ON THE SINGULARITY FORMATION FOR THE NON LINEAR SCHRÖDINGER EQUATION

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These notes are an introduction to the qualitative description of singularity formation for the nonlinear Schrödinger equation. Part of the material was presented during the 2008 Clay summer school on Nonlinear Evolution Equations at the ETH Zurich. The manuscript has been enriched with additions in 2012 in order to give a more accurate view on this very active research field and present a number of open problems.

We consider the semi linear Schrödinger equation

$$(NLS) \quad \left\{ \begin{array}{l} iu_t = -\Delta u - |u|^{p-1}u, \quad (t,x) \in [0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x), \quad u_0 : \mathbb{R}^N \to \mathbb{C} \end{array} \right. \tag{0.1}$$

with $u_0 \in H^1 = \{u, \nabla u \in L^2(\mathbb{R}^N)\}$ in dimension $N \geq 1$ and for energy subcritical nonlinearities:

$$1 < p < 2^* - 1 \quad \text{with} \quad 2^* = \left\{ \begin{array}{ll} +\infty & \text{for} \quad N = 1,2 \\ \frac{2N}{N-2} & \text{for} \quad N \geq 3 \end{array} \right. . \tag{0.2}$$

where 2^* is the Sobolev exponent of the injection $\dot{H}^1 \hookrightarrow L^{2^*}$. The case $p = 3$ appears in various areas of physics: for the propagation of waves in non linear media and optical fibers for $N = 1$, the focusing of laser beams for $N = 2$, the Bose-Einstein condensation phenomenon for $N = 3$, see the monograph [106] for a more systematic introduction to this physical aspect of the problem.

Our aim is to develop tools for the qualitative description of the flow for data in the energy space H^1, and this includes long time existence, scattering or formation of singularities. The possibility of finite time blow up corresponding to a self focusing of the nonlinear wave and the concentration of energy will be of particular interest to us. Note that (NLS) is an infinite dimensional Hamiltonian system without any space localization property and infinite speed of propagation. It is in this context together with the critical generalized (gKdV) equation[1] one of the few examples where blow up is known to occur. For (NLS), an elementary proof of existence of blow up solutions is known since the 60's but is based on energy constraints and is not constructive. In particular, *no qualitative information of any type on the blow up dynamics is obtained this way.* In fact, the theory of global existence or blow up for (NLS) as known up to now is intimately connected to the theory of ground states or solitons which are special periodic in time solutions to the Hamiltonian system. A central question is the stability of these solutions and the description of the flow around them which has attracted a considerable amount of work for the past thirty years.

These notes are organized as follows.

[1] see (4.22).

In the first section, we recall the main standard results about subcritical non linear Schrödinger equations and in particular the existence and orbital stability of soliton like solutions which relies on nowadays standard variational tools. In section 2, we introduce the blow up problem and present some of the very few general results known on the singularity formation in this case, and this includes old results from the 50' and very recent ones. Section 3 focuses onto the mass critical problem $p = 1 + \frac{4}{N}$ and we extend in the critical blow up regime the subcritical variational theory of ground states. In section 4, we present the state of the art on the question of description of the flow near the ground state for mass critical problems, including recent complete answers for the generalized (gKdV) problem. In section 5, we present a detailed proof of the pioneering result obtained in collaboration with F.Merle in [71], [72] on the derivation of the sharp log-log upper bound on blow up rate for a suitable class of initial data near the ground state solitary wave.

We expect the presentation to be essentially self contained provided the prior knowledge of standard tools in the study of non linear PDE's.

1. The subcritical problem

We recall in this section the main classical facts regarding the global well posedness in the energy space of (NLS), and the main variational tools at the heart of the proof of the existence and stability of special periodic solutions: the ground state solitary waves.

1.1. Global well posedness in the subcritical case.

Let us consider the general non linear Schrödinger equation:

$$\left\{ \begin{array}{l} iu_t = -\Delta u - |u|^{p-1}u \\ u(0,x) = u_0(x) \in H^1 \end{array} \right. \tag{1.1}$$

with p satisfying the energy subcriticality assumption (0.2). The local well posedness of (1.1) in H^1 is a result of Ginibre, Velo, [23], see also [31]. Thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in \mathcal{C}([0,T), H^1)$. Moreover, the life time of the solution can be proved to be lower bounded by a function depending on the H^1 size of the solution only, $T(u_0) \geq f(\|u_0\|_{H^1})$, and hence there holds the blow up alternative:

$$T < +\infty \ \text{ implies } \ \lim_{t \to T} \|u(t)\|_{H^1} = +\infty. \tag{1.2}$$

We refer to [11] for a complete introduction to the Cauchy theory. To prove the global existence of the solution, it thus suffices to control the size of the solution in H^1. This is achieved in some cases using the invariants of the flow. Indeed, the following H^1 quantities are conserved:

- L^2-norm:
$$\int |u(t,x)|^2 = \int |u_0(x)|^2; \tag{1.3}$$
- Energy -or Hamiltonian-:
$$E(u(t,x)) = \frac{1}{2} \int |\nabla u(t,x)|^2 - \frac{1}{p+1} \int |u(t,x)|^{p+1} = E(u_0); \tag{1.4}$$
- Momentum:
$$Im \left(\int \nabla u \overline{u}(t,x) \right) = Im \left(\int \nabla u_0 \overline{u_0}(x) \right). \tag{1.5}$$

Note that the growth condition on the non linearity (0.2) ensures from Sobolev embedding that the energy is well defined, and this is why H^1 is referred to as the energy space. These invariants are related to the group of symmetry of (1.1) in H^1:

- Space-time translation invariance: if $u(t,x)$ solves (1.1), then so does $u(t+t_0, x+x_0)$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$.
- Phase invariance: if $u(t,x)$ solves (1.1), then so does $u(t,x)e^{i\gamma}$, $\gamma \in \mathbb{R}$.
- Scaling invariance: if $u(t,x)$ solves (1.1), then so does $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$, $\lambda > 0$.
- Galilean invariance: if $u(t,x)$ solves (1.1), then so does $u(t, x-\beta t)e^{i\frac{\beta}{2}\cdot(x-\frac{\beta}{2}t)}$, $\beta \in \mathbb{R}^N$.

Let us point out that this group of H^1 symmetries is the same like for the *linear* Schrödinger equation -up to the conformal invariance to which we will come back later-.

The *critical space* is a fundamental phenomenological number for the analysis and is defined as the number of derivatives in L^2 which are left invariant by the scaling symmetry of the flow:

$$|u_\lambda(t)|_{\dot{H}^{s_c}} = |u(\lambda^2 t)|_{\dot{H}^{s_c}} \quad \text{for} \quad s_c = \frac{N}{2} - \frac{2}{p-1}. \tag{1.6}$$

Observe that $s_c < 1$ from (0.2).

A direct consequence of the Cauchy theory, the conservation laws and Sobolev embeddings is the celebrated global existence result:

Theorem 1.1 (Global wellposedness in the subcritical case)**.** *Let $N \geq 1$ and $1 < p < 1 + \frac{4}{N}$ -equivalently $s_c < 0$-, then all solutions to (1.1) are global and bounded in H^1.*

Proof of Theorem 1.1. By L^2 conservation: $|u(t)|_{L^2} = |u_0|_{L^2}$. Moreover, the Gagliardo-Nirenberg interpolation estimate:

$$\forall v \in H^1, \quad \int |v|^{p+1} \leq C(N,p) \left(\int |\nabla v|^2\right)^{\frac{N(p-1)}{4}} \left(\int |v|^2\right)^{\frac{p+1}{2} - \frac{N(p-1)}{4}}. \tag{1.7}$$

applied to $v = u(t)$ implies using the conservation of the energy and the L^2 norm:

$$\forall t \in [0,T), \quad E_0 \geq \frac{1}{2}\left[\int |\nabla v|^2 - C(u_0)\left(\int |\nabla v|^2\right)^{\frac{N(p-1)}{4}}\right].$$

The subcriticality assumption $p < 1 + \frac{4}{N}$ now implies an a priori bound on the H^1 norm which concludes the proof of Theorem 1.1. □

The critical exponent

$$p = 1 + \frac{4}{N} \quad \text{ie} \quad s_c = 0$$

arises from this analysis and corresponds to the so-called L^2 *or mass critical case*. It is the smallest power nonlinearity for which blow up can occur and corresponds to an exact balance between the kinetic and potential energies under the constraint of conserved L^2 mass. The L^2 supercritical -and energy subcritical cases- correspond to

$$1 + \frac{4}{N} < p < 2^* - 1 \quad \text{ie} \quad 0 < s_c < 1.$$

1.2. **The solitary wave.** A fundamental feature of the focusing (NLS) problem is the existence of time periodic solutions. Indeed,

$$u(t,x)=\phi(x)e^{it}$$

is an H^1 solution to (1.1) iff ϕ solves the nonlinear elliptic equation:

$$\Delta\phi-\phi+\phi|\phi|^{p-1}=0,\quad \phi\in H^1(\mathbb{R}^N) \tag{1.8}$$

There are various ways to construct solutions to (1.8), the simplest one being to look for radial solutions via a shooting method, [4].

Proposition 1.2 (Existence of solitary waves)**.** *(i) For $N=1$, all solutions to (1.8) are translates of*

$$Q(x)=\left(\frac{p+1}{2\cosh^2\left(\frac{(p-1)x}{2}\right)}\right)^{p-1}. \tag{1.9}$$

(ii) For $N\geq 2$, there exist a sequence of radial solutions $(Q_n)_{n\geq 0}$ with increasing L^2 norm such that Q_n vanishes n times on $\mathbb{R}^N$.

The exact structure of the set of solutions to (1.8) is not known in dimension $N\geq 2$. An important rigidity property however which combines nonlinear elliptic techniques and ODE techniques is *the uniqueness of the nonnegative* solution to (1.8).

Proposition 1.3 (Uniqueness of the ground state)**.** *All solutions to*

$$\Delta\phi-\phi+\phi|\phi|^{p-1}=0,\quad \phi\in H^1(\mathbb{R}^N),\quad \phi(x)>0 \tag{1.10}$$

are a translate of an exponentially decreasing $\mathcal{C}^2$ radial profile $Q(r)$ ([22]) which is the unique nonnegative radially symmetric solution to (1.8) ([42]). Q is the so called ground state solution.

The uniqueness is thus the consequence of two facts. A positive decaying at infinity solution to (1.10) is necessarily radially symmetric with respect to a point, this is a very deep and non trivial result due to Gidas, Ni, Nirenberg [22] which relies on the maximum principle. And then there is uniqueness of the radial decaying positive solution in the ODE sense. The original -and delicate- proof of this last fact by Kwong [42] has been revisited by MacLeod [52] and is very nicely presented in the Appendix of Tao [107]. We also refer to [48] for a beautiful extension of uniqueness methods to nonlocal problems where the ODE approach fails.

Let us now observe that we may let the full group of symmetries of (1.1) act on the solitary wave $u(t,x)=Q(x)e^{it}$ to get a $2N+2$ parameters family of solitary waves: for $(\lambda_0,x_0,\gamma_0,\beta)\in\mathbb{R}_+^*\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$,

$$u(t,x)=\lambda_0^{\frac{2}{p-1}}Q(\lambda_0(x+x_0)-\lambda_0^2\beta t)e^{i\lambda_0^2t}e^{i\gamma_0}e^{i\frac{\beta}{2}\cdot(\lambda_0(x+x_0)-\lambda_0^2\beta t)}.$$

These waves are moving according to the free Galilean motion and oscillating at a phase related to their size: the larger the λ_0, the wilder the oscillations in time. An explicit computation reveals that the solitary wave can be made arbitrarily small in H^1 in the subcritical regime $s_c<0$ only.

1.3. **Orbital stability of the ground states in the subcritical case.** We address in this section the question of the stability of the ground state solitary wave $u(t,x) = Q(x)e^{it}$, $Q > 0$, as a solution to (1.1) in the mass subcritical case

$$1 < p < 1 + \frac{4}{N}, \quad s_c < 0. \tag{1.11}$$

Let us first observe that two trivial instabilities are given by the symmetries of the equation:

- Scaling instability: $\forall \lambda > 0$, the solution to (1.1) with initial data $u_0(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x)$ is $u(t,x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) e^{i\lambda^2 t}$.
- Galilean instability: $\forall \beta > 0$, the solution to (1.1) with initial data $u_0(x) = Q(x)e^{i\beta}$ is $u(t,x) = Q(x - \beta t)e^{it + \frac{\beta}{2}\cdot(x - \frac{\beta}{2}t)}$.

In both cases,

$$\sup_{t\in\mathbb{R}} |u(t,x) - Q(x)e^{it}| > |Q(x)|$$

and thus the solution does not stay uniformly close to Q. Cazenave and Lions [12] proved that these trivial instabilities are the only ones in the energy subcritical setting: this is the celebrated *orbital stability* of the ground state solitary wave.

