \qquad (4.15)$$

using integration by parts. Thus

$$\partial_t^2 V(t) = 16E(u(t)) - \frac{4\lambda}{p+1} \left(4 - n(p-1)\right) \int_{\mathbb{R}^n} |u|^{p+1} dx. \qquad (4.16)$$

Another useful calculation is the following. Set

$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 + \frac{8t^2\lambda}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx.$$

Then we have:

$$\begin{aligned}
K(t) &= \|xu\|_{L^2}^2 + 4t^2 \|\nabla u\|_{L^2}^2 - 4t \int_{\mathbb{R}^n} x \cdot p \ dx + \frac{8t^2\lambda}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx \\
&= \int_{\mathbb{R}^n} a(x)\rho(x) dx + 8t^2 E(u(t)) - 2t \int_{\mathbb{R}^n} \nabla a \cdot p \ dx \\
&= \int_{\mathbb{R}^n} a(x)\rho(x) dx + 8t^2 E(u_0) - 2t \int_{\mathbb{R}^n} \nabla a \cdot p \ dx, \qquad (4.17)
\end{aligned}$$

with $a(x) = |x|^2$. However

$$\partial_t \int_{\mathbb{R}^n} a(x)\rho(x) dx = \int_{\mathbb{R}^n} \nabla a \cdot p \ dx$$

and thus

$$\partial_t K(t) = -2t \int_{\mathbb{R}^n} \partial_j a(x) \partial_t p^j dx + 16tE(u_0) = -2t \partial_t M(t) + 16tE(u_0).$$

If we use (4.14) we have that

$$\partial_t K(t) = \frac{4\lambda t}{p+1} \left(4 - n(p-1)\right) \int_{\mathbb{R}^n} |u|^{p+1} dx.$$

Notice that for $p = 1 + \frac{4}{n}$, the quantity $K(t)$ is conserved.

4.2. **Two particle Morawetz inequalities.** We now define the two-particle Morawetz action

$$M(t)=\int_{\mathbb{R}^n_y}|u(y)|^2 M_y(t)\ dy$$

and differentiate with respect to time. Using the identity above and the local conservation of mass law we obtain four terms

$$\begin{aligned}\partial_t M(t) &= \int_{\mathbb{R}^n_y}|u(y)|^2\partial_t M_y(t)\ dy+\int_{\mathbb{R}^n_y}\partial_t\rho(y)M_y(t)\ dy\\ &= \int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}|u(y)|^2\Delta_x a(x-y)\big(-\frac{1}{2}\Delta\rho+\lambda\frac{p-1}{p+1}|u|^{p+1}\big)dxdy\\ &+2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}|u(y)|^2\big(\nabla_{x,j}\nabla^{x,k}a(x-y)\big)\Re\big(\nabla^{x,j}\overline{u}\nabla_{x,k}u\big)dxdy\\ &-2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla^{y,j}p_j(y)\nabla_{x,k}a(x-y)p^k(x)dxdy\\ &= I+II+III+2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}p_j(y)\nabla^{y,j}\nabla_{x,k}a(x-y)p^k(x)dxdy\end{aligned}$$

by integration by parts with respect to the $y-$variable. Since

$$\nabla^{y,j}\nabla_{x,k}a(x-y)=-\nabla^{x,j}\nabla_{x,k}a(x-y)$$

we obtain that

$$\begin{aligned}\partial_t M(t) &= I+II+III-2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla^{x,j}\nabla_{x,k}a(x-y)p_j(y)p^k(x)dxdy \qquad (4.18)\\ &= I+II+III+IV\end{aligned}$$

where

$$I=\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}|u(y)|^2\Delta_x a(x-y)\big(-\frac{1}{2}\Delta\rho\big)dxdy,$$

$$II=\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}|u(y)|^2\Delta_x a(x-y)\big(\lambda\frac{p-1}{p+1}|u|^{p+1}\big)dxdy,$$

$$III=2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}|u(y)|^2\big(\nabla_{x,j}\nabla^{x,k}a(x-y)\big)\Re\big(\nabla^{x,j}\overline{u}\nabla_{x,k}u\big)dxdy,$$

$$IV=-2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla^{x,j}\nabla_{x,k}a(x-y)p_j(y)p^k(x)dxdy.$$

Claim: $III+IV\geq 0$. Assume the claim. Since $\Delta_x a(x-y)=\frac{n-1}{|x-y|}$ we have that

$$\partial_t M(t)\geq\frac{n-1}{2}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\frac{|u(y)|^2}{|x-y|}\big(-\Delta\rho\big)dxdy+(n-1)\lambda\frac{p-1}{p+1}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\frac{|u(y)|^2}{|x-y|}|u(x)|^{p+1}dxdy.$$

But recall that on one hand we have that $-\Delta=D^2$ and on the other that the distributional Fourier transform of $\frac{1}{|x|}$ for any $n\geq 2$ is $\frac{c}{|\xi|^{n-1}}$ where c is a positive constant depending only on n. Thus we can define

$$D^{-(n-1)}f(x)=c\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|}dy$$

and express the first term as

$$\frac{n-1}{2}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\frac{|u(y)|^2}{|x-y|}(-\Delta\rho)dxdy = c\frac{n-1}{2} < D^{-(n-1)}|u|^2,\ D^2|u|^2 >= C\|D^{-\frac{n-3}{2}}|u|^2\|^2_{L^2_x}$$

by the usual properties of the Fourier transform for positive and real functions. Integrating from 0 to t we obtain the theorem in the case that $n\geq 2$.

Proof of the claim: Notice that

$$III{+}IV = 2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x{-}y)\Big(|u(y)|^2\Re\big(\nabla^{x,j}\overline{u}(x)\nabla_{x,k}u(x)\big){-}p^j(y)p_k(x)\Big)dxdy$$

$$= 2\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x{-}y)\Big(\frac{\rho(y)}{\rho(x)}\Re\big(u(x)(\nabla^{x,j}\overline{u}(x))\overline{u}(x)(\nabla_{x,k}u(x))\big){-}p^j(y)p_k(x)\Big)dxdy.$$

Since

$$\nabla_{x,j}\nabla_{x,k}a(x-y) = \nabla_{y,j}\nabla_{y,k}a(y-x)$$

by exchanging the roles of x and y we obtain the same inequality and thus

$$III{+}IV = \int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x{-}y)\Big(\frac{\rho(y)}{\rho(x)}\Re\big(u(x)(\nabla^{x,j}\overline{u}(x))\overline{u}(x)(\nabla_{x,k}u(x))\big){-}p^j(y)p_k(x)$$

$$+\frac{\rho(x)}{\rho(y)}\Re\big(u(y)(\nabla^{y,j}\overline{u}(y))\overline{u}(y)(\nabla_{y,k}u(y))\big) - p^j(x)p_k(y)\Big)dxdy.$$