Theorem 1.4 (Orbital stability of the ground state, [12])**.** *Let $N \geq 1$ and p satisfy (1.11). For all $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that the following holds true. Let $u_0 \in H^1$ with*

$$\|u_0 - Q\|_{H^1} < \delta(\varepsilon),$$

then there exist a translation shift $x(t) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N)$ and a phase shift $\gamma(t) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ such that:

$$\forall t \in \mathbb{R}, \quad \|u(t,x) - Q(x - x(t))e^{i\gamma(t)}\|_{H^1} < \varepsilon.$$

The strength -and the weakness- of the proof is that it relies only on the conservation laws and the *variational characterization of the ground state solitary wave.* This study falls into the classical sets of *concentration compactness techniques* as introduced by Lions in [50],[51]. Given $\lambda > 0$, we let

$$Q_\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x).$$

The following variational result immediately implies Theorem 1.4:

Proposition 1.5 (Description of the minimizing sequences)**.** *Let $N \geq 1$ and p satisfy (1.11). Let $M > 0$ be fixed.*
(i) Variational characterization of Q: The minimization problem

$$I(M) = \inf_{|u|_{L^2} = M} E(u) \tag{1.12}$$

is attained on the family

$$Q_{\lambda(M)}(\cdot - x_0)e^{i\gamma_0}, \quad x_0 \in \mathbb{R}^N, \gamma_0 \in \mathbb{R},$$

where $\lambda(M)$ is the unique scaling such that $|Q_{\lambda(M)}|_{L^2} = M$.
(ii) Description of the minimizing sequences: Any minimizing sequence v_n to (1.12) is relatively compact in H^1 up to translation and phase shifts, that is up to a subsequence:

$$v_n(\cdot + x_n)e^{i\gamma_n} \to Q_{\lambda(M)} \quad \text{in} \quad H^1.$$

The fact that Proposition 1.5 implies Theorem 1.4 is now a simple consequence of the conservation laws and is left to the reader. The next section is devoted to the proof of Proposition 1.5.

1.4. **The concentration compactness argument.** The first key to the proof of Proposition 1.5 is the description of the lack of compactness in $\mathbb{R}^N$ of the Sobolev injection $H^1 \hookrightarrow L^{p+1}$, $2 \leq p+1 < 2^*$. This description is a consequence of Lions' concentration compactness Lemma. Let us recall that the injection is compact on a smooth bounded domain. Note also that the injection is still compact when restricted to radial functions in dimension $N \geq 2$. Here one uses the estimate:

$$u^2(r) = -\int_r^{+\infty} u(s)u'(s)ds \quad \text{and thus} \quad |u|_{L^\infty(r\geq R)} \leq \frac{C}{R^{\frac{N-1}{2}}}|\nabla u|_{L^2}^{\frac{1}{2}}|u|_{L^2}^{\frac{1}{2}}$$

so that any H^1 bounded sequence of radially symmetric functions is L^{p+1} compact. This would considerably simplify the proof of Proposition 1.5 when restricting the problem to radially symmetric functions. In general, there holds the following:

Proposition 1.6 (Description of the lack of compactness of $H^1 \hookrightarrow L^q$)**.** *Let a sequence $u_n \in H^1$ with*

$$|u_n|_{L^2} = M, \quad |\nabla u_n|_{L^2} \leq C, \tag{1.13}$$

Then there exists a subsequence u_{n_k} such that one of the following three scenari occurs:

(i) Compactness: $\exists y_k \in \mathbb{R}^N$ such that

$$\forall 2 \leq q < 2^*, \quad u_{n_k}(\cdot + y_k) \to u \quad \text{in} \quad L^q. \tag{1.14}$$

(ii) Vanishing:

$$\forall 2 < q < 2^*, \quad u_{n_k} \to 0 \quad \text{in} \quad L^q. \tag{1.15}$$

(iii) Dichotomy: $\exists v_k, w_k$, $\exists 0 < \alpha < 1$ such that $\forall 2 \leq q < 2^*$:

$$\begin{cases} Supp(v_k) \cap Supp(w_k) = \emptyset, \quad dist(Supp(v_k), Supp(w_k)) \to +\infty, \\ \|v_k\|_{H^1} + \|w_k\|_{H^1} \leq C, \\ \|v_k\|_{L^2} \to \alpha M, \quad \|w_k\|_{L^2} \to (1-\alpha)M, \\ \lim_{k\to+\infty} \left|\int |u_{n_k}|^q - \int |v_k|^q - \int |w_k|^q\right| = 0, \\ \liminf_{k\to+\infty} \int |\nabla u_{n_k}|^2 - \int |\nabla v_k|^2 - \int |\nabla w_k|^2 \geq 0. \end{cases} \tag{1.16}$$

Remark 1.7. *The key in the dichotomy case is that there is no loss of potential energy during the splitting in space of u_{n_k} into two bumps v_k, w_k which support go away from each other, while on the other hand only a lower semi continuity bound can be derived for the kinetic energy.*

Remark 1.8. *The case dichotomy corresponds to the localization of the first bubble of concentration. One can then continue the extraction iteratively and obtain the profile decomposition of the sequence u_n, see P. Gerard* [21]*, Hmidi, Keraani* [28] *for a very elegant proof.*

The proof of Proposition 1.6 is given in Appendix A. We now show how the description of the lack of compactness of the Sobolev injection is a powerful tool for the study of variational problems.

Proof of Proposition 1.5. **step1** Computation of $I(M)$. Let $I(M)$ be given by (1.12). We claim that

$$-\infty < I(M) = M^{\frac{2(1-s_c)}{|s_c|}} I(1) < 0. \tag{1.17}$$

Indeed, $I(M) > -\infty$ follows directly form the Gagliardo-Nirenberg inequality (1.7) and the subcriticality condition (1.11). The computation of the nonpositive value of the infimum follows from the scaling properties of the problem. First, given $u \in H^1$ with $\|u\|_{L^2} = 1$, we use the L^2 scaling

$$v_\lambda(x) = \lambda^{\frac{N}{2}} u(\lambda x)$$

to get:

$$E(v_\lambda) = \lambda^2\left[\frac{1}{2}\int|\nabla u|^2 - \frac{1}{(p+1)\lambda^{(p-1)|s_c|}}\int|u|^{p+1}\right].$$

Letting $\lambda\to 0$ yields $I(1)<0$. The homogeneity in M of $I(M)$ is derived using the scaling of the equation

$$v_\lambda(x) = \lambda^{\frac{2}{p-1}}u(\lambda x), \quad \|v_\lambda\|_{L^2} = \lambda^{|s_c|}\|u\|_{L^2}, \quad E(v_\lambda) = \lambda^{2(1-s_c)}E(u),$$

which yields the claim.
Let now u_n be a minimizing sequence for $I(M)$. Then u_n is bounded in H^1 from (1.7) and satisfies the assumptions of Proposition 1.5, and we now examine the various scenario:

step 2 Vanishing cannot occur. Otherwise, from (1.15):

$$I(M) = \lim_{k\to+\infty} E(u_{n_k}) \geq \liminf_{k\to+\infty}\frac{1}{2}\int|\nabla u_{n_k}|^2 \geq 0$$

which contradicts (1.17).

step 3 Dichotomy cannot occur. Otherwise, from (1.16), we have sequences v_k, w_k and $0<\alpha<1$ such that

$$\|v_k\|_{L^2} = \alpha M, \quad \|w_k\|_{L^2} = (1-\alpha)M$$

and

$$I(M) \geq \liminf_{k\to+\infty} E(v_k) + \liminf_{k\to+\infty} E(w_k).$$

In particular, this implies:

$$I(M) \geq I(\alpha M) + I((1-\alpha)M) \tag{1.18}$$

and thus from (1.17):

$$1 \leq \alpha^{\frac{2(1-s_c)}{|s_c|}} + (1-\alpha)^{\frac{2(1-s_c)}{|s_c|}} \quad \text{for some} \quad 0<\alpha<1.$$

Now a straightforward convexity argument implies from $\frac{2(1-s_c)}{|s_c|} > 1$ that $\alpha = 0$ or $\alpha = 1$, a contradiction.

step 4 Conclusion. We conclude that only compactness occurs ie

$$u_{n_k}(\cdot + x_k) \to u \quad \text{in} \quad L^{p+1}.$$

Observe then from the strong L^{p+1} convergence and the lower semicontinuity of the $\dot{H}^1$ norm that u attains the infimum:

$$\|u\|_{L^2} = M, \quad E(u) = I(M).$$

It thus remains to characterize the infimum. We claim that:

$$u(x) = Q_{\lambda(M)}(\cdot + x_0)e^{i\gamma_0} \tag{1.19}$$

which concludes the proof of Proposition 1.6.
Proof of (1.19): First observe from $\int|\nabla|u||^2 \leq \int|\nabla u|^2$ that $v=|u|$ is a minimizer with $v\geq 0$. From standard Euler Lagrange theory, v solves

$$\Delta v + v|v|^{p-1} = \mu v, \quad v\in H^1.$$

The Lagrange multiplier, which a priori depends on v, can be computed by multiplying the equation by v and then $y \cdot \nabla v$ (Pohozaev integration) leading to:

$$\mu = \mu(M) = \frac{N+2-p(N-2)}{2M\left(\frac{N(p-1)}{4}-1\right)} I(M) > 0.$$

We now observe by rescaling that $w(x) = \lambda^{\frac{2}{p-1}} v(\lambda x)$ with $\lambda = \sqrt{\mu}$ satisfies

$$\Delta w - w + w|w|^{p-1} = 0, \quad w \in H^1(\mathbb{R}^N), \quad w \geq 0,$$

and w non zero. From the uniqueness statement of Proposition 1.3, this yields:

$$w(x) = Q(x - x_0),$$

and hence $v(x) = Q_{\lambda(M)}(x - x_0)$. This implies in particular that v does not vanish which together with $\int |\nabla u|^2 = \int |\nabla |u||^2$ -because they both are minimizers- implies[2]

$$u(x) = |u(x)|e^{i\gamma_0} = Q_{\lambda(M)}(x - x_0)e^{i\gamma_0},$$

and (1.19) is proved. □

Furthers comments

1. More general nonlinearities: The proof we have presented reproduces the original argument by Cazenave, Lions [12] and heavily relies onto the specific scaling properties of the nonlinearity. The advantage of this argument is to completely avoid the linearization near the ground state, but the prize to pay is the proof of global estimates like (1.18) which may be non trivial in the absence of symmetries. Another approach to stability proceeds by brute force linearization and the derivation of suitable coercivity properties of the linearized operator close to the ground state as for example done in Grillakis, Shatah, Strauss [26] to treat more general nonlinearities. We also refer to [44], [46], [47] for analogue results for gravitational kinetic equations which display a similar structure.

2. Asymptotic stability: An important question is to know whether, when stability holds, asymptotic stability also holds, that is do solutions asymptotically converge to the ground state in some local norm in space as $t \to +\infty$? This kind of property corresponds to a form of asymptotic irreversibility of the flow. This is an extremely delicate problem which has attracted a considerable amount of work for the past ten years. For some specific type of nonlinearities, asymptotic stability holds due to a fine tuning mechanism known as the "Fermi Golden Rule", see Soffer, Weinstein [105], Rodnianski, Soffer, Schlag [102], Sulem, Buslaev [10], Sigal, Zhou [20]. However, the case of pure power is still open because essentially small solitons are delicate to deal with. Indeed, in the pure power case, a soliton Q_λ can be made arbitrarily small in H^1 and not disperse. Moreover, one should keep in mind that the asymptotic stability is *false* in the completely integrable case $N = 1$, $p = 3$, see [112].

3. Generic long time dynamics: In general, one expects the long time behavior of the solution to correspond to a splitting of the solution into a non dispersive part corresponding to a sum of decoupled solitary waves moving at different speeds and a radiative part which disperses -ie goes to 0 in L^∞ say-. Such a general behavior

[2] see for example[49].

has been proved in the integrable case for the KdV system

$$(KdV) \quad \begin{cases} u_t + (u_{xx} + u^2)_x = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x), & u_0 : \mathbb{R}^N \to \mathbb{R}, \end{cases}$$

but complete integrability plays a very specific role here. See Rodnianski, Soffer, Schlag [102], Martel, Merle, Tsai [63], for the case of non integrable (NLS) systems but with specific nonlinearities. One should think here that in general, even the simpler question of the orbital stability of the multisolitary wave in the pure power case for (NLS) is open.

2. The blow up problem

We focus in this section on the NLS problem (1.1) with mass critical/super critical and energy subcritical nonlinearities, or equivalently according to (1.6):

$$0 \le s_c < 1, \quad 1 + \frac{4}{N} \le p < 2^* - 1.$$

Our aim is to collect old and new results regarding the qualitative description of blow up solutions which involves so far many open problems.

2.1. **Existence of blow up solutions: the virial law.** The Cauchy theory ensures global existence for small data in H^1 but for large data, the Gagliardo Nirenberg inequality (1.7) does not suffice anymore to ensure global existence. A well known global obstructive argument known as the virial law allows one to very easily prove the existence of finite time blow up solutions.

Theorem 2.1 (Virial blow up for $E_0 < 0$)**.** *Let $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$ with*

$$E_0 < 0,$$

then the corresponding solution to (1.1) blows up in finite time $0 < T < +\infty$.

Proof of Theorem 2.1. Integrating by parts in (1.1), we find:

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 4N(p-1)E_0 - \frac{16 s_c}{N - 2s_c} \int |\nabla u|^2 \le 4N(p-1)E_0 \quad (2.1)$$

from $s_c \ge 0$. Hence from $E_0 < 0$, the positive quantity $\int |x|^2 |u(t,x)|^2 dx$ lies below an inverted parabola and hence the solution cannot exist for all times. □

This blow up argument is extraordinary because it provides a blow up criterion based essentially on a pure Hamiltonian information $E_0 < 0$ which applies to arbitrarily large initial data in H^1. In particular, it exhibits an *open* region of the energy space -up to extra integrability condition- where blow up is proven to be a stable phenomenon. While it may seem at first hand to be very specific to the (NLS) problem, this kind of convexity argument is very common for parabolic or wave type problems, see for example [30], kinetic problems [25], or even compressible Euler equations, [104]. However, it has has two major weaknesses:
(i) It heavily relies on a very specific algebra and hence is very unstable by perturbation of the equation. It thus is completely unable to predict blow up even in situations where it is strongly expected. A typical case is for example (NLS) on a domain with Dirichlet boundary conditions, [96].
(ii) More fundamentally, this argument is *purely obstructive* in nature and says very little a priori on the singularity formation. In fact the blow up time formally predicted which is the time of vanishing of the variance $\int |x|^2 |u|^2$ is almost never correct, solutions generically blow up before.

2.2. **Scaling lower bound on blow up rate.** In the setting of arbitrarily large initial data, little is known regarding the description of the singularity formation. This is mainly a consequence of the fact that the virial blow up argument does not provide any insight into the blow up dynamics. More generally, the a priori control of the blow up speed $|\nabla u(t)|_{L^2}$ which plays a fundamental role for the classification of blow up dynamics for example for the heat or the wave equation, is poorly understood. However a general lower bound on the blow up rate holds as a very simple consequence of the scaling invariance of the problem:

Proposition 2.2 (Scaling lower bound on blow up rate)**.** *Let $N \geq 1$, $0 \leq s_c < 1$. Let $u_0 \in H^1$ such that the corresponding solution $u(t)$ to (1.1) blows up in finite time $0 < T < +\infty$, then there holds:*

$$\forall t \in [0,T), \quad |\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{(T-t)^{\frac{1-s_c}{2}}}. \tag{2.2}$$

Proof of Proposition 2.2. We give the proof for $s_c = 0$ which is elementary and based on the scaling invariance of the equation and the local well posedness theory in H^1. The proof for $s_c > 0$ is similar and requires the Cauchy theory in $\dot{H}^{s_c} \cap \dot{H}^1$, see [76]. Consider for fixed $t \in [0,T)$

$$v^t(\tau,z) = |\nabla u(t)|_{L^2}^{-\frac{N}{2}} u\left(t + |\nabla u(t)|_{L^2}^{-2}\tau, |\nabla u(t)|_{L^2}^{-1} z\right).$$

v^t is a solution to (1.1) by scaling invariance. We have $|\nabla v^t(0)|_{L^2} = 1$, $|v^t|_{L^2} = |u_0|_{L^2}$, and thus by the resolution of the Cauchy problem locally in time by fixed point argument, there exists $\tau_0 > 0$ independent of t such that v^t is defined on $[0,\tau_0]$. Therefore, $t + |\nabla u(t)|_{L^2}^{-2}\tau_0 \leq T$ which is the desired result. □

One can ask for the sharpness of the bound (2.2), or equivalently for the existence of self similar solutions in the energy space, i.e. solutions which blow according to the scaling law

$$|\nabla u(t)|_{L^2} \sim \frac{1}{(T-t)^{\frac{1-s_c}{2}}}. \tag{2.3}$$

For $s_c = 0$, it is an important open problem, [7]. It is however proved in [97], [74] that the lower bound (2.2) is *not sharp* for data near the ground state in connection with the log log law, see Theorem 4.3. On the contrary, for $s_c > 0$, a stable self similar blow up regime in the sense of (2.3) is observed numerically, [106], and a rigorous derivation of these solutions is obtained in collaboration with Merle and Szeftel in [78] for slightly super critical problems:

Theorem 2.3 (Existence and stability of self similar solutions, [78])**.** *Let $1 \leq N \leq 5$ and $0 < s_c \ll 1$. Then there exists an open set of initial data $u_0 \in H^1$ such that the corresponding solution to (1.1) blows up with in finite time $T = T(u_0) < +\infty$ with the self similar speed:*

$$|\nabla u(t)|_{L^2} \sim \frac{1}{(T-t)^{\frac{1-s_c}{2}}}.$$

The extension of this result to the full critical range $s_c < 1$ is an important open problem, in particular to address the physical case $N = p = 3$, $s_c = \frac{1}{2}$, but is confronted to the construction and the understanding of the stationary self similar profiles which is poorly understood, see [78] for a further discussion.

2.3. **On concentration of the critical norm.** A second general phenomenon of finite blow up solutions is the *concentration of the critical norm.* The first result of this type goes back to Merle, Tsutsumi, [81] in the radial case, and generalized by Nawa, [92], for the mass critical NLS.

Theorem 2.4 (L^2 concentration phenomenon for $s_c = 0$, [81], [92]). *Let $s_c = 0$. Let $u_0 \in H^1$ such that the corresponding solution $u(t)$ to (1.1) blows up in finite time $0 < T < +\infty$. Then there exists $x(t) \in \mathcal{C}^0([0,T)\mathbb{R}^N)$ such that:*

$$\forall R > 0, \quad \liminf_{t \to T} \int_{|x-x(t)| \le R} |u(t,x)|^2 dx \ge \int Q^2. \tag{2.4}$$

Theorem 2.4 relies on the sharp variational characterization of the geound state solitary wave Q and we therefore postpone the proof to section 3.1. We refer to [108] for an extension to critical regularity $u_0 \in L^2$. Two natural questions following Theorem 2.4 are still open in the general case:
(i) Does the function $x(t)$ have a limit as $t \to T$ defining then at least one exact blow up point in space where L^2 concentration takes place?
(i) Which is the exact amount of mass focused by the blow up dynamic?

An explicit construction of blow up solutions due to Merle, [64], is the following: let k points $(x_i)_{1 \le i \le k} \in \mathbb{R}^N$, then there exists a blow up solution $u(t)$ which blows up in finite time $0 < T < +\infty$ exactly at these k points and accumulates exactly the mass:

$$|u(t)|^2 \rightharpoonup \Sigma_{1 \le i \le k} |Q|^2_{L^2} \delta_{x=x_i} \quad \text{as} \quad t \to T,$$

in the sense of measures. A general conjecture concerning L^2 concentration is formulated in [75] and states that a blow up solution focuses a quantized and universal amount of mass at a finite number of points in $\mathbb{R}^N$, the rest of the L^2 mass being purely dispersed. The exact statement which is directly related to the soliton resolution conjecture is the following:

Conjecture (*): *Let $u(t) \in H^1$ be a solution to (1.1) which blows up in finite time $0 < T < +\infty$. Then there exist $(x_i)_{1 \le i \le L} \in \mathbb{R}^N$ with $L \le \frac{\int |u_0|^2}{\int Q^2}$, and $u^* \in L^2$ such that: $\forall R > 0$,*

$$u(t) \to u^* \quad \text{in} \quad L^2(\mathbb{R}^N - \bigcup_{1 \le i \le L} B(x_i, R))$$

$$\text{and} \quad |u(t)|^2 \rightharpoonup \Sigma_{1 \le i \le L} m_i \delta_{x=x_i} + |u^*|^2 \quad \text{with} \quad m_i \in [\int Q^2, +\infty).$$

Let us now address the same question of the behavior of the critical norm for the super critical NLS $0 < s_c < 1$. There is no simple a priori lower bound like for (2.2) for the critical norm $|u(t)|_{\dot{H}^{s_c}}$ which is invariant by the scaling symmetry of the flow. Moreover, a major difference between the mass critical problem and the super critical problem is that the critical norm is conserved by the flow for $s_c = 0$ only, and this leads to dramatic differences in the blow up dynamics. We for example proved in [76] that for radial data the critical norm not only concentrates at blow up, it explodes:

Theorem 2.5 (Blow up of the critical norm, [76]). *Let $0 < s_c < 1$, $p < 5$ and $N \ge 2$. There exists a universal constant $\gamma = \gamma(N,p) > 0$ such that the following holds true. Let $u_0 \in H^1$ with radial symmetry and assume that the corresponding* *solution to* (1.1) *blows up in finite time* $T < +\infty$. *Then there holds the lower bound for* t *close enough to* T:

$$|u(t)|_{\dot{H}^{s_c}} \geq |\log(T-t)|^{\gamma(N,p)}.$$

Related results were proved for the Navier Stokes equation [16], and are a first step towards the understanding of the formation of the blow up bubble. Note that the logarithmic lower bound can be proved to be sharp in some regimes, [78], but there also exist regimes where the critical norm blows up polynomially, [80]. The regimes $N = 1, 2$ with $p \geq 5$ are still open, as well as the general non radial case. The proof relies on the quantification of a Liouville type theorem, see [38] for recent extensions to the wave equation.

2.4. **A sharp upper bound on blow up rate.** We now address the question of upper bounds on blow up rate for general solutions. A simple observation by Merle is that for $0 < s_c < 1$, the brute force time integration of the virial law (2.1) not only implies finite time blow up for $E_0 < 0$, it also immediately yields an upper bound on the blow up rate for any finite time blow up solution:

Theorem 2.6 (General upper bound on blow up rate). *Let* $0 < s_c < 1$ *and* $u_0 \in \Sigma$ *such that the corresponding solution to* (1.1) *blows up in finite time* $0 < T < +\infty$, *then:*

$$\int_0^T (T-t)|\nabla u(t)|^2_{L^2} dt < +\infty. \tag{2.5}$$

Note that in particular on a subsequence

$$|\nabla u(t_n)|_{L^2}(T-t_n) \to 0 \quad \text{as} \quad t_n \to T.$$

Interestingly enough, this bound fails for $s_c = 0$, see (3.10), and in fact *there exists no known upper bound on blow up rate in the mass critical case* which is one of the reason why the mass critical problem is in some sense more degenerate[3]. For $0 < s_c < 1$, we observed in collaboration with Merle and Szeftel [80] that a relatively elementary argument based on a localization of the virial identity as initiated in [76] implies an improved upper bound for u_0 radial.

Theorem 2.7 (Sharp upper bound on blow up rate for radial data, [80]). *Let*

$$N \geq 2, \quad 0 < s_c < 1, \quad p < 5.$$

Let the interpolation number[4]

$$\alpha = \frac{5-p}{(p-1)(N-1)}. \tag{2.6}$$

Let $u_0 \in H^1$ *with radial symmetry and assume that the corresponding solution* $u \in \mathcal{C}([0,T), H^1)$ *of* (1.1) *blows up in finite time* $T < +\infty$. *Then there holds the space time upper bound:*

$$\int_t^T (T-\tau)|\nabla u(\tau)|^2_{L^2} d\tau \leq C(u_0)(T-t)^{\frac{2\alpha}{1+\alpha}}. \tag{2.7}$$

This implies in particular

$$|\nabla u(t_n)|_{L^2} \lesssim \frac{1}{(T-t_n)^{\frac{1}{1+\alpha}}}$$

[3]The example of the (gKdV) problem and Theorem 4.9 indicate that there may be no bound...
[4]Observe that $0 < \alpha < 1$.

on a subsequence $t_n \to T$. Note that it would be very interesting to obtain the pointwise bound for all times.

Before proving Theorem 2.7 which relies on a sharp localization of the virial law, let us say that we do not know if the bound (2.5) is sharp. However, we claim that the general bound for radial data (2.7) *is indeed sharp* and saturated on a new class of blow up solutions: the collapsing ring profiles.

Theorem 2.8 (Collapsing ring solutions, [80])**.** *Let*

$$N \geq 2, \quad 0 < s_c < 1, \quad p < 5,$$

let $0 < \alpha < 1$ *be given by* (2.6) *and the Galilean shift:*

$$\beta_\infty = \sqrt{\frac{5-p}{p+3}}.$$

Let Q *be the one dimensional mass subcritical ground state* (1.9)*. Then there exists a time* $\underline{t} < 0$ *and a solution* $u \in \mathcal{C}([\underline{t}, 0), H^1)$ *of* (1.1) *with radial symmetry which blows up at time* $T = 0$ *according to the following dynamics. There exist geometrical parameters* $(r(t), \lambda(t), \gamma(t)) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}$ *such that:*

$$u(t,r) - \frac{1}{\lambda^{\frac{2}{p-1}}(t)} \left[Qe^{-i\beta_\infty y}\right] \left(\frac{r - r(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to 0 \quad in \quad L^2(\mathbb{R}^N). \tag{2.8}$$

The blow up speed, the radius of concentration and the phase drift are given by the asymptotic laws:

$$r(t) \sim |t|^{\frac{\alpha}{1+\alpha}}, \quad \lambda(t) \sim |t|^{\frac{1}{1+\alpha}}, \quad \gamma(t) \sim |t|^{-\frac{1-\alpha}{1+\alpha}} \quad as \quad t \uparrow 0. \tag{2.9}$$

Moreover, the blow up speed admits the equivalent:

$$|\nabla u(t)|_{L^2} \sim \frac{1}{(T-t)^{\frac{1}{1+\alpha}}} \quad as \quad t \uparrow 0. \tag{2.10}$$

Comments on the result

1. Standing and collapsing ring: The construction of ring solutions started in [98], [100] for $p = 5$ in dimension $N \geq 2$ where we constructed *standing ring* blow up solutions which concentrate on a standing sphere $r = 1$ at the speed given by the log-log law (4.14). The idea is that the geometry of the blow up set given by a standing sphere allows one to reduce the leading order blow up dynamics to the one dimension quintic NLS which is the mass critical one for $p = 5$. This has been further extended to other geometries in higher dimensions [29], [114]. Then in the breakthrough paper [17], Fibich, Gavish and Wang extended formally the construction to $3 < p < 5$ in dimension $N = 2$ and observed numerically the collapsing ring solutions which existence is made rigorous in [80]. Note that the collapsing ring is expected to be stable by radial perturbation of the data, but this is still an open problem.