Now set $z_1 = \overline{u}(x)\nabla_{x,k}u(x)$ and $z_2 = \overline{u}(x)\nabla^{x,j}u(x)$ and apply the identity

$$\Re(z_1\bar{z}_2) = \Re(z_1)\Re(z_2) + \Im(z_1)\Im(z_2)$$

to obtain

$$\Re\big(u(x)(\nabla^{x,j}\overline{u}(x))\overline{u}(x)(\nabla_{x,k}u(x))\big) = \Re\big(\overline{u}(x)\nabla_{x,k}u(x)\big)\Re\big(\overline{u}(x)\nabla^{x,j}u(x)\big)$$

$$+\Im\big(\overline{u}(x)\nabla_{x,k}u(x)\big)\Im\big(\overline{u}(x)\nabla^{x,j}u(x)\big) = \frac{1}{4}\nabla_{x,k}\rho(x)\nabla^{x,j}\rho(x) + p_k(x)p^j(x)$$

and similarly

$$\Re\big(u(y)(\nabla^{y,j}\overline{u}(y))\overline{u}(y)(\nabla_{y,k}u(y))\big) = \frac{1}{4}\nabla_{y,k}\rho(y)\nabla^{y,j}\rho(y) + p_k(y)p^j(y).$$

Thus

$$III + IV = \frac{1}{4}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x-y)\frac{\rho(y)}{\rho(x)}\nabla_{x,k}\rho(x)\nabla^{x,j}\rho(x)dxdy$$

$$+\frac{1}{4}\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{y,j}\nabla^{y,k}a(x-y)\frac{\rho(x)}{\rho(y)}\nabla_{y,k}\rho(y)\nabla^{y,j}\rho(y)dxdy$$

$$+\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{y,j}\nabla^{y,k}a(x{-}y)\Big(\frac{\rho(y)}{\rho(x)}p_k(x)p^j(x){+}\frac{\rho(x)}{\rho(y)}p_k(y)p^j(y){-}p_k(x)p^j(y){-}p_k(y)p^j(x)\Big)dxdy.$$

Since the matrix $\nabla_{x,j}\nabla^{x,k}a(x-y) = \nabla_{y,j}\nabla^{y,k}a(x-y)$ is positive definite, the first two integrals are positive. Thus,

$$III + IV \geq$$

$$\int_{\mathbb{R}^n_y\times\mathbb{R}^n_x}\nabla_{x,j}\nabla^{x,k}a(x{-}y)\Big(\frac{\rho(y)}{\rho(x)}p_k(x)p^j(x){+}\frac{\rho(x)}{\rho(y)}p_k(y)p^j(y){-}p_k(x)p^j(y){-}p_k(y)p^j(x)\Big)dxdy.$$

Now if we define the two point vector

$$J_k(x,y) = \sqrt{\frac{\rho(y)}{\rho(x)}}p_k(x) - \sqrt{\frac{\rho(x)}{\rho(y)}}p_k(y)$$

we obtain that

$$III + IV \geq \int_{\mathbb{R}^n_y \times \mathbb{R}^n_x} \nabla_{x,j}\nabla^{x,k} a(x-y) J^j(x,y) J_k(x,y) dxdy \geq 0$$

and we are done.

The proof when $n = 1$ is easier. First, an easy computation shows that if $a(x,y) = |x-y|$ then $\partial_{xx} a(x,y) = 2\delta(x-y)$. In this case from (4.18) we obtain

$$\partial_t M(t) = \int_{\mathbb{R}_y \times \mathbb{R}_x} |u(y)|^2 2\delta(x-y)\big(-\frac{1}{2}\rho_{xx}\big)dxdy + 2\int_{\mathbb{R}} |u(x)|^2\big(\lambda\frac{p-1}{p+1}|u(x)|^{p+1}\big)dx$$

$$+4\int_{\mathbb{R}} |u(x)|^2|u_x|^2 dx - 4\int_{\mathbb{R}} p^2(x)dx.$$

But

$$\int_{\mathbb{R}_y \times \mathbb{R}_x} |u(y)|^2 2\delta(x-y)\big(-\frac{1}{2}\rho_{xx}\big)dxdy = \int_{\mathbb{R}} \Big(\partial_x |u(x)|^2\Big)^2 dx.$$

In addition a simple calculation shows that

$$|u(x)|^2|u_x|^2 = \Big(\Re(\overline{u}u_x)\Big)^2 + \Big(\Im(\overline{u}u_x)\Big)^2 = \frac{1}{4}\Big(\partial_x|u|^2\Big)^2 + p^2(x).$$

Thus

$$4|u(x)|^2|u_x|^2 - 4p^2(x) = \Big(\partial_x|u|^2\Big)^2$$

and the identity becomes

$$\partial_t M(t) = 2\int_{\mathbb{R}} \Big(\partial_x|u|^2\Big)^2 dx + 2\int_{\mathbb{R}} |u(x)|^2\big(\lambda\frac{p-1}{p+1}|u(x)|^{p+1}\big)dx \tag{4.19}$$

which finishes the proof of the theorem. □

5. Applications.

In this section we present a few applications of the decay estimates that were established in Section 4.

5.1. Blow-up for the energy sub-critical and mass (super)-critical problem . We show a criterion for blow-up for the energy subcritical and mass critical or super-critical

$$1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$$

focusing ($\lambda = 1$) problem which is due to Zakharov and Glassey. In our presentation we follow [21]. In addition we assume that our data have some decay (which will be specified below).

From the lwp theory we have a well-defined solution in $(0, T^\star)$ of the following initial value problem:

$$\begin{cases} iu_t + \Delta u + |u|^{p-1}u = 0, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n,\ |x|^2 dx), \end{cases} \tag{5.1}$$

for any $1 + \frac{4}{n} \leq p < 1 + \frac{4}{n-2}$ when $n \geq 3$, and for $1 + \frac{4}{n} \leq p < \infty$ when $n = 1, 2$.