2. Mass concentration: The ring solutions have a quite unexpected blow up behavior. Indeed, despite the fact that the problem is mass super critical, the structure (2.8) coupled with the speeds (2.9) imply the *concentration of the* L^2 *mass*

$$|u(t)|^2 \rightharpoonup |Q|_{L^2}^2 \delta_{x=0} \quad \text{as} \quad t \uparrow 0. \tag{2.11}$$

A contrario the self similar blow up solutions of Theorem 2.3 constructed in [78] have a strong limit in L^2 at blow up time. In fact, by rescaling, we can let the amount of concentrated mass in (2.11) be arbitrary, and hence the expected quantization of Conjecture (*) for the mass critical problem does not hold here. In some sense, the proof of Theorem 2.8 amounts showing that *in the ring regime*, the super critical problem can be treated as a mass critical problem. Moreoever, this is the first construction of blow up solutions for a large set of super critical regimes including the physical one $N = p = 3$.

We now turn to the proof of the sharp upper bound (2.7) which relies on a suitably localized virial identity in the continuation of [76].

Proof of Theorem 2.7. **step 1** Localized virial identity. Let $N \geq 2$, $0 < s_c < 1$ and $u \in \mathcal{C}([0,T), H^1)$ be a radially symmetric finite time blow up solution $0 < T < +\infty$. Pick a time $t_0 < T$ and a radius $0 < R = R(t_0) \ll 1$ to be chosen. Let $\chi \in \mathcal{D}(\mathbb{R}^N)$ and recall the localized virial identity[5] for radial solutions:

$$\frac{1}{2}\frac{d}{d\tau}\int \chi|u|^2 = Im\left(\int \nabla\chi \cdot \nabla u\overline{u}\right), \tag{2.12}$$

$$\frac{1}{2}\frac{d}{d\tau}Im\left(\int \nabla\chi \cdot \nabla u\overline{u}\right) = \int \chi''|\nabla u|^2 - \frac{1}{4}\int \Delta^2\chi|u|^2 - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int \Delta\chi|u|^{p+1}.$$

Applying with $\chi = \psi_R = R^2\psi(\frac{x}{R})$ where $\psi(x) = \frac{|x|^2}{2}$ for $|x| \leq 2$ and $\psi(x) = 0$ for $|x| \geq 3$, we get:

$$\begin{aligned} & \frac{1}{2}\frac{d}{d\tau}Im\left(\int \nabla\psi_R \cdot \nabla u\overline{u}\right) \\ = & \int \psi''(\frac{x}{R})|\nabla u|^2 - \frac{1}{4R^2}\int \Delta^2\psi(\frac{x}{R})|u|^2 - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int \Delta\psi(\frac{x}{R})|u|^{p+1} \\ \leq & \int |\nabla u|^2 - N\left(\frac{1}{2} - \frac{1}{p+1}\right)\int |u|^{p+1} + C\left[\frac{1}{R^2}\int_{2R\leq|x|\leq 3R}|v|^2 + \int_{|x|\geq R}|u|^{p+1}\right]. \end{aligned}$$

Now from the conservation of the energy:

$$\int |u|^{p+1} = \frac{p+1}{2}\int |\nabla u|^2 - (p+1)E(u_0)$$

from which

$$\int |\nabla u|^2 - N\left(\frac{1}{2} - \frac{1}{p+1}\right)\int |u|^{p+1} = \frac{N(p-1)}{2}E(u_0) - \frac{2s_c}{N-2s_c}\int |\nabla u|^2,$$

and thus:

$$\begin{aligned} & \frac{2s_c}{N-2s_c}\int |\nabla u|^2 + \frac{1}{2}\frac{d}{d\tau}Im\left(\int \nabla\psi_R \cdot \nabla u\overline{u}\right) \\ \lesssim & \left[|E_0| + \int_{|x|\geq R}|u|^{p+1} + \frac{1}{R^2}\int_{2R\leq|x|\leq 3R}|u|^2\right] \qquad (2.13) \\ \leq & \ C(u_0)\left[1 + \frac{1}{R^2} + \int_{|x|\geq R}|u|^{p+1}\right] \end{aligned}$$

from the energy and L^2 norm conservations.

[5] see [76] for further details.

step 2 Radial Gagliardo-Nirenberg interpolation estimate. In order to control the outer nonlinear term in (2.13), we recall the radial interpolation bound:

$$\|u\|_{L^\infty(r\geq R)} \leq \frac{\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}}}{R^{\frac{N-1}{2}}},$$

which together with the L^2 conservation law ensures:

$$\begin{aligned}\int_{|x|\geq R}|u|^{p+1} &\leq \|u\|^{p-1}_{L^\infty(r\geq R)}\int|u|^2 \leq \frac{C(u_0)}{R^{\frac{(N-1)(p-1)}{2}}}\|\nabla u\|_{L^2}^{\frac{p-1}{2}}\\ &\leq \delta\frac{2s_c}{N-2s_c}\int|\nabla u|^2+\frac{C}{\delta R^{\frac{2(N-1)(p-1)}{(5-p)}}}\\ &= \delta\frac{2s_c}{N-2s_c}\int|\nabla u|^2+\frac{C}{\delta R^{\frac{2}{\alpha}}}\end{aligned}$$

where we used Hölder for $p<5$ and the definition of α (2.6). Injecting this into (2.13) yields for $\delta>0$ small enough using $R\ll 1$ and $0<\alpha<1$:

$$\frac{s_c}{N-2s_c}\int|\nabla u|^2+\frac{d}{d\tau}Im\left(\int\nabla\psi_R\cdot\nabla u\overline{u}\right)\leq\frac{C(u_0,p)}{R^{\frac{2}{\alpha}}} \tag{2.14}$$

step 3 Time integration. We now integrate (2.14) twice in time on $[t_0,t_2]$ using (2.12). This yields up to constants using Fubini in time:

$$\begin{aligned}&\int\psi_R|u(t_2)|^2+\int_{t_0}^{t_2}(t_2-t)\|\nabla u(t)\|_{L^2}^2dt\\ \lesssim\ &\frac{(t_2-t_0)^2}{R^{\frac{2}{\alpha}}}+(t_2-t_0)\left|Im\left(\int\nabla\psi_R\cdot\nabla u\overline{u}\right)(t_0)\right|+\int\psi_R|u(t_0)|^2\\ \leq\ &C(u_0)\left[\frac{(t_2-t_0)^2}{R^{\frac{2}{\alpha}}}+R(t_2-t_0)\|\nabla u(t_0)\|_{L^2}+R^2\|u_0\|_{L^2}^2\right]\end{aligned}$$

We now let $t\to T$. We conclude that the integral in the left hand side converges and

$$\int_{t_0}^{T}(T-t)|\nabla u(t)|_{L^2}^2dt\leq C(u_0)\left[\frac{(T-t_0)^2}{R^{\frac{2}{\alpha}}}+R(T-t_0)\|\nabla u(t_0)\|_{L^2}+R^2\right]. \tag{2.15}$$

We now optimize in R by choosing:

$$\frac{(T-t_0)^2}{R^{\frac{2}{\alpha}}}=R^2\ \text{ ie }\ R(t_0)=(T-t_0)^{\frac{\alpha}{1+\alpha}}.$$

(2.15) now becomes:

$$\begin{aligned}\int_{t_0}^{T}(T-t)\|\nabla u(t)\|_{L^2}^2dt &\leq C(u_0)\left[(T-t_0)^{\frac{2\alpha}{1+\alpha}}+(T-t_0)^{\frac{\alpha}{1+\alpha}}(T-t_0)\|\nabla u(t_0)\|_{L^2}\right]\\ &\leq C(u_0)(T-t_0)^{\frac{2\alpha}{1+\alpha}}+(T-t_0)^2\|\nabla u(t_0)\|_{L^2}^2.\end{aligned} \tag{2.16}$$

In order to integrate this differential inequality, let

$$g(t_0)=\int_{t_0}^{T}(T-t)\|\nabla u(t)\|_{L^2}^2dt, \tag{2.17}$$

then (2.16) means:

$$g(t)\leq C(T-t)^{\frac{2\alpha}{1+\alpha}}-(T-t)g'(t)$$

ie

$$\left(\frac{g}{T-t}\right)' = \frac{1}{(T-t)^2}((T-t)g'+g) \leq \frac{1}{(T-t)^{2-\frac{2\alpha}{1+\alpha}}}.$$

Integrating this in time yields

$$\frac{g(t)}{T-t} \leq C(u_0) + \frac{1}{(T-t)^{1-\frac{2\alpha}{1+\alpha}}} \quad \text{ie} \quad g(t) \leq C(u_0)(T-t)^{\frac{2\alpha}{1+\alpha}}$$

for t close enough to T, which together with (2.17) yields (2.7). □

2.5. **More blow up problems.** The study of singularity formation for nonlinear dispersive equations has experienced a substantial acceleration since the end of the 1990's in particular in the continuation of the pioneering breakthrough works by Merle and Zaag on the nonlinear heat equation [83], [84], [85], and Martel and Merle on the mass critical (gKdV) problem [55], [56], [57], [58], [59]. The analysis has spread to various other problems and led to the development of new tools. It is not the aim of these notes to give a complete account of the existing literature, but we would like to point out the deep unity between some of these recent works. One particularly active direction or research is on *energy critical models* $s_c = 1$ which surprisingly enough display a similar structure like the mass critical problem, even though essential new phenomenons occur. This includes energy critical wave or heat problems, or more geometric problems like wave and Schrödinger maps for which the sole existence of blow up solutions in the critical regimes has been a long standing open problem. Among the key results obtained in the past ten years, let us mention some dynamical constructions: the first construction of blow up solutions for the energy critical wave map problem by Krieger, Schlag, Tataru [41], the derivation of the stable regime for the wave map jointly with Rodnianski [99], the first construction of blow up bubble for the Schrödinger map problem and the discovery of the rotational instability jointly with Merle and Rodnianski [77]. Moreover, a new generation of *classification theorems* have occurred in the direction of the multi solitary wave resolution conjecture, see in particular Duyckaerts, Kenig, Merle [15] for the energy critical nonlinear wave equation and the spectacular series of works by Merle and Zaag [86], [87], [88], [89], [90] which give the first complete classification of all blow up regimes for a nonlinear wave equation.

3. The mass critical problem

We focus in this section and for the rest of these notes onto the L^2 critical case

$$p = 1 + \frac{4}{N}, \quad s_c = 0.$$

which is the smallest power nonlinearity for which blow up occurs. We will show that a large part of the orbital stability theory developed for subcritical problems still applies in some generalized sense and provides some essential information on the structure of the blow up bubble.

3.1. **Variational characterization of the ground state.** The minimization problem (1.12) is no longer adapted to the critical problem due to the L^2 scaling invariance

$$u_\lambda(t,x) = \lambda^{\frac{N}{2}} u(\lambda^2 t, \lambda x). \tag{3.1}$$

Indeed, one easily proves that $I(M) = 0$ for $M << 1$ and $I(M) = -\infty$ for $M \gg 1$. In fact, as observed by Weinstein [111], the L^2 criticality of (1.1) corresponds to an exact balance between the kinetic and potential energies which can be quantified through the knowledge of the sharp constant in the Gagliardo-Nirenberg inequality (1.7).

Proposition 3.1 (Sharp Gagliardo-Nirenberg estimate, [111]). *Let the H^1 functional:*

$$J(v) = \frac{(\int |\nabla v|^2)(\int |v|^2)^{\frac{2}{N}}}{\int |v|^{2+\frac{4}{N}}}. \tag{3.2}$$

The minimization problem

$$\min_{v\in H^1,\quad v\neq 0} J(v)$$

is attained on the three parameters family:

$$\lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0)e^{i\gamma_0}, \quad (\lambda_0, x_0, \gamma_0) \in \mathbb{R}_*^+ \times \mathbb{R}^N \times \mathbb{R},$$

where Q is the unique ground state solution to:

$$\begin{cases} \Delta Q - Q + Q^{1+\frac{4}{N}} = 0, \quad Q > 0, \quad Q \text{ radial} \\ Q(r) \to 0 \quad \text{as} \quad r \to +\infty. \end{cases} \tag{3.3}$$

In particular, there holds the following Gagliardo-Nirenberg inequality with best constant:

$$\forall v \in H^1, \quad E(v) \geq \frac{1}{2}\int |\nabla v|^2 \left(1 - \left(\frac{|v|_{L^2}}{|Q|_{L^2}}\right)^{\frac{4}{N}}\right). \tag{3.4}$$

While $E(Q) = I(M) < 0$ in the subcritical case, we have in the critical case [6]

$$E(Q) = 0.$$

A reformulation of (3.4) which is very useful is the following variational characterization of Q:

Proposition 3.2 (Variational characterization of the ground state). *Let $v \in H^1$ such that*

$$\int |v|^2 = \int Q^2 \quad \text{and} \quad E(v) = 0,$$

then

$$v(x) = \lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0)e^{i\gamma_0},$$

for some parameters $\lambda_0 \in \mathbb{R}_+^*$, $x_0 \in \mathbb{R}^N$, $\gamma_0 \in \mathbb{R}$.

To sum up, the situation is as follows: let $v \in H^1$, then if $|v|_{L^2} < |Q|_{L^2}$, the kinetic energy dominates the potential energy and (3.4) yields $E(v) > C(v)\int |\nabla v|^2$ and the energy is in particular non negative; at the critical mass level $|v|_{L^2} = |Q|_{L^2}$, the only zero energy function is Q up to the symmetries of scaling, phase and translation which generate the three dimensional manifold of minimizers of (3.2). For $|v|_{L^2} > |Q|_{L^2}$, the sign of the energy is no longer prescribed.

Remark 3.3. *Remark that on the contrary to the subcritical case, the scaling (3.1) leaves the L^2 norm invariant and hence there are no small solitary waves in the critical case.*

A simple consequence of the sharp lower bound (3.4) is the concentration of the mass at blow up given by Theorem 2.4.

[6]This can be seen for example by multiplying the Q equation by $\frac{N}{2}Q + y \cdot \nabla Q$ and integrating by parts.

Proof of Theorem 2.4. The proof is purely variational. We prove the result in the radial case for $N \geq 2$. The general case follows from concentration compactness techniques, see [91], [28]. Let $u_0 \in H^1$ radial and assume that the corresponding solution $u(t)$ to (1.1) blows up at time $0 < T < +\infty$, or equivalently:

$$\lim_{t \to T} |\nabla u(t)|_{L^2} = +\infty. \tag{3.5}$$

We need to prove (2.4) and argue by contradiction: assume that for some $R > 0$ and $\varepsilon > 0$, there holds on some sequence $t_n \to T$,

$$\lim_{n \to +\infty} \int_{|y| \leq R} |u(t_n, y)|^2 dy \leq \int Q^2 - \varepsilon. \tag{3.6}$$

Let us rescale the solution by its size and set:

$$\lambda(t_n) = \frac{1}{|\nabla u(t_n)|_{L^2}}, \quad v_n(y) = \lambda^{\frac{N}{2}}(t_n) u(t_n, \lambda(t_n) y),$$

then from explicit computation:

$$|\nabla v_n|_{L^2} = 1 \text{ and } E(v_n) = \lambda^2(t_n) E(u). \tag{3.7}$$

First observe that v_n is H^1 bounded and we may assume on a sequence $n \to +\infty$:

$$v_n \rightharpoonup V \text{ in } H^1.$$

We first claim that V is non zero. Indeed, from (3.5), (3.7) and the conservation of the energy for $u(t)$, $E(v_n) \to 0$ as $n \to +\infty$, and thus:

$$\frac{1}{2 + \frac{4}{N}} \int |v_n|^{2 + \frac{4}{N}} = \frac{1}{2} \int |\nabla v_n|^2 - E(v_n) = \frac{1}{2} - E(v_n) \to \frac{1}{2} \text{ as } n \to +\infty.$$

Now from the compact embedding of $H^1_{radial} \hookrightarrow L^{2+\frac{4}{N}}$, $v_n \to V$ in $L^{2+\frac{4}{N}}$ up to a subsequence, and thus $\frac{1}{2+\frac{4}{N}} \int |V|^{2+\frac{4}{N}} \geq \frac{1}{2}$ and V is non zero. Moreover, from the weak H^1 convergence and the strong $L^{2+\frac{4}{N}}$ convergence,

$$E(V) \leq \liminf_{n \to +\infty} E(v_n) = 0.$$

Last, we have from (3.5), (3.6) and the weak H^1 convergence: $\forall A > 0$

$$\begin{aligned} \int_{|y| \leq A} |V(y)|^2 dy &\leq \liminf_{n \to +\infty} \int_{|y| \leq A} |v_n(y)|^2 dy \leq \lim_{n \to +\infty} \int_{|y| \leq \frac{R}{\lambda(t_n)}} |v(t_n, y)|^2 dy \\ &= \lim_{n \to +\infty} \int_{|x| \leq R} |u(t_n, x)|^2 dx \leq \int Q^2 - \varepsilon. \end{aligned}$$

Thus $\int |V|^2 \leq \int Q^2 - \varepsilon$ which together with V non zero and $E(V) \leq 0$ contradicts the sharp Gagliardo-Nirenberg inequality (3.4). □

The proof in the non radial case has been simplified by Hmidi, Keraani [28], which derived the following optimal result from concentration compactness - more precisely profile decomposition- techniques:

Lemma 3.4. *Let a sequence $u_n \in H^1$ with*

$$\limsup_{n \to +\infty} |\nabla u_n|_{L^2} \leq |\nabla Q|_{L^2}, \quad \limsup_{n \to +\infty} |u_n|_{L^{2+\frac{4}{N}}} \geq |Q|_{L^{2+\frac{4}{N}}},$$

then there exists $x_n \in \mathbb{R}^N$ and $V \in H^1$ such that up to a subsequence:

$$v_n(\cdot + x_n) \rightharpoonup V \quad \textit{in} \quad H^1 \quad \textit{with} \quad |V|_{L^2} \geq |Q|_{L^2}.$$

3.2. **The sharp global wellposedness criterion.** A generalization of Theorem 1.1 has been obtained by Weinstein [111]:

Theorem 3.5 (Global well posedness for subcritical mass, [111])**.** *Let $u_0 \in H^1$ with $|u_0|_{L^2} < |Q|_{L^2}$, the corresponding solution $u(t)$ to (1.1) is global and bounded in H^1. More precisely, the solution scatters as $t \pm \infty$.*

Proof of Theorem 3.5. From the conservation of the L^2 norm, $|u(t)|_{L^2} < |Q|_{L^2}$ for all $t \in [0, T)$, and thus an a priori bound on $|u(t)|_{H^1}$ follows from the conservation of the energy and the sharp Gagliardo-Nirenberg inequality (3.4) applied to $v = u(t)$. The scattering claim is easily proved for $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$ using the explicit pseudo conformal symmetry: if $u(t, x)$ is a solution to (1.1), then so is

$$v(t, x) = \frac{1}{|t|^{\frac{N}{2}}} u(\frac{-1}{t}, \frac{x}{t}) e^{i\frac{|x|^2}{4t}}. \tag{3.8}$$

The pseudo conformal symmetry is a well known symmetry of the linear Schrödinger flow and a symmetry of the nonlinear problem in the mass critical case only. It is moreover an L^2 isometry and thus applying Weinstein's criterion to v ensures that v has a limit in Σ as $t \uparrow 0$, and hence u scatters as $t \to +\infty$ as readily seen on (3.8). The case when $u_0 \in L^2$ only is considerably more delicate and relies on the rigidity theorem approach developed by Kenig, Merle [33], see Killip, Tao, Visan, Li, Zhang [35], [36], [37] and references therein, Dodson [14]. □

A spectacular feature is that Weinstein's criterion for global existence is sharp. On the one hand, from (3.3),

$$W(t, x) = Q(x)e^{it}$$

is a gobal solution to (1.1) with critical mass $|W|_{L^2} = |Q|_{L^2}$ which does not disperse. One should thus think of $|Q|_{L^2}$ as the minimal amount of mass required to avoid complete dispersion of the wave, and the solitary wave is the *smallest non linear object* for which dispersion and concentration exactly balance each other.

Observe now that the pseudo conformal symmetry (3.8) applied to the solitary wave solution $u(t, x) = Q(x)e^{it}$ yields the *explicit minimal mass blow up element*:

$$S(t, x) = \frac{1}{|t|^{\frac{N}{2}}} Q(\frac{x}{t}) e^{-i\frac{|x|^2}{4t} + \frac{i}{t}} \tag{3.9}$$

which scatters as $t \to -\infty$, and blows up at the origin at the speed

$$|\nabla S(t)|_{L^2} \sim \frac{1}{|t|} \tag{3.10}$$

by concentrating its mass:

$$|S(t)|^2 \rightharpoonup |Q|_{L^2}^2 \delta_{x=0} \quad \text{as} \quad t \uparrow 0. \tag{3.11}$$

Remark 3.6. *For the mass critical NLS, the sharp threshold for global existence and for scattering are therefore the same. This in fact an exceptional case induced by the Laplace operator and the Galilean symmetry -which is again an L^2 isometry-. For a more general dispersion of the type $(-\Delta)^\alpha$, these threshold are not the same,* [39].

3.3. **Orbital stability of the ground state.** More can be said on the structure of the singularity formation, and in particular on the blow up profile for initial data with L^2 mass just above the critical mass required for blow up:

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \ \text{ with } \ \int Q^2 \leq \int |u_0|^2 \leq \int Q^2 + \alpha^*\} \tag{3.12}$$

for some parameter $\alpha^* > 0$ small enough. This situation is moreover conjectured to locally describe the generic blow up dynamic around one blow up point.

Let us recall that $E(Q) = 0$ together with the virial blow up result of Theorem 2.1 imply the instability of the solitary wave $Q(x)e^{it}$. We claim however that the orbital stability of Q may be retrieved in some sense according to the following generalization of Theorem 1.4:

Theorem 3.7 (Orbital stability in the critical case)**.** *Let $N \geq 1$. For all $\alpha^* > 0$ small enough, there exists $\delta(\alpha^*)$ with $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$ such that the following holds true. Let $u_0 \in H^1$ with*

$$\int |u_0|^2 \leq \int Q^2 + \alpha^*, \quad E(u) \leq \alpha^* \int |\nabla u|^2, \tag{3.13}$$

and let $u(t)$ be the corresponding solution to (1.1) with life time $0 < T \leq +\infty$, then there exist $(x(t), \gamma(t)) \in \mathcal{C}^0([0,T), \mathbb{R}^N \times \mathbb{R})$ such that:

$$\forall t \in [0,T), \quad \|\lambda^{\frac{N}{2}}(t) u(t, \lambda(t)x + x(t)) e^{-i\gamma(t)} - Q\|_{H^1} < \delta(\alpha^*). \tag{3.14}$$

Note that a finite time blow up solution with small super critical mass automatically satisfies (3.13) near blow up time, and hence it is closed to the ground state in H^1 up to the set of H^1 symmetries. This property is again purely based on the conservation laws and the variational characterization of Q, and not on refined properties of the flow.

Proof of Theorem 3.7. Equivalently, we need to prove the following: let a sequence $u_n \in H^1$ with

$$\|u_n\|_{L^2} \to \|Q\|_{L^2}, \quad \limsup_{n \to +\infty} \frac{E(u_n)}{\|\nabla u_n\|^2_{L^2}} \leq 0, \tag{3.15}$$

let

$$v_n = \lambda_n^{\frac{N}{2}} u(\lambda_n x) \ \text{ with } \ \lambda_n = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u_n\|_{L^2}}, \tag{3.16}$$

then there exist $x_n \in \mathbb{R}^N$, $\gamma_n \in \mathbb{R}$ such that:

$$v_n(\cdot + x_n) e^{i\gamma_n} \to Q \ \text{ in } \ H^1 \ \text{ as } \ n \to +\infty. \tag{3.17}$$

Indeed, observe from (3.15) and (3.16) that

$$\|v_n\|_{L^2} \to \|Q\|_{L^2}, \quad \|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}, \quad \limsup_{n \to +\infty} E(v_n) \leq 0.$$

We now apply Proposition 1.6 to v_n. If vanishing occurs, then up to a subsequence, we have for n large enough:

$$E(v_n) \geq \frac{\|\nabla Q\|^2_{L^2}}{4}$$

which contradicts $\limsup_{n \to +\infty} E(v_n) \leq 0$. If dichotomy occurs, then there exist w_k, z_k and $0 < \alpha < 1$ such that

$$\|w_k\|_{L^2} \to \alpha \|Q\|_{L^2}, \quad \|z_k\|_{L^2} \to (1-\alpha)\|Q\|_{L^2} \ \text{ and } \ 0 \geq \limsup_{k \to +\infty} (E(w_k) + E(z_k)).$$

But from the sharp Gagliardo-Nirenberg inequality (3.4) applied to w_k and z_k, this implies

$$\|\nabla w_k\|_{L^2} + \|\nabla z_k\|_{L^2} \to 0 \quad \text{as} \quad k \to +\infty$$

and thus

$$\|v_{n_k}\|_{L^{2+\frac{4}{N}}} \to 0 \quad \text{as} \quad k \to +\infty,$$

and we are back to the vanishing case. Hence compactness occurs and

$$v_n(\cdot + x_n) \to V \quad \text{strongly in} \quad L^{2+\frac{4}{N}}, L^2$$

up to a subsequence. But then $E(v) \leq 0$ and $\|V\|_{L^2} = \|Q\|_{L^2}$ imply from (3.4) and Proposition 3.2 that $V(x) = Q(x + x_0)e^{i\gamma_0}$. This in turns implies $E(V) = 0$ and thus $|\nabla v_n(\cdot + x_n)|^2_{L^2} \to |\nabla Q|^2_{L^2}$ which impies (3.17). □

4. **Dynamical construction of blow up solutions**

We give in this section an overview on the known results on singularity formation in the mass critical case which go beyond the pure variational analysis of the previous section and rely on an explicit construction of blow up solutions for data near the ground state. This kind of question still attracts a considerable amount of interest, and we shall not be able to give a complete overview of the existing literature in these notes. We shall only give some key results in connection in particular with the question of the description of the flow near the ground state solitary wave which is the first nonlinear object.