Recall from Section 4 that for the variance, which was introduced as follows:

$$V(t) = \int_{\mathbb{R}^n} |x|^2 |u(x,t)|^2 dx,$$

we calculated that (see (4.15) and (4.16) and expressions leading to them):

$$\partial_t V(t) = 2M(t), \tag{5.2}$$

where

$$M(t) = \int_{\mathbb{R}^n} \vec{x} \cdot \vec{p}\ dx = \int_{\mathbb{R}^n} \vec{x} \cdot \Im(\bar{u} \nabla u)\ dx,$$

and

$$\partial_t^2 V(t) = 16E(u(t)) + \frac{4}{p+1}\left(4 - n(p-1)\right) \int_{\mathbb{R}^n} |u|^{p+1} dx. \tag{5.3}$$

Hence (5.3) together with conservation of energy and the fact that $p \geq 1 + \frac{4}{n}$, implies:

$$\partial_t^2 V(t) \leq 16E(u_0),$$

which we can integrate twice to obtain:

$$\begin{aligned} V(t) &\leq 8t^2 E(u_0) + tV'(0) + V(0) \\ &= 8t^2 E(u_0) + 2tM(0) + V(0) \\ &= 8t^2 E(u_0) + 4t \int_{\mathbb{R}^n} \vec{x} \cdot \Im(\overline{u_0} \nabla u_0)\ dx + \|xu_0\|_{L^2}^2. \end{aligned} \tag{5.4}$$

Since

$$u_0 \in \Sigma = H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n,\ |x|^2 dx),$$

the coefficients of the second degree polynomial in t on the right hand side of (5.4) are finite. Now if the initial data have negative energy, that is if

$$E(u_0) < 0,$$

the coefficient of t^2 is negative. On the other hand, for all times

$$V(t) = \int_{\mathbb{R}^n} |x|^2 |u(x,t)|^2 dx \geq 0.$$

Therefore $V(t)$ starts with a positive value $V(0)$ and at some finite time the second order polynomial $V(t)$ will cross the horizontal axis. Thus $T^\star$ is finite. By the blow-up alternative of the lwp theory this gives that

$$\lim_{t \to T^\star} \|u(t)\|_{H^1} = \infty,$$

if in addition to $u_0 \in H^1$, we have that $\|xu_0\|_{L^2} < \infty$ and $E(u_0) < 0$.

Remark 5.1. *We make a few comments:*

(1) *Note that the assumption $E(u_0) < 0$ is a sufficient condition for finite-time blow-up, but it is not necessary. One can actually prove that for any $E_0 > 0$ there exists u_0 with $E(u_0) = E_0$ and $T^\star < \infty$. For details consult* [4].

(2) *One can reasonably ask whether she can prove the same result for H^1 data? The authors in* [38] *prove such a result with the additional assumption of radial symmetry for any $n \geq 2$. For the L^2-critical case ($p = 1 + \frac{4}{n}$) the radial assumption is not needed. See the papers* [39, 20, 37] *for details.*

(3) *Many results have been devoted to the rate of the blow-up for the focusing problem. A variant of the local well-posedness theory provides the following result:*
If $u_0 \in H^1$ and $T^\star < \infty$, then there exists a $\delta > 0$ such that for all $0 \leq t < T^\star$ we have that
$$\|\nabla u(t)\|_{L^2} \geq \frac{\delta}{(T^\star - t)^{\frac{1}{p-1} - \frac{n-2}{4}}}.$$
Note that the above gives a lower estimate but not an upper estimate. The authors in [33] *have provided an upper estimate for the L^2-critical case that is very close to the one above.*

5.2. **Global Well-Posedness for the L^2-critical problem.** We have seen that in the mass-critical case when $p = 1 + \frac{4}{n}$ the local existence time depends not only on the norm of the initial data but also on the profile. This prevents the use of the conservation of mass law in order to extend the solutions globally, even in the defocusing case ($\lambda = -1$).

5.2.1. *Defocusing problem under the finite variance assumption.* In the case when $\lambda < 0$, the conjecture was (for a long time) that $T^\star = \infty$. Although the conjecture is proven to be true in [11, 12, 13], in these notes we present a positive answer to an easier problem where we consider the corresponding problem for H^1 data (that can be large), but in addition we assume finiteness of the variance. This scenario can be analyzed using methods of Section 4 and as such it fits well into the flow of our presentation.

Recall that
$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 + \frac{8t^2}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx$$
is a conserved quantity for $p = 1 + \frac{4}{n}$. Thus
$$K(t) = \|(x + 2it\nabla)u\|_{L^2}^2 + \frac{8t^2}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx = \|xu_0\|_{L^2}^2.$$
We approximate the data with an H^1 sequence such that $u_{0,n} \to u_0$ in L^2 and have finite variance. The corresponding solutions satisfy $u_n \in C(\mathbb{R}, H^1(\mathbb{R}^n))$ and $xu_n \in C(\mathbb{R}, L^2(\mathbb{R}^n))$. The conservation law for $K(t)$ implies that
$$\frac{8t^2}{p+1}\int_{\mathbb{R}^n} |u_n|^{p+1} dx \leq C$$
and thus
$$\int_{\mathbb{R}^n} |u_n|^{2+\frac{4}{n}} dx \leq \frac{C}{t^2}$$
for all $t \in (0, T^\star)$. By continuous dependence this implies that
$$\int_{\mathbb{R}^n} |u(x,t)|^{2+\frac{4}{n}} dx \leq \frac{C}{t^2}$$

for a.a. $t \in (0, T^\star)$. Thus if $T^\star < \infty$ one can integrate the above quantity from any $t < T^\star$ to $T^\star$ and obtain that

$$\int_t^{T^\star} \int_{\mathbb{R}^n} |u(x,t)|^{2+\frac{4}{n}} dx dt < C.$$

Since on the other hand we have that

$$u \in L_t^{2+\frac{4}{n}}((0,t); L_x^{2+\frac{4}{n}})$$

we conclude

$$L_t^{2+\frac{4}{n}}((0,T^\star); L_x^{2+\frac{4}{n}}) < \infty.$$

But this contradicts the blow-up alternative for this problem and thus $T^\star = \infty$. Actually since the L^2 Strichartz norm $L_t^{2+\frac{4}{n}} L_t^{2+\frac{4}{n}}$ is bounded we also have scattering (more on that later).

5.2.2. *Focusing problem.* Now let us derive a global well-posedness condition for the focusing equation

$$iu_t + \Delta u + |u|^{\frac{4}{n}} u = 0. \tag{5.5}$$

We have already seen that for small enough L^2 data the problem, focusing or defocusing, has global solutions. We have also mentioned the result in [14] that gives a sharp criterion for global existence for the focusing problem. Here we reproduce the result in [51] which states that if one assumes small L^2 data (but not arbitrarily small), which are, in addition, in H^1, global well-posedness follows by discovering the sharp constant of the Gagliardo-Nirenberg inequality.

More precisely since

$$\|u(t)\|^{2+\frac{4}{n}}_{L^{2+\frac{4}{n}}} \leq C\|\nabla u(t)\|^2_{L^2} \|u(t)\|^{\frac{4}{n}}_{L^2} = C\|\nabla u(t)\|^2_{L^2} \|u_0\|^{\frac{4}{n}}_{L^2},$$

one can easily see that the energy functional

$$E(u)(t) = \frac{1}{2} \int |\nabla u(t)|^2 dx - \frac{1}{2+\frac{4}{n}} \int |u(t)|^{2+\frac{4}{n}} dx$$

is bounded from below as follows

$$E(u(t)) = E(u_0) \geq \|\nabla u(t)\|^2_{L^2} \left(\frac{1}{2} - C\|u_0\|^{\frac{4}{n}}_{L^2} \right). \tag{5.6}$$

Thus for $\|u_0\|_{L^2} < \eta$, η a fixed number, we have that

$$\|\nabla u(t)\|_{L^2} + \|u(t)\|_{L^2} \leq C_{M(u_0),E(u_0)} < \infty.$$

By the blow-up alternative of the H^1 theory we see that $T_{\max} = \infty$.

The question remains what is the optimal η. It was conjectured that, even with L^2 data, the optimal η is the mass of the ground state Q, which is the solution to the elliptic equation:

$$-Q + \Delta Q = |Q|^{\frac{4}{n}} Q,$$

that can be obtained by using the the ansatz $u(x,t) = e^{it}Q(x)$ in (5.5). It is shown that Q is unique, positive, spherically symmetric and very smooth (see [4] for exact references). Also Q satisfies certain identities (Pohozaev's identities) that can be obtained by multiplying the elliptic equation by $\bar{u}$ and $x \cdot \nabla u$ and take the real part respectively. In particular the identities imply that $E(Q) = 0$. In [51] Weinstein discovered that the mass of the ground state is related to the best constant of the Gagliardo-Nirenberg inequality. More precisely by minimizing the functional

$$J(u) = \frac{\|\nabla u(t)\|^2_{L^2} \|u\|^{\frac{4}{n}}_{L^2}}{\|u\|^{2+\frac{4}{n}}_{L_x^{2+\frac{4}{n}}}},$$

Weinstein showed that the best constant of the Galgiardo-Nirenberg inequality

$$\frac{1}{2+\frac{4}{n}} \|u(t)\|^{2+\frac{4}{n}}_{L^{2+\frac{4}{n}}} \leq \frac{C}{2} \|\nabla u(t)\|^2_{L^2} \|u(t)\|^{\frac{4}{n}}_{L^2},$$

is

$$C = \|Q\|^{-\frac{4}{n}}_{L^2}.$$

Hence we can revisit (5.6) to obtain

$$E(u_0) \geq \frac{1}{2} \|\nabla u(t)\|^2_{L^2} \left(1 - \frac{\|u_0\|^{\frac{4}{n}}_{L^2}}{\|Q\|^{\frac{4}{n}}_{L^2}} \right).$$

Therefore, if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, we have a global solution.

Moreover the condition is sharp in the sense that for any $\eta > \|Q\|_{L^2}$, there exists $u_0 \in H^1$ such that $\|u_0\|_{L^2} = \eta$, and $u(t)$ blows-up in finite time. To see that, set

$$\gamma = \frac{\eta}{\|Q\| L^2} > 1,$$

and consider $u_0 = \gamma Q$. Then $\|u_0\|_{L^2} = \eta$ and

$$E(u_0) = \gamma^{2+\frac{4}{n}} E(Q) - \frac{\gamma^{2+\frac{4}{n}} - \gamma^2}{2} \|\nabla Q\|^2_{L^2} = -\frac{\gamma^{2+\frac{4}{n}} - \gamma^2}{2} \|\nabla Q\|^2_{L^2} < 0.$$

Since $u_0 = \gamma Q \in \Sigma$ and $E(u_0) < 0$, by the Zakharov-Glassey argument we have blow-up in finite time.

Remark 5.2. *As consequence of the pseudo-conformal transformation*

$$u(x,t) \to (1-t)^{-\frac{n}{2}} e^{-\frac{i|x|^2}{4(1-t)}} u(\frac{t}{1-t}, \frac{x}{1-t}),$$

we actually have blow-up even for $\eta = \|Q\|_{L^2}$. We cite [4] *for the details. It is interesting that the blow-up rate is $\frac{1}{t}$ and thus at least in the L^2-critical case the lower estimate we gave is not optimal for all blow-up solutions.*

5.3. **Blow-up for the L^2-critical problem.** We now prove that for the focusing L^2-critical problem, the mass at the origin concentrates the mass of the ground state. We assume radial H^1 data with $n \geq 2$. Both assumptions (radiality and dimension) have been removed but the proof is more elaborate. For the detials of the H^1 theory see [4] and the references therein.

Theorem 5.3. *Consider* (5.5) *with $u_0 \in H^1(\mathbb{R}^n) \cap \{radial\}$ in dimensions $n \geq 2$. Let ρ be any function $(0,\infty) \to (0,\infty)$ such that $\lim_{s\downarrow 0} \rho(s) = \infty$ and that $\lim_{s\downarrow 0} s^{\frac{1}{2}} \rho(s) = 0$. If u is the maximal solution of* (5.5) *and $T^\star < \infty$ then*

$$\liminf_{t \uparrow T^\star} \|u(t)\|_{L^2(\Omega_t)} \geq \|Q\|_{L^2},$$

where

$$\Omega_t = \Big(x \in \mathbb{R}^n : \ |x| < |T^\star - t|^{\frac{1}{2}} \rho(T^\star - t) \Big).$$

To prove the theorem we note that a result of W. Strauss states that a radial bounded sequence of functions in H^1 contains a subsequence that converges strongly in L^p for $2 < p < \frac{2n}{n-2}$. Now set

$$\lambda(t) = \frac{1}{\|\nabla u(t)\|_{L^2}}$$

so that

$$\lim_{t \uparrow T^\star} \lambda(t) = 0.$$

We claim that

$$\liminf_{t \uparrow T^\star} \|u(t)\|_{L^2(|x|<\lambda(t)\rho(T^\star - t))} \geq \|Q\|_{L^2}.$$

The result then follows since ρ is arbitrary and $\|\nabla u(t)\|_{L^2} \geq \frac{\delta}{(T^\star - t)^{\frac{1}{2}}}$.

We prove the claim by contradiction. Assume there exists $t_n \uparrow T^\star$ such that

$$\lim_{n \to \infty} \|u(t)\|_{L^2(|x|<\lambda(t_n)\rho(T^\star - t_n))} < \|Q\|_{L^2}.$$

Set

$$v_n(t) = \lambda(t)^{\frac{n}{2}} u(t_n, \lambda(t_n) x).$$

Clearly

$$\|v_{t_n}\|_{L^2} = 1,$$

$$\|\nabla v_n\|_{L^2} = 1, \tag{5.7}$$

$$E(v_n) = \lambda(t_n)^2 E(u(t_n)) = \lambda(t_n)^2 E(u_0).$$

In particular

$$E(v_n) = \frac{1}{2} - \frac{1}{2 + \frac{4}{n}} \|v_n\|_{L^{2+\frac{4}{n}}}^{2+\frac{4}{n}},$$

and

$$\lim_{n \to \infty} E(v_n) = 0.$$

Thus

$$\lim_{n \to \infty} \|v_n\|_{L^{2+\frac{4}{n}}}^{2+\frac{4}{n}} \to \frac{2 + \frac{4}{n}}{2} \neq 0. \tag{5.8}$$