4.1. **Minimal mass blow up.** Initial data $u_0 \in H^1$ with subcritical mass $|u_0|_{L^2} < |Q|_{L^2}$ generate global bounded solutions from Theorem 3.5. Moreoever, there exists an explicit minimal mass blow up element $S(t)$ induced by the pseudo conformal symmetry (3.8) and explicitly given by (3.9). The existence of the minimal element plays a distinguished role in the Kenig Merle approach to global existence [33]. An essential feature of (3.9) is that $S(t)$ is *compact* up to the symmetries of the flow, meaning that all the mass is put into the singularity formation. The basic intuition is that such a behavior is very special, and minimal elements should be classified[7]. This was proved using the pseudo conformal symmetry in a seminal work by Merle:

Theorem 4.1 (Classification of the minimal mass blow up solution, [66]). *Let $u_0 \in H^1$ with*

$$|u_0|_{L^2} = |Q|_{L^2}.$$

Assume that the corresponding solution to (1.1) blows up in finite time $0 < T < +\infty$. Then

$$u(t) = S(t)$$

up to the symmetries.

Before giving the proof of Merle's classification Theorem, let us say that the question of the existence of minimal elements in various settings has been a long standing open problem, mostly due to the fact that the existence of the minimal element for NLS relies entirely on the exceptional pseudo conformal symmetry. Merle in [67] considered the inhomogeneous problem

$$i\partial_t u + \Delta u + k(x)u|u|^2 = 0, \quad x \in \mathbb{R}^2$$

which breaks the full symmetry group, and obtains for non smooth k *non existence* results of minimal elements. A contrario and more recently, a sharp criterion for the

[7]This is a dispersive intuition which for example is completely false in the parabolic setting, [5].

existence and uniqueness of minimal solutions is derived in collaboration with Szeftel in [101] which relies on a dynamical construction and new Lypapounov rigidity functionals at the minimal mass level. A further extension to non local dispersion can be found in [39] which shows that minimal mass blow up is in fact the generic situation, and has little to do with the pseudo conformal symmetry, see also [2] for an extension to curved backgrounds, and Theorem 4.6 for the case of the critical (gKdV).

Proof of Theorem 4.1. This is the first proof of classification of minimal elements in the Schrödinger setting. We advise the reader to compare it with the proof of the Liouville theorem in [33] and observe the deep unity of both arguments. The original proof by Merle [66] has been further simplified by Banica [1] and Hmidi, Keraani [27], and it is the proof we present now.

step 1 Compactness of the flow in H^1 up to scaling. Let u as in the hypothesis of the Theorem with blow up time $0 < T < \infty$. Let

$$\lambda(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}} \to 0 \ \text{ as } \ t \to T.$$

Then

$$v(t,x) = \lambda^{\frac{N}{2}}(t) u(t, \lambda(t)x + x(t))$$

satisfies:

$$|\nabla v(t)|_{L^2} = |\nabla Q|_{L^2}, \ \lim_{t\to T} E(v) = 0, \ |v(t)|_{L2} = |Q|_{L^2}.$$

Arguing as for the proof of Theorem 3.7, we conclude from standard concentration compactness techniques and the variational characterization of the ground state that:

$$v(t, x + x(t))e^{i\gamma(t)} \to Q \ \text{ in } \ H^1 \ \text{ as } \ t \to T. \tag{4.1}$$

step 2 A refined Cauchy-Schwarz for critical mass functions. For $|w|_{L^2} < |Q|_{L^2}$, the energy controls the kinetic energy from (3.4). This controls fails for $|w|_{L^2} = |Q|_{L^2}$ but can be retrieved in some weak sense. Indeed, Banica observed the following: let a smooth real valued ψ and $w \in H^1$ with $|w|_{L^2} = |Q|_{L^2}$, then:

$$|Im(\nabla\psi \cdot \nabla w\overline{w})|^2 \lesssim \sqrt{E(w)} \left(\int |\nabla\psi|^2|w|^2\right)^{\frac{1}{2}}. \tag{4.2}$$

Indeed, for any $a > 0$,

$$|we^{ia\psi}|_{L^2} = |Q|_{L^2} \ \text{ and thus } \ E(we^{ia\psi}) \geq 0$$

and the result follows by expanding in a.

step 3 L^2 compactness of u and control of the concentration point. We now claim that u is L^2 compact: $\forall \varepsilon > 0$, $\exists R > 0$ such that

$$\forall t \in [0,T), \ \int_{|x|\geq R} |u(t,x)|^2 dx < \varepsilon. \tag{4.3}$$

Indeed, pick ε large enough, For $R > 0$, let $\chi_R(x) = \chi(\frac{x}{R})$ where χ is a smooth radial cut off function with $\chi(r) = 0$ for $r \leq \frac{1}{2}$, $\chi(r) = 1$ for $r \geq 1$. Then integrating by parts in (1.1) and using (4.2), we get:

$$\left|\frac{1}{2}\frac{d}{dt}\int \chi_R|u|^2\right| = |Im(\nabla\chi_R \cdot \nabla u\overline{u})| \leq C\sqrt{E(u)}\left(\int |\nabla\chi_R|^2|u|^2\right)^{\frac{1}{2}} \leq \frac{C}{R}\sqrt{E_0}|u_0|_{L^2}$$

where we used the conservation of energy and L^2 norm in the last step. Integrating in time on $[0,T]$ and using $T<+\infty$ yields (4.3).
Now observe that (4.1) and (4.3) automatically imply a localization of the concentration point:

$$\forall t\in[0,T),\quad |x(t)|\leq C(u_0). \tag{4.4}$$

step 4 $u\in\Sigma$. From (4.4) and up to a translation in space, we may consider a sequence of times $t_n\to T$ such that

$$x(t_n)\to 0\in\mathbb{R}^N.$$

From (4.1), (4.3):

$$|u(t_n,x)|^2\rightharpoonup\left(\int|Q|^2\right)\delta_0\ \text{ as }\ t_n\to T. \tag{4.5}$$

This means that at time T, all the mass is at the origin. Even though there is no finite speed of propagation for (NLS), the idea is to integrate backwards from the singularity to conclude that this implies that there was not much mass initially at infinity, that is

$$u_0\in\Sigma=H^1\cap\{xu\}\in L^2. \tag{4.6}$$

This step is very important and corresponds to a non trivial *gain of regularity* for the asymptotic object which is a direct consequence of its non dispersive behavior. Let a smooth radial cut off function $\psi(r)=r^2$ for $r\leq 1$, $\psi(r)=8$ for $r\geq 2$ and such that $|\nabla\psi|^2\leq C\psi$. Let $A>0$ and $\psi_A(r)=A^2\psi(\frac{r}{A})$, then:

$$|\nabla\psi_A|^2\lesssim\psi_A. \tag{4.7}$$

Then integrating by parts in (1.1), we have using (4.2) and (4.7):

$$\begin{aligned}\left|\frac{1}{2}\frac{d}{dt}\int\psi_A|u|^2\right| &= |Im(\nabla\psi_A\cdot\nabla u\overline{u})|\lesssim\sqrt{E_0}\left(\int|\nabla\psi_A|^2|u|^2\right)^{\frac{1}{2}}\\ &\lesssim \sqrt{E_0}\left(\int\psi_A|u|^2\right)^{\frac{1}{2}}\end{aligned}$$

or equivalently:

$$\left|\frac{d}{dt}\sqrt{\int\psi_A|u|^2}\right|\lesssim\sqrt{E_0}. \tag{4.8}$$

Now observe from (4.5) that

$$\int\psi_A|u(t_n)|^2\to 0\ \text{ as }\ t_n\to T.$$

Integrating (4.8) on $[t,t_n]$ and letting $t_n\to T$, we thus get:

$$\forall t\in[0,T),\quad\sqrt{\int\psi_A|u(t)|^2}\leq C(E_0)(T-t).$$

Note that the right hand side of the above expression is independent of A. We may thus let $A\to\infty$ and conclude to an even more precise version of (4.6):

$$\forall t\in[0,T),\quad u(t)\in\Sigma\ \text{ with }\ \int|x|^2|u(t,x)|^2dx\to 0\ \text{ as }\ t\to T. \tag{4.9}$$

step 5 Pseudo-conformal transformation. The conclusion of the proof is pure magic. It relies on the following completely general fact. Let $u(t)$ be a solution to

(1.1) leaving on $[0,T)$, then

$$v(t,x)=\left(\frac{T}{T+t}\right)^{\frac N2}u\left(\frac{tT}{T+t},\frac{Tx}{T+t}\right)e^{i\frac{|x|^2}{4(T+t)}}$$

is a solution to (1.1) with

$$|v|_{L^2}=|u|_{L^2}\ \text{ and }\ E(v)=\frac18\lim_{t\to T}\int|x|^2|u(t,x)|^2dx.$$

Applying this to u and using (4.9), this implies that

$$|v|_{L^2}=|u|_{L^2}=|Q|_{L^2}\ \text{ and }\ E(v)=0.$$

From Proposition 3.2, $v=Q$ up to the symmetries of the flow, and this concludes the proof of Theorem 4.1. □

4.2. **Log log blow up.** The only explicit blow up solution we have encountered so far is the minimal mass blow up bubble (3.9). This bubble is intrinsically unstable because a mass subcritical perturbation leads to a globally defined solution. The question of the description of *stable* blow up bubbles has attracted a considerable attention which started in the 80's with the development of sharp numerical methods and the prediction of the "log-log law" for NLS by Landman, Papanicoalou, Sulem, Sulem [43].

To simplify the presentation, let us restrict our attention with mass just above the minimal required for singularity formation

$$u_0\in\mathcal{B}_{\alpha^*}=\left\{u_0\in H^1\ \text{ with }\ |Q|_{L^2}<|u_0|_{L^2}<|u_0|_{L^2}+\alpha^*\right\},\quad 0<\alpha^*\ll1. \tag{4.10}$$

A general and fundamental open problem is to completely describe the flow for such initial data which in some sense corresponds according to the scattering statement of Theorem 3.5 to the first non linear zone. The generalized orbital stability statement of Theorem 3.7 ensures that under (4.10), if u blows up at $T<+\infty$. then for t close enough to T, the solution must admit a nonlinear decomposition

$$u(t,x)=\frac{1}{\lambda(t)^{\frac N2}}(Q+\varepsilon)(t,\frac{x-x(t)}{\lambda(t)})e^{i\gamma(t)}, \tag{4.11}$$

where

$$|\varepsilon(t)|_{H^1}\le\delta(\alpha^*),\quad \lambda(t)\sim\frac{1}{|\nabla u(t)|_{L^2}}. \tag{4.12}$$

This decomposition implies that in any blow up regime, the ground state solitary wave Q is a good approximation of the blow up profile, and this is the starting point for a perturbative analysis. The sharp description of the blow up bubble now relies on the extraction of the finite dimensional and possibly universal dynamic for the evolution of the geometrical parameters $(\lambda(t),x(t),\gamma(t))$ which is coupled to the infinite dimensional dispersive dynamic driving the small excess of mass $\varepsilon(t)$.

Remark 4.2. *An illuminating computation is to reformulate* (3.9) *for the minimal blow up element in terms of* (4.11)*:*

$$\lambda(t)=|t|,\quad \varepsilon(t,y)=Q(y)\left(e^{-i\frac{b(t)|y|^2}{4}}-1\right),\quad b(t)=|t|.$$

All possible regimes of $\lambda(t)$ are not known, but some progress has been done on the understanding of stable and threshold dynamics. The following Theorem summarizes the series of results obtained in [71], [72], [73], [74], [75], [97]:

Theorem 4.3 ([71], [72], [73], [74], [75], [97])**.** *Let $N \leq 5$. There exists a universal constant $\alpha^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and $u \in \mathcal{C}([0,T), H^1)$, $0 < T \leq +\infty$ be the corresponding solution to* (1.1).
(i) Sharp L^2 concentration: Assume $T < +\infty$, then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathcal{C}^1([0,T), \mathbb{R}_+^* \times \mathbb{R}^N \times \mathbb{R})$ and an asymptotic profile $u^* \in L^2$ such that

$$u(t) - \frac{1}{\lambda(t)^{\frac{N}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \quad \text{in} \quad L^2 \quad \text{as} \quad t \to T, \tag{4.13}$$

and the blow up point is finite:

$$x(t) \to x(T) \in \mathbb{R}^N \quad \text{as} \quad t \to T.$$

(ii) Classification of the speed: Under (i), the solution is either in the log-log regime

$$\lambda(t)\sqrt{\frac{\log|\log(T-t)|}{T-t}} \to \sqrt{2\pi} \quad \text{as} \quad t \to T \tag{4.14}$$

and then the asymptotic profile is not smooth:

$$u^* \notin H^1 \quad \text{and} \quad u^* \notin L^p \quad \text{for} \quad p > 2, \tag{4.15}$$

or there holds the sharp lower bound

$$\lambda(t) \lesssim C(u_0)(T - t) \tag{4.16}$$

and the improved regularity:

$$u^* \in H^1. \tag{4.17}$$

(iii) Sufficient condition for log-log blow up: Assume $E_0 < 0$, then the solution blows un finite time $T < +\infty$ in the log log regime (4.14).
(iv) H^1 stability of the log log blow up: More generally, the set of initial data in $\mathcal{B}_{\alpha^*}$ such that the corresponding solution to (1.1) blows up in finite time with the log-log law (4.14) is open in H^1.