Since v_n is a bounded H^1 sequence there exists a subsequence which we still denote by v_n that converges weakly to w in H^1 and strongly by Strauss' result in $L^{2+\frac{4}{n}}$. By the properties of weak and strong limits and (5.8) we have that

$$E(w) \leq 0, \qquad w \neq 0.$$

By the sharp Gagliardo-Nirenberg inequality this means

$$\|w\|_{L^2} \geq \|Q\|_{L^2}.$$

Now given $M > 0$ we have that

$$\begin{aligned}\|w\|_{L^2(|x|<M)} &= \lim_{n\to\infty} \|v_n\|_{L^2(|x|<M)} \\ &= \lim_{n\to\infty} \|u(t_n)\|_{L^2(|x|<M\lambda(t_n))} \\ &\leq \liminf_{n\to\infty} \|u(t_n)\|_{L^2(|x|<\lambda(t_n)\rho(T^\star - t))},\end{aligned}$$

since $\rho(s) \to \infty$ as $s \downarrow 0$. But since M was arbitrary, we obtain

$$\liminf_{n\to\infty} \|u(t_n)\|_{L^2(|x|<\lambda(t_n)\rho(T^\star - t))} \geq \|w\|_{L^2} > \|Q\|_{L^2},$$

reaching the contradiction.

5.4. **Quantum scattering in the energy space.** Consider the defocusing L^2-super-critical problem

$$\begin{cases} iu_t + \Delta u - |u|^{p-1}u = 0, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n), \end{cases} \tag{5.9}$$

for any $1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$.
We define the set of initial values u_0 which have a scattering state at $+\infty$ (by time reversibility all the statements are equivalent at $-\infty$):

$$\mathcal{R}_+ = (u_0 \in H^1 : T^\star = \infty, \quad u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t) \ \mathit{exists}\ ). \tag{5.10}$$

Now define the operator

$$U : \mathcal{R}_+ \to H^1.$$

This operator sends u_0 to the scattering state u_+. If this operator is injective then we can define the wave operator

$$\Omega_+ = U^{-1} : U(\mathcal{R}_+) \to \mathcal{R}_+$$

which sends the scattering state u_+ to u_0. Thus the first problem of scattering is the existence of wave operator:
• *Existence of wave operators. For each u_+ there exists unique $u_0 \in H^1$ such that $u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t)$.*
If the wave operator is also surjective we say that we have asymptotic completeness (thus in this case the wave operator is invertible):
• *Asymptotic completeness. For every $u_0 \in H^1$ there exists u_+ such that $u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t)$.*
Both statements make rigorous the idea that we have scattering if, as time goes to infinity, the nonlinear solution of the NLS behaves like the solution of the linear equation.
Using the decay estimates of section 4 we can solve the scattering problem for every $p > 1 + \frac{4}{n}$. Well-defined wave operators for this range of p is easy and it is almost a byproduct of the local theory. But asymptotic completeness is hard. In dimensions $n \geq 3$ this was proved in [18] and for $n = 1, 2$ in [35, 36]. The proofs are complicated since they were achieved before the interaction Morawetz estimates. Using the interaction Morawetz estimates we can prove the scattering properties in two simple steps. To make the presentation clear we will only show the $n = 3$ case with the cubic nonlinearity. But keep in mind that the interaction Morawetz estimates give global a priori control on quantities of the form

$$\|u\|_{L^q_t L^r_x} \leq C_{M(u_0),E(u_0)},$$

for certain q and r in all dimensions. It turns out that in the L^2-supercritical case this is enough to give scattering for any $p > 1 + \frac{4}{n}$ and n. Finally for completeness we also outline the wave operator question.

Theorem 5.4. *For every $u_+ \in H^1(\mathbb{R}^3)$ there exists unique $u_0 \in H^1(\mathbb{R}^3)$ such that the maximal solution $u \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ of $iu_t + \Delta u = |u|^2 u$, satisfies*

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_+\|_{H^1(\mathbb{R}^3)} = 0.$$

Proof: For $u_+ \in H^1$ define the map

$$\mathcal{A}(u)(t) = e^{it\Delta}u_+ + i\int_t^\infty e^{i(t-s)\Delta}(|u|^2u)(s)ds.$$

What is the motivation behind this map? Recall that

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}(|u|^2u)(s)ds,$$

$$e^{-it\Delta}u(t) = u_0 - i\int_0^t e^{-is\Delta}(|u|^2u)(s)ds. \tag{5.11}$$

If the problem scatters we have that $\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_+\|_{H^1} = 0$ and thus

$$u_+ = u_0 - i\int_0^\infty e^{-is\Delta}(|u|^2u)(s)ds \tag{5.12}$$

in H^1 sense. Now subtracting (5.12) from (5.11) we have that

$$u(t) = e^{it\Delta}u_+ + i\int_t^\infty e^{i(t-s)\Delta}(|u|^2u)(s)ds.$$

By Strichartz estimates we have that

$$\|e^{it\Delta}u_+\|_{L_t^qW_x^{1,r}} \lesssim \|u_+\|_{H^1} < \infty.$$

By the monotone convergence theorem there exists $T = T(u_+)$ large enough such that for $q < \infty$ we have

$$\|e^{it\Delta}u_+\|_{L_t^qW_x^{1,r}} \lesssim \epsilon.$$

The trick here is to use the smallness assumption to iterate the map in the interval (T, ∞). But our local theory was performed in the norms

$$\|u\|_{\mathcal{S}^1(I\times\mathbb{R}^n)} = \|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} + \|\nabla u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)}$$

where

$$\|u\|_{\mathcal{S}^0(I\times\mathbb{R}^n)} = \sup_{(q,r)-admissible} \|u\|_{L^q_{t\in I}L^r_x}.$$

But this norms contain L_t^∞. So momentarily we will go to the smaller space

$$X = L_t^5L_x^5 \cap L_t^{\frac{10}{3}}W_x^{1,\frac{10}{3}}.$$

For this norm we also have that for large T

$$\|e^{it\Delta}u_+\|_{X_{[T,\infty)}} \lesssim \epsilon.$$

Furthermore Strichartz estimates show that

$$\|\mathcal{A}(u)\|_{X_{[T,\infty)}} \lesssim \epsilon + \|u\|^3_{X_{[T,\infty)}}.$$

The main step here is Sobolev embedding

$$\|f\|_{L^5_t L^5_x} \lesssim \|f\|_{L^5_t W^{1,\frac{30}{11}}_x}$$

where the pair $(5, \frac{30}{11})$ is Strichartz admissible. The details are as follows: Notice that the dual pair of $(\frac{10}{3}, \frac{10}{3})$ is $(\frac{10}{7}, \frac{10}{7})$.

$$\begin{aligned}
\|u\|_{L^5_t L^5_x} &\lesssim \|e^{it\Delta}u_+\|_{L^5_t L^5_x} + \|\int_T^\infty e^{i(t-s)\Delta}\Big(|u|^2 u(s)\Big)ds\|_{L^5_t L^5_x} \\
&\lesssim \|e^{it\Delta}u_+\|_{L^5_t W^{1\frac{30}{13}}_x} + \|\int_T^\infty e^{i(t-s)\Delta}\Big(|u|^2 u(s)\Big)ds\|_{L^5_t W^{1,\frac{30}{13}}_x} \\
&\lesssim \epsilon + \|u^3\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + \|(\nabla u)u^2\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} \\
&\lesssim \epsilon + \|u\|^2_{L^5_t L^5_x}\|u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} + \|u\|^2_{L^5_t L^5_x}\|\nabla u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \\
&\lesssim \epsilon + \|u\|^2_{L^5_t L^5_x}\|u\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}} \lesssim \epsilon + \|u\|^3_{X_{[T,\infty)}}.
\end{aligned}$$

Similarly we derive

$$\|\nabla u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \lesssim \epsilon + \|u\|^2_{L^5_t L^5_x}\|\nabla u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \lesssim \epsilon + \|u\|^3_{X_{[T,\infty)}}.$$

and

$$\|u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \lesssim \epsilon + \|u\|^2_{L^5_t L^5_x}\|u\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \lesssim \epsilon + \|u\|^3_{X_{[T,\infty)}}.$$

Thus for T large enough we have that

$$\|u\|_{X_{[T,\infty)}} \lesssim \epsilon.$$

More precisely to obtain the last claim one has to estimate $\|\mathcal{A}\|_X$, $\|\mathcal{A}(u) - \mathcal{A}(v)\|_X$ and prove that the map $\mathcal{A}$ is a contraction. Thanks to the ϵ we derive simultaneously this property along with the estimate

$$\|u\|_{X_{[T,\infty)}} \lesssim \epsilon.$$

It remains to show that the solution is in $C([T,\infty); H^1(\mathbb{R}^3))$. But by Strichartz again and using any admissible pair we have

$$\|u\|_{L^q_{t\in[T,\infty)} W^{1,r}_x} \lesssim \|u_+\|_{H^1} + \|u\|^3_{X_{[T,\infty)}} \lesssim \|u_+\|_{H^1}.$$

In particular $\psi = u(T) \in H^1$ and we have a strong H^1 solution of the equation with initial data $u(T) = \psi$. But we know that the solutions of this equation are global and thus $u(0)$ is well-defined. Finally

$$e^{-it\Delta}u(t) - u_+ = i\int_t^\infty e^{-is\Delta}(|u|^2 u)(s)ds,$$

$$\nabla\Big(e^{-it\Delta}u(t) - u_+\Big) = i\int_t^\infty e^{-is\Delta}\Big(\nabla(|u|^2 u)\Big)(s)ds,$$

$$\|e^{-it\Delta}u(t) - u_+\|_{H^1} \lesssim \|\nabla u\|_{L^{\frac{10}{3}}_{[t,\infty)} L_x^{\frac{10}{3}}}\|u\|^2_{L^5_{[t,\infty)} L^5_x} \lesssim \|u\|^3_{X_{[T,\infty)}}$$

But for T large enough we have that $\|u\|_{X_{[T,\infty)}} \lesssim \epsilon$ and thus

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_+\|_{H^1} = 0.$$

Therefore $u(0) = u_0 \in H^1$ satisfies the assumptions of the theorem. We end with asymptotic completeness.

Theorem 5.5. *If $u_0 \in H^1(\mathbb{R}^3)$ and if $u \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ where u is the solution of $iu_t + \Delta u = |u|^2 u$, then there exists u_+ such that*

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_+\| = 0.$$

The proof is based on a simple proposition assuming the interaction Morawetz estimates. This was the hardest part in the earlier proofs of quantum scattering.

Proposition 5.6. *Let u be a global H^1 solution of the cubic defocusing equation on $\mathbb{R}^3$. Then*

$$\|u\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C.$$

Proof: We know that $\|u\|_{L^4_t L^4_x} \leq C$ for energy solutions. Thus we can pick ϵ small to be determined later and a finite number of intervals $\{I_k\}_{k=1,2,\ldots,M}$, with $M < \infty$ such that

$$\|u\|_{L^4_{t\in I_k} L^4_x} \leq \epsilon$$

for all k. If we apply the Strichartz estimates on each I_k we obtain for some $\alpha < 1$

$$\|u\|_{\mathcal{S}^1(I_k)} \lesssim \|u(0)\|_{H^1} + \|u\|^{2\alpha}_{L^4_{t\in I_k} L^4_x} \|u\|^{3-2\alpha}_{\mathcal{S}^1(I_k)}, \tag{5.13}$$

$$\|u\|_{\mathcal{S}^1(I_k)} \lesssim \|u(0)\|_{H^1} + \epsilon^{2\alpha} \|u\|^{3-2\alpha}_{\mathcal{S}^1(I_k)}.$$

We can pick ϵ so small such that

$$\|u\|_{\mathcal{S}^1(I_k)} \leq K.$$

Since the number of intervals are finite and the conclusion can be made for all $I_k's$ the proposition follows.

Remarks. 1. Where do we use the condition $p > 1 + \frac{4}{n}$? This is a delicate matter. It is not hard to see that the interaction Morawetz estimates are global estimates of Strichartz type but are not L^2 scale invariant. If one inspects the right hand side of the interaction inequalities, a simple scaling argument shows that these are $H^{\frac{1}{4}}$ invariant estimates. Thus only in the case that $p > 1 + \frac{4}{n}$ we can take advantage of an non L^2 estimate such as $L^4_t L^4_x$. This is the heart of the matter in proving (5.13). In the case that $p = 1 + \frac{4}{n}$ we need to have a global L^2 Strichartz estimate like $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}$ in dimensions 3. Estimates of this sort can never come from Morawetz estimates due to scaling.
2. Notice that the proposition gives a global decay estimate for the nonlinear solution.

Let's finish the proof of asymptotic completeness. Note that

$$e^{-it\Delta}u(t) = u_0 - i\int_0^t e^{-is\Delta}(|u|^2 u)(s)ds,$$

$$e^{-i\tau\Delta}u(\tau) = u_0 - i\int_0^\tau e^{-is\Delta}(|u|^2 u)(s)ds.$$

Thus

$$\|e^{-it\Delta}u(t) - e^{-i\tau\Delta}u(\tau)\|_{H^1} = \|u(t) - e^{-i(t-\tau)\Delta}u(\tau)\|_{H^1} \lesssim \|u\|^3_{\mathcal{S}^1_{(t,\tau)}} \leq C$$

again by Strichartz estimates. Thus as $t, \tau \to \infty$ we have that

$$\|e^{-it\Delta}u(t) - e^{-i\tau\Delta}u(\tau)\|_{H^1} \to 0.$$

By completeness of H^1 there exists $u_+ \in H^1$ such that $e^{-it\Delta}u(t) \to u_+$ in H^1 as $t \to \infty$. In particular in H^1 we have

$$u_+ = u_0 - i\int_0^\infty e^{-is\Delta}(|u|^2u)(s)ds$$

and thus

$$\|e^{-it\Delta}u(t) - u_+\|_{H^1} \lesssim \|u\|^3_{\mathcal{S}^1_{(t,\infty)}}.$$

As $t \to \infty$ the conclusion follows.