Comments on the result

1. The log log law. The log log law (4.14) of stable blow up was first proposed in the pioneering formal and numerical work [43]. The first rigorous construction of such a solution is due to Galina Perelman [95] in dimension $N = 1$. The proof of Theorem 4.3 involves a mild coercivity property of the linearized operator close to Q, see the Spectral Property 5.6, which is proved in dimension $N = 1$ in [71] and checked numerically in an elementary way in [18] for $N \leq 5$. Here we face the difficulty that there is no explicit formula for the ground state in dimensions $N \geq 2$.

2. Upper bound on the blow up speed: There exists *no upper bound of no type* on the blow up speed $|\nabla u(t)|_{L^2}$ in the mass critical case, even for data $u_0 \in \mathcal{B}_{\alpha^*}$ only. The lower bound (4.16) is sharp and saturated by the minimal blow up element $S(t)$. The derivation of slower blow up, which through the pseudo conformal symmetry is equivalent to the construction of infinite time grow up solutions, is linked to the description of the flow near the ground state which is still incomplete for (NLS). The intuition is led here by the recent classification results obtained for the mass critical KdV problem which we present in section 4.5.

3. Quantization of the blow up mass: The strong convergence (4.13) gives a complete description of the blow up bubble in the scaling invariance space and implies in particular that the mass which is put into the singularity formation is quantized

$$|u(t)|^2 \rightharpoonup |Q|_{L^2}^2 \delta_{x=x(T)} + |u^*|^2 \quad \text{as} \quad t \to T, \quad |u^*|^2 \in L^2$$

which shows the validity of the conjecture (*) for near minimal mass blow up solutions. This kind of general asymptotic simplification theorem started in the dispersive setting in the pioneering works by Martel and Merle [55] , and was recently propagated to impressive classification result -without assumption of size on the data- for energy critical wave equations [15]. Underlying the convergence (4.13) is the asymptotic stability statement of the solitary wave as the universal attractor of all blow up solutions which in the language (4.11) means

$$\varepsilon(t,x) \to 0 \quad \text{as} \quad t \to T \quad \text{in} \quad L^2_{\text{loc}}.$$

In fact, there are steps in the proof of Theorem 4.3 and the derivation of either upper bounds or lower bounds on the blow up rate is intimately connected to the question of dispersion for the excess of mass $\varepsilon(t,x)$.

4. Asymptotic profile: The regularity of the asymptotic profile u^* sees the change of regime because in the stable log log regime, the singular and regular parts of the solution are very much coupled, while they are more separated in any other regimes.

4.3. **Threshold dynamics.** We still consider small super critical mass initial data $u_0 \in \mathcal{B}_{\alpha^*}$. Theorem 4.3 describes the stable log log blow up. The explicit minimal mass blow up given by (4.3) does not belong to this class and is unstable. Bourgain and Wang [8] observed however that $S(t)$ can be stabilized on a finite codimensional manifold, and they do so by integrating the flow backwards from the singularity. The excess of mass in this regime corresponds to a *flat and smooth* asymptotic profile. More precisely, let $N = 1, 2$, fix the origin as the blow up point and let a limiting profile $u^* \in H^1$ such that

$$\frac{d^i}{dx^i} u^*(0) = 0, \quad 1 \le i \le A, \quad A \gg 1, \tag{4.18}$$

then one can build a solution to (1.1) which blows up at $t = 0$ at $x = 0$ and satisfies:

$$u(t) - S(t) \to u^* \quad \text{in} \quad H^1 \quad \text{as} \quad t \uparrow 0. \tag{4.19}$$

We refer to [40] for a further discussion on the manifold construction. Note that this produces blow up solutions with super critical mass $|u_0|_{L^2} > |Q|_{L^2}$ which saturate the lower bound (4.16):

$$|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}.$$

Also for small L^2 perturbation of $S(-1)$, the Bourgain Wang solution blows up at $t = 0$ but is global and scatters as $t \to -\infty$, simply because $S(t)$ scatters as $t \to -\infty$, and scattering is an L^2 stable behavior[8].

We proved in collaboration with Merle and Szeftel in [79] that these solutions sit on the border between the two open sets of solutions which scatter to the left as $t \to -\infty$ and respectively are global to the right and scatter as $t \to +\infty$, and blow up in finite time *in the log log regime*.

[8]This is a simple consequence of Strichartz estimates and the L^2 critical Cauchy theory of Cazenave-Weissler [13].

Theorem 4.4 (Strong instability of Bourgain Wang solutions, [79]). *Let $N = 1, 2$. Let u^* be a smooth radially symmetric satisfying the degeneracy at blow up point* (4.18)*. Let $u^0_{BW} \in \mathcal{C}((-\infty, 0), H^1)$ be the corresponding Bourgain-Wang. solution. Then there exists a continuous map*

$$\Gamma : [-1, 1] \to \Sigma$$

such that the following holds true. Given $\eta \in [-1, 1]$, let $u_\eta(t)$ be the solution to (1.1) *with data $u_\eta(-1) = \Gamma(\eta)$, then:*

- *$\Gamma(0) = u^0_{BW}(-1)$ ie $\forall t < 0$, $u_{\eta=0}(t) = u^0_{BW}(t)$ is the Bourgain Wang solution on $(-\infty, 0)$ with blow up profile $S(t)$ and regular part u^*;*
- *$\forall \eta \in (0, 1]$, $u_\eta \in \mathcal{C}(\mathbb{R}, \Sigma)$ is global in time and scatters forward and backwards;*
- *$\forall \eta \in [-1, 0)$, $u_\eta \in \mathcal{C}((-\infty, T^*_\eta), \Sigma)$ scatters to the left and blows up in finite time $T^*_\eta < 0$ on the right in the log-log regime* (4.14) *with*

$$T^*_\eta \to 0 \quad as \quad \eta \to 0. \tag{4.20}$$

Note that this theorem describes the flow near the Bourgain Wang solution along one instability solution. A major open problem in the field is to describe the flow near the ground state Q. Theorem 4.4 is a first step towards the description of the flow near the Bourgain Wang solutions which itself is a very interesting open problem.

4.4. **Structural instability of the log-log law.** Another model with fundamental physical relevance, [106], is the Zakharov system in dimensions $N = 2, 3$:

$$\begin{cases} iu_t = -\Delta u + nu \\ \frac{1}{c_0^2} n_{tt} = \Delta n + \Delta |u|^2 \end{cases} \tag{4.21}$$

for some fixed constant $0 < c_0 < +\infty$. In the limit $c_0 \to +\infty$, we formally recover (1.1). In dimension $N = 2$, this system displays a variational structure like (1.1), even though the scaling symmetry is destroyed by the wave coupling. In particular, a virial law in the spirit of (2.1) holds and yields finite time blow up for radial non positive energy initial data, see Merle [69]. Moreover, a one parameter family of blow up solutions has been constructed as a continuation of the exact $S(t)$ solution for (1.1), see Glangetas, Merle, [24]. These explicit solutions have blow up speed:

$$|\nabla u(t)|_{L^2} \sim \frac{C(u_0)}{T - t}$$

and appear to be *stable* from numerics, see Papanicolaou, Sulem, Sulem, Wang, [94]. Now from Merle, [68], *all finite time blow up solutions to (4.21) satisfy*

$$|\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{T - t}.$$

In particular, there will be no log-log blow up solutions for (4.21). This fact suggests that in some sense, the Zakharov system provides a much more stable and robust blow up dynamics than its asymptotic limit (NLS). This fact enlightens the belief that the log-log law heavily relies on the specific algebraic structure of (1.1), and some non linear degeneracy properties will indeed be at the heart of our understanding of the blow up dynamics. Let us insist that the fine study of the singularity formation for the Zakharov system is mostly open, and in some sense it is the first towards the understanding of more physical and complicated systems related to Maxwell's equations.

4.5. **Classification of the flow near Q: the case of the generalized KdV.** We present in this section the recent series of results [62], [61], [60] which give a complete description of the flow near the ground for an L^2 critical problem: the generalized KdV equation

$$(gKdV) \quad \begin{cases} \partial_t u + (u_{xx} + u^5)_x = 0 \\ u_{|t=0} = u_0 \end{cases}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{4.22}$$

This problems admits the same L^2 norm and energy conservation laws like (NLS), and the same mass critical scaling. The solitary wave is here a traveling wave solution

$$u(t,x) = Q(x-t)$$

where Q is the one dimensional ground state

$$Q(x) = \left(\frac{3}{\mathrm{ch}^2(2x)}\right)^{\frac{1}{4}}.$$

This model problem has been thoroughly studied by Martel and Merle in the pioneering breakthrough works [55], [56],[57],[58], [59]

as a toy model for which the pseudo conformal symmetry and the associated virial algebra are lost. The long standing open problem of the existence of blow up solutions was solved in [70], but the structure of the singularity formation was still only poorly understood. We give in the series of works [62], [61], [60] a complete description of the flow near the ground state and expect that the obtained picture is canonical.

More precisely, let the set of initial data

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{y>0} y^{10}\varepsilon_0^2 < 1 \right\},$$

and consider the L^2 tube around the family of solitary waves

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 \text{ with } \inf_{\lambda_0>0,\, x_0\in\mathbb{R}} \|u - \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right)\|_{L^2} < \alpha^* \right\}.$$

We first claim the rigidity of the dynamics for data in $\mathcal{A}$:

Theorem 4.5 (Rigidity of the flow in $\mathcal{A}$, [62])**.** *Let $0 < \alpha_0 \ll \alpha^* \ll 1$ and $u_0 \in \mathcal{A}$. Let $u \in \mathcal{C}([0,T), H^1)$ be the corresponding solution to* (4.22)*. Then one of the following three scenarios occurs:*

(Blow up)*: the solution blows up in finite time $0 < T < +\infty$ in the universal regime*

$$|u(t)|_{H^1} = \frac{\ell(u_0) + o(1)}{T-t} \quad as \quad t \to T, \quad \ell(u_0) > 0. \tag{4.23}$$

(Soliton)*: the solution is global $T = +\infty$ and converges asymptotically to a solitary wave.*

(Exit)*: the solution leaves the tube $\mathcal{T}_{\alpha^*}$ at some time $0 < t_u^* < +\infty$.*

Moreover, the scenarios (Blow up) *and* (Exit) *are stable by small perturbation of the data in $\mathcal{A}$.*

In other words, we obtain a complete classification of solutions with data in $\mathcal{A}$ which remain close in the L^2 critical sense to the manifold of solitary waves. It remains to understand the long time dynamics in the (Exit) regime. The first step is the *existence and uniqueness* of a minimal blow up element which is the generalization of the $S(t)$ dynamics for (NLS):

Theorem 4.6 (Existence and uniqueness of the minimal mass blow up element, [61])**.**
(i) Existence. *There exists a solution* $\tilde{S}(t) \in \mathcal{C}((0,+\infty), H^1)$ *to* (4.22) *with minimal mass* $\|\tilde{S}(t)\|_{L^2} = \|Q\|_{L^2}$ *which blows up backward at the origin at the speed*

$$|\nabla \tilde{S}(t)|_{L^2} \sim \frac{1}{t} \quad as \quad t \downarrow 0,$$

and is globally defined on the right in time.
(ii) Uniqueness. *Let* $u_0 \in H^1$ *with* $\|u_0\|_{L^2} = \|Q\|_{L^2}$ *and assume that the corresponding solution* $u(t)$ *to* (4.22) *blows up in finite time. Then*

$$u \equiv S$$

up to the symmetries of the flow.

In other words, we recover Merle's result in the absence of pseudo conformal symmetry, and the proof is here completely dynamical and deeply related to the analysis of the inhomogeneous NLS model in [101]. We now claim that $\tilde{S}$ is the *universal attractor* of all solutions in the (Exit) regime.

Theorem 4.7 (Description of the (Exit) scenario, [61])**.** *Let* $u(t)$ *be a solution of* (4.22) *corresponding to the* (Exit) *scenario in Theorem 4.6 and let* $t_u^* \gg 1$ *be the corresponding exit time. Then there exist* $\tau^* = \tau^*(\alpha^*)$ *(independent of* u*) and* (λ_u^*, x_u^*) *such that*

$$\left\|(\lambda_u^*)^{\frac{1}{2}} u\left(t_u^*, \lambda_u^* x + x_u^*\right) - \tilde{S}(\tau^*, x)\right\|_{L^2} \leq \delta_I(\alpha_0),$$

where $\delta_I(\alpha_0) \to 0$ *as* $\alpha_0 \to 0$.