More remarks. What about energy scattering for $p \leq 1 + \frac{4}{n}$. The critical case has been solved in [11, 12, 13]. For $p < 1 + \frac{4}{n}$ the problem is completely open. We have already mentioned that scattering makes rigorous the intuition that as time increases, for a defocusing problem, the nonlinearity $|u|^{p-1}u$ becomes negligible. From this observation one expects that the bigger the power of p the better chance the solution has to scatter. Thus the question: Is there any threshold p_0 with $1 < p_0 \leq 1 + \frac{4}{n}$ such that energy scattering does fail? The answer is yes and $p_0 = 1 + \frac{2}{n}$. This is in [43] for higher dimensions and in [1] for dimension one. More precisely using the pseudo-conformal conservation law and decay estimates that we discuss later in the notes, they showed that for any $1 < p \leq p_0$, $U(-t)u(t)$ doesn't converge even in L^2. Thus the wave operators cannot exist in any reasonable set. The problem remains open for

$$1 + \frac{2}{n} < p < 1 + \frac{4}{n}$$

and for general energy data. For partial results see [24] and the references therein.

5.5. **Quantum scattering in the Σ space.** If we are willing to abandon the energy space can we improve scattering in the range $1 + \frac{2}{n} < p < 1 + \frac{4}{n}$? Recall that

$$\Sigma = H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n,\ |x|^2dx).$$

We will not go into the details but a few comments can clarify the situation. Exactly like the energy case it is enough to prove that

$$\|u\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C.$$

How one can obtain this estimate for different values of p? First recall that for

$$K(t) = \|(x + 2it\nabla)u\|^2_{L^2} + \frac{8t^2}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx$$

we have that

$$K(t) - K(0) = \int_0^t \theta(s)ds,$$

where

$$\theta(t) = \frac{4t}{p+1}\left(4 - n(p-1)\right)\int_{\mathbb{R}^n}|u|^{p+1}dx.$$

Using this quantity and a simple analysis one can obtain the following proposition:

Proposition 5.7. *Consider the defocusing NLS*

$$\begin{cases} iu_t + \Delta u = |u|^{p-1}u \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^n). \end{cases} \tag{5.14}$$

for any $1 < p < 1 + \frac{4}{n-2}$, $n \geq 3$ *(*$1 < p < \infty$ *for* $n = 1, 2$*). If in addition* $\|xu_0\|_{L^2} < \infty$ *and*

$$u \in C^0_t(\mathbb{R}; H^1(\mathbb{R}^n))$$

solves (5.14)*, then we have:*
i) If $p > 1 + \frac{4}{n}$ *then for any* $2 \leq r \leq \frac{2n}{n-2}$ *(*$2 \leq r \leq \infty$ *if* $n = 1$, $2 \leq r < \infty$ *if* $n = 2$*)*

$$\|u(t)\|_{L^r} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{r})}$$

for all $t \in \mathbb{R}^n$.
ii) If $p < 1 + \frac{4}{n}$ *then for any* $2 \leq r \leq \frac{2n}{n-2}$ *(*$2 \leq r \leq \infty$ *if* $n = 1$, $2 \leq r < \infty$ *if* $n = 2$*)*

$$\|u(t)\|_{L^r} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{r})(1-\theta(r))}$$

where

$$\theta(r) = \begin{cases} 0 & \text{if } 2 \leq r \leq p+1 \\ \frac{[r-(p+1)][4-n(p-1)]}{(r-2)[(n+2)-p(n-1)]} & \text{if } r > p+1. \end{cases}$$

Remarks. 1. Notice that for $p \geq 1 + \frac{4}{n}$ the decay is as strong as the linear equation. Recall here the basic $L^1 - L^\infty$ estimate of the linear problem and its interpolation with Plancherel's theorem.
2. Using these estimates and the standard theory we have developed one can prove that global solutions defined in the Σ space obey

$$\|u\|_{\mathcal{S}^1(\mathbb{R}\times\mathbb{R}^3)} \leq C,$$

for any

$$1 + \frac{2 - n + \sqrt{n^2 + 12n + 4}}{2n} < p < 1 + \frac{4}{n-2}.$$

The existence of wave operators and asymptotic completeness follows easily. Of course

$$1 + \frac{2}{n} < 1 + \frac{2 - n + \sqrt{n^2 + 12n + 4}}{2n} < 1 + \frac{4}{n}.$$

3. The existence of the wave operators can go below the above threshold in all dimensions. Indeed one can cover the full range $p > 1 + \frac{2}{n}$. The subject is rather technical and we refer to [4] for more details.

References

[1] J. E. Barab, *Nonexistence of asymptotically free solutions for nonlinear Schrödinger equation*, J. Math Phys. 25 (1984), pp. 3270–3273.
[2] J. Bourgain, *Global solutions of nonlinear Schrödinger equations,* American Mathematical Society, Providence, RI, 1999.
[3] J. Bourgain, *Global well-posedness of defocusing 3D critical NLS in the radial case,* J. Amer. Math. Soc., 12 (1999), 145–171.
[4] T. Cazenave, *Semilinear Schrödinger equations,* CLN 10, eds: AMS, 2003.
[5] T. Cazenave, and F. B. Weissler, *Some remarks on the nonlinear Schrödinger equation in critical case,* Nonlinear semigroups, partial differential equations and attractors (Washington DC 1987), 18–29. Lecture Notes in Mathematics, 1394. Springer, Berlin, 1989.

[6] J. Colliander, M. Grillakis, and N. Tzirakis, *Commutators and correlations estimates with applications to NLS,* Comm. Pure Appl. Math. 62 (2009), no. 7, 920–968.
[7] J. Colliander, M. Grillakis, and N. Tzirakis, *Remarks on global a priori estimates for the nonlinear Schrödinger equation,* Proc. Amer. Math. Soc. 138 (2010), no. 12, 4359–4371.
[8] J. Colliander, J. Holmer, M. Visan and X. Zhang, *Global existence and scattering for rough solutions to generalized nonlinear Schrödinger equations on* $\mathbb{R}$*,* Commun. Pure Appl. Anal. 7 (2008), no. 3, 467–489.
[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Global existence and scattering for rough solutions to a nonlinear Schrödinger equation on* $\mathbb{R}^3$, Comm. Pure Appl. Math., 57 (2004), no. 8, 987–1014.
[10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global well-posedness and scattering in the energy space for the critical nonlinear Schrödinger equation in* $\mathbb{R}^3$*,* Ann. of Math. (2), 167 (2008), no. 3, 767–865.
[11] B. Dodson, *Global well-posedness and scattering for the defocusing,* L^2-*critical, nonlinear Schrödinger equation when* $d \geq 3$*,* preprint, http://arxiv.org/abs/0912.2467.
[12] B. Dodson, *Global well-posedness and scattering for the defocusing,* L^2-*critical, nonlinear Schrödinger equation when* $d = 2$*,* preprint, http://arxiv.org/abs/1006.1375.
[13] B. Dodson, *Global well-posedness and scattering for the defocusing,* L^2-*critical, nonlinear Schrödinger equation when* $d = 1$*,* preprint, http://arxiv.org/abs/1010.0040.
[14] B. Dodson, *Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state,* preprint, http://arxiv.org/abs/1104.1114.
[15] L. C. Evans Partial Differential Equations: Second Edition (Graduate Studies in Mathematics).
[16] J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Schrödinger equation revisited,* Ann. Inst. H. Poincaré Anal. Non Linéaire, 2 (1985), no. 4, 309–327.
[17] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of Hartree equations,* in Nonlinear Wave Equations, Y. Guo Ed. Contemporary Mathematics 263, AMS 2000.
[18] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of nonlinear Schrödinger equations,* J. Math. Pure Appl., 64 (1985), 363-401.
[19] J. Ginibre, and G. Velo, *Quadratic Morawetz inequalities and asymptotic completeness in the energy space for nonlinear Schrödinger and Hartree equations,* Quart. Appl. Math. 68 (2010), 113–134.
[20] L. Glangetas and F. Merle, *A geometrical approach of existence of blow up solutions in* H^1 *for nonlinear Schrödinger equation,* Rep. No. R95031, Laboratoire d'Analyse Numérique, Univ. Pierre Marie Curie (1995).
[21] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations,* J. Math. Phys. 18 (1977), 1794–1797.
[22] M. Grillakis, *On nonlinear Schrödinger equations,* Commun. Partial Differential Equations 25, no. 9-10, (2005), 1827-1844.
[23] N. Hayashi and Y. Tsutsumi, *Scattering theory for the Hartree type equations,* Ann. Inst. H. Poincaré Phys. Théor. 46 (1987), 187–213.
[24] J. Holmer, and N. Tzirakis *Asymptotically linear solutions in* H^1 *of the 2D defocusing nonlinear Schrödinger and Hartree equations*, J. Hyperbolic Differ. Equ. 7 (2010), no. 1, 117–138.
[25] T.Kato, *On nonlinear Schrödinger equations II.* H^s *solutions and unconditionally well-posedness,* J. Anal. Math. 67 (1995), 281-306.
[26] T.Kato, *On nonlinear Schrödinger equations,* Ann. Inst. H. Poincare Phys. Theor. 46 (1987), 113–129.
[27] M. Keel, and T. Tao, *Endpoint Strichartz estimates,* Amer. J. Math. 120 (1998), no. 5, 955–980.
[28] C. E. Kenig and F. Merle, *Global well-posedness, scattering and blow up for the energy-critical, focusing, nonlinear Schrödinger equation in the radial case,* Invent. Math. 166 (2006), 645–675.
[29] R. Killip, and M. Visan *The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher,* Amer. J. Math. 132 (2010), 361–424.
[30] R. Killip and M. Visan, *Energy supercritical NLS: Critical* $\dot{H}^s-$ *bounds imply scattering,* http://arxiv.org/abs/0812.2084.

[31] J. E. Lin, and W. A. Strauss, *Decay and scattering of solutions of a nonlinear Schrödinger equation,* J. Funct. Anal. 30 (1978), no.2, 245–263.

[32] F. Linares and G. Ponce, *Introduction to Nonlinear Dispersive equations,* UTX, Springer-Verlag, 2009.

[33] F. Merle, and P. Raphael *Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation,* Geom. Funct. Anal. 13 (2003), no. 3, 591-642.

[34] C. Morawetz, *Time decay for the nonlinear Klein-Gordon equation,* Proc. Roy. Soc. A, 306 (1968), 291–296.

[35] K. Nakanishi, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2,* J. Funct. Anal. 169 (1999), 201–225.

[36] K. Nakanishi, *Energy scattering for Hartree equations,* Math. Res. Lett. 6, no 1, (1999),107–118.

[37] H. Nawa, *Asymptotic and limiting profiles of blowup solutions of the nonlinear Schrdinger equation with critical power,* Comm. Pure Appl. Math. 52 (1999), no. 2, 193–270.

[38] T Ogawa, and Y. Tsutsumi, *Blow-up of H^1 solutions for the nonlinear Schrödinger equation,* J. Differential Equations 92 (1991), 317–330.

[39] T Ogawa, and Y. Tsutsumi, *Blow-up of H^1 solutions for the one dimensional nonlinear Schrödinger equation with critical power nonlinearity,* Proc. Amer. Math. Soc. 111 (1991), 487–496.

[40] F. Planchon, and L. Vega, *Bilinear Virial Identities and Applications*, Ann. Sci. c. Norm. Supr. (4) 42 (2009), no. 2, 261–290.

[41] M. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations,* Duke Math. J. 44 (1977), 705–714.

[42] E. Ryckman and M. Visan, *Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$,* Amer. J. Math. 129 (2007), 1–60.

[43] W. Strauss, *Nonlinear scattering theory*, Scattering Theory in Mathematical Physics, ed. La Vita and Marchard, Reidel, (1974), 53–78.

[44] C.Sulem, and P-L. Sulem, *The nonlinear Schrödinger equation. Self-focusing and wave collapse,* Springer-Verlag, New York, 1999.

[45] E. M. Stein, *Singular integrals and differentiability properties of functions,* Princeton Univ. Press, 1967.

[46] T. Tao, *Nonlinear dispersive equations. Local and global analysis* CBMS 106, eds: AMS, 2006.

[47] T. Tao, M. Visan, and X. Zhang *The nonlinear Schrödinger equation with combined power-type nonlinearities,* Comm. Partial Differential Equations 32 (2007), no. 7-9, 1281–1343.

[48] T. Tao, M. Visan, and X. Zhang, *Global well-posedness and scattering for the mass-critical nonlinear Schrödinger equation for radial data in high dimensions,* Duke Math. J. 140 (2007), 165–202.

[49] Y. Tsutsumi, *L^2 solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. 30 (1987) 115–125.

[50] M. Visan, *The defocusing energy-critical nonlinear Schrodinger equation in dimensions five and higher,* Ph. D. Thesis UCLA, 2006.

[51] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates,* Comm. Math. Phys. 87 (1983), 567–576.

N. Pavlović, Department of Mathematics, University of Texas at Austin.

E-mail address: `natasa@math.utexas.edu`

N. Tzirakis, University of Illinois at Urbana-Champaign.

E-mail address: `tzirakis@math.uiuc.edu`