In fact a solution at the (Exit) time acquires a *specific* profile with a large defocusing spreading $\lambda_u^* \gg 1$ -coherent with dispersion-. Understanding the flow for u after the (Exit) is now equivalent to controlling the flow of $\tilde{S}(t)$ for large times. For (NLS), we can see on the formula (3.9) that $S(t)$ blows up at $t = 0$ and *scatters* as $t \to +\infty$. For (gKdV), we know from Theorem 4.6 that $\tilde{S}(t)$ is global as $t \to +\infty$, but scattering is not known. We however expect that this holds true, in which case because scattering is open in L^2 thanks to the Kenig, Ponce, Vega L^2 critical theory [34], we obtain the following:

Corollary 4.8. *Assume that* $S(t)$ *scatters as* $t \to +\infty$*. Then any solution in the* (Exit) *scenario is global for positive time and scatters as* $t \to +\infty$*.*

It is important to notice that the above results rely on the *explicit* computation of the solution in the various regimes, and not on algebraic virial type identities. Indeed we introduce the nonlinear decomposition of the flow

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{1}{2}}}(Q+\varepsilon)\left(t, \frac{x - x(t)}{\lambda(t)}\right)$$

and show that to leading order, $\lambda(t)$ obeys the dynamical system

$$\lambda_{tt} = 0, \quad \lambda(0) = 1. \tag{4.24}$$

The three regimes (Exit), (Blow up), (Soliton) now correspond respectively to $\lambda_t(0) > 0$, $\lambda_t(0) < 0$ and the threshold dynamic $\lambda_t(0) = 0$.

Our last result shows that the universality of the leading order ODE (4.24) is valid *under the decay assumption* $u_0 \in \mathcal{A}$ *only*, and indeed the tail of slowly decaying data can interact with the solitary wave which for (KdV) is moving to the right, and this may lead to new exotic singular regimes:

Theorem 4.9 (Exotic blow up regimes, [60]).
(i) Blow up in finite time: *for any* $\nu > \frac{11}{13}$, *there exists* $u \in C((0,1], H^1)$ *solution to* (4.22) *which blows up at* $t = 0$ *with speed*

$$\|u_x(t)\|_{L^2} \sim t^{-\nu} \quad as \quad t \to 0^+. \tag{4.25}$$

(ii) Blow up in infinite time: *there exists* $u \in C([1, +\infty), H^1)$ *solution of* (4.22) *growing up at* $+\infty$ *with speed*

$$\|u_x(t)\|_{L^2} \sim e^t \quad as \quad t \to +\infty. \tag{4.26}$$

For any $\nu > 0$, *there exists* $u \in C([1, +\infty), H^1)$ *solution of* (4.22) *blowing up at* $+\infty$ *with*

$$\|u_x(t)\|_{L^2} \sim t^{\nu} \quad as \quad t \to +\infty. \tag{4.27}$$

Such solutions can be constructed arbitrarily close in H^1 *to the ground state solitary wave.*

Note that this implies in particular that blow up can be arbitrarily slow.

We expect that the (KdV) picture is fairly general, and Theorem 4.4 is a first step towards a similar description for the mass critical NLS. Let also mention that in super critical regimes and large dimensions, Nakanishi and Schlag have obtained a related classification of the flow near the solitary wave which in particular involves a complete description of the scattering zone and its boundary.

5. The log log upper bound on blow up rate

Our aim in this section is to present a self contained proof of the first result contained in Theorem 4.3 for the mass critical problem and for small super critical mass initial data.

Theorem 5.1 ([71],[72]). *Let* $N \leq 4$. *There exist universal constants* $\alpha^*, C^* > 0$ *such that the following holds true. Given* $u_0 \in \mathcal{B}_{\alpha^*}$ *with*

$$E_G(u) = E(u) - \frac{1}{2}\left(\frac{Im(\int \nabla u \overline{u})}{|u|_{L^2}}\right)^2 < 0, \tag{5.1}$$

then the corresponding solution $u(t)$ *to (1.1) blows up in finite time* $0 < T < +\infty$ *and there holds for* t *close to* T*:*

$$|\nabla u(t)|_{L^2} \leq C^* \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}. \tag{5.2}$$

This theorem is the first fundamental improvement on the virial law: it not only shows blow up in finite time of non positive energy solutions, it also gives an upper bound on the blow up rate which in particular rules out the $S(t)$ type of dynamic. Moreover the steps of the proof are in some sense canonical for our study.

The heart of our analysis will be to exhibit as a consequence of dispersive properties of (1.1) close to Q *strong rigidity constraints for the dynamics of non positive energy solutions. These will in turn imply monotonicity properties, that is the existence of a Lyapounov function. The corresponding estimates will then allow us to prove blow up in a dynamical way and the sharp upper bound on the blow up speed will follow.*

5.1. **Existence of the geometrical decomposition.** Let an initial data $u_0 \in \mathcal{B}(\alpha^*)$ with $E_G(u_0) < 0$. First observe that up to a fixed Galilean transform, we may equivalently assume

$$E(u_0) < 0 \ \text{ and } \ Im(\nabla u \overline{u_0}) = 0. \tag{5.3}$$

Proposition 3.7 thus applies and implies for $t \in [0, T)$ the existence of a geometrical decomposition

$$u(t,x) = \frac{1}{\lambda_0^{\frac{N}{2}}(t)}(Q+\varepsilon_0)(t, \frac{x - x_0(t)}{\lambda_0(t)})e^{i\gamma_0(t)}, \quad \|\varepsilon_0\|_{H^1} \leq \delta(\alpha^*).$$

Let us observe that this geometrical decomposition is by no mean unique. Nevertheless, one can freeze and regularize this decomposition by choosing a set of orthogonality conditions on the excess of mass: this is the modulation argument which will be examined later on. Let us so far assume that we have a smooth decomposition of the solution: $\forall t \in [0, T)$,

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\varepsilon)(t, \frac{x - x(t)}{\lambda(t)})e^{i\gamma(t)} \tag{5.4}$$

with

$$\lambda(t) \sim \frac{C}{|\nabla u(t)|_{L^2}} \ \text{ and } \ |\varepsilon(t)|_{H^1} \leq \delta(\alpha^*) \to 0 \ \text{ as } \ \alpha^* \to 0.$$

To study the blow up dynamic is now equivalent to understanding the coupling between the finite dimensional dynamic which governs the evolution of the geometrical parameters $(\lambda(t), \gamma(t), x(t))$ *and the infinite dimensional dispersive dynamic which drives the excess of mass* $\varepsilon(t)$.

To enlighten the main issues, let us rewrite (1.1) in the so-called rescaled variables. Let us introduce the rescaled time:

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)}.$$

It is elementary to check that whatever is the blow up behavior of $u(t)$, one always has:

$$s([0, T)) = \mathbb{R}^+.$$

Let us set:

$$v(s,y) = e^{i\gamma(t)}\lambda(t)^{\frac{N}{2}}u(\lambda(t)x + x(t)).$$

For a given function f, we introduce the generator of L^2 scaling

$$\Lambda f = \frac{N}{2}f + y \cdot \nabla f$$

then from direct computation, $u(t,x)$ solves (1.1) on $[0,T)$ iff $v(s,y)$ solves: $\forall s \geq 0$,

$$iv_s + \Delta v - v + v|v|^{\frac{4}{N}} = i\frac{\lambda_s}{\lambda}\Lambda v + i\frac{x_s}{\lambda} \cdot \nabla v + \tilde{\gamma}_s v, \tag{5.5}$$

where $\tilde{\gamma} = -\gamma - s$. Now $v(s,y) = Q(y) + \varepsilon(s,y)$ and we linearize (5.5) close to Q. The obtained system has the form:

$$i\varepsilon_s + L\varepsilon = i\frac{\lambda_s}{\lambda}\Lambda Q + \gamma_s Q + i\frac{x_s}{\lambda} \cdot \nabla Q + R(\varepsilon), \tag{5.6}$$

$R(\varepsilon)$ formally quadratic in ε, and $L = (L_+, L_-)$ is the matrix linearized operator closed to Q which has components:

$$L_+ = -\Delta + 1 - \left(1 + \frac{4}{N}\right)Q^{\frac{4}{N}}, \quad L_- = -\Delta + 1 - Q^{\frac{4}{N}}.$$

A standard approach is to think of equation (5.6) in the following way: it is essentially a linear equation forced by terms depending on the law for the geometrical parameters. The classical study of this kind of system relies on the understanding of the dispersive properties of the propagator e^{isL} of the linearized operator close to Q. In particular, one needs to exhibit its spectral structure. This has been partially done by Weinstein, [110], using the variational characterization of Q. The result is the following: L is a non self adjoint operator with a generalized eigenspace at zero. The eigenmodes are explicit and generated by the symmetries of the problem:

$$L_+(\Lambda Q) = -2Q \ \ \text{(scaling invariance)}, \ \ L_+(\nabla Q) = 0 \ \ \text{(translation invariance)},$$

$$L_-(Q) = 0 \ \ \text{(phase invariance)}, \ \ L_-(yQ) = -2\nabla Q \ \ \text{(Galilean invariance)}.$$

An additional relation is induced by the pseudo-conformal symmetry:

$$L_-(|y|^2 Q) = -4\Lambda Q,$$

and this in turns implies the existence of an additional mode ρ solution to

$$L_+\rho = -|y|^2 Q.$$

These explicit directions induce "growing" solutions to the homogeneous linear equation $i\partial_s\varepsilon + L\varepsilon = 0$. More precisely, there exists a (2N+3) dimensional space S spanned by the above directions such that $H^1 = M \oplus S$ with $|e^{isL}\varepsilon|_{H^1} \leq C$ for $\varepsilon \in M$ and $|e^{isL}\varepsilon|_{H^1} \sim s^3$ for $\varepsilon \in S$. As each symmetry is at the heart of a growing direction, a first idea is to use the symmetries from modulation theory to a priori ensure that ε is orthogonal to S. Roughly speaking, the strategy to construct blow up solutions is then: chose the parameters λ, γ, x so as to get good a priori dispersive estimates on ε in order to build it from a fixed point scheme. Now the fundamental problem is that one has (2N+2) symmetries, but (2N+3) bad modes in the set S. Both constructions in [8] and [95] develop non trivial strategies to overcome this intrinsic difficulty of the problem.

Our strategy will be more non linear. On the basis of the decomposition (5.4), we will prove bounds on ε induced by the virial structure (2.1). The proof will rely on non linear degeneracies of the structure of (1.1) around Q. Using then the Hamiltonian information $E_0 < 0$, we will inject these estimates into the finite dimensional dynamic which governs $\lambda(t)$ -which measures the size of the solution- and prove rigidity properties of Lyapounov type. This will then allow us to prove finite time blow up together with the control of the blow up speed.

5.2. **Choice of the blow up profile.** Before exhibiting the modulation theory type of arguments, we present in this subsection a formal discussion regarding explicit solutions of equation (5.5) which is inspired from a discussion in [106]. This corresponds to a finite dimensional reduction of the problem which actually computes the leading order terms of the solution.

First, let us observe that the key geometrical parameter is λ which measures the size of the solution. Let us then set

$$-\frac{\lambda_s}{\lambda} = b$$

and look for solutions to a simpler version of (5.5):

$$iv_s + \Delta v - v + ib\left(\frac{N}{2}v + y\cdot\nabla v\right) + v|v|^{\frac{4}{N}} = 0.$$

From the orbital stability property, we want solutions which remain close to Q in H^1. Let us look for solutions of the form $v(s,y) = Q_{b(s)}(y)$ where the mappings $b \to Q_b$ and the law for $b(s)$ are the unknown. We think of b as remaining uniformly small and $Q_{b=0} = Q$. Injecting this ansatz into the equation, we get:

$$i\frac{db}{ds}\left(\frac{\partial \overline{Q}_b}{\partial b}\right) + \Delta \overline{Q}_{b(s)} - \overline{Q}_{b(s)} + ib(s)\left(\frac{N}{2}\overline{Q}_{b(s)} + y\cdot\nabla\overline{Q}_{b(s)}\right) + \overline{Q}_{b(s)}|\overline{Q}_{b(s)}|^{\frac{4}{N}} = 0.$$

To handle the linear group, we let $\overline{P}_{b(s)} = e^{i\frac{b(s)}{4}|y|^2}\overline{Q}_{b(s)}$ and solve:

$$i\frac{db}{ds}\left(\frac{\partial \overline{P}_b}{\partial b}\right) + \Delta \overline{P}_{b(s)} - \overline{P}_{b(s)} + \left(\frac{db}{ds} + b^2(s)\right)\frac{|y|^2}{4}\overline{P}_{b(s)} + \overline{P}_{b(s)}|\overline{P}_{b(s)}|^{\frac{4}{N}} = 0. \quad (5.7)$$

A remarkable fact related to the specific algebraic structure of (1.1) around Q is that (5.7) admits three solutions:

- The first one is $(b(s), \overline{P}_{b(s)}) = (0, Q)$, that is the ground state itself. This is just a consequence of the scaling invariance.
- The second one is $(b(s), \overline{P}_{b(s)}) = (\frac{1}{s}, Q)$. This non trivial solution is a rewriting of the explicit critical mass blow up solution $S(t)$ and is induced by the pseudo-conformal symmetry.
- The third one is given by $(b(s), \overline{P}_{b(s)}) = (b, \overline{P}_b)$ for some fixed non zero constant b and $\overline{P}_b$ satisfies:

$$\Delta \overline{P}_b - \overline{P}_b + \frac{b^2}{4}|y|^2\overline{P}_b + \overline{P}_b|\overline{P}_b|^{\frac{4}{N}} = 0. \quad (5.8)$$

  This corresponds to self similar profiles. Indeed, recall that $b = -\frac{\lambda_s}{\lambda}$, so if b is frozen, we have from $\frac{ds}{dt} = \frac{1}{\lambda^2}$:

$$b = -\frac{\lambda_s}{\lambda} = -\lambda\lambda_t \ \text{ ie } \ \lambda(t) = \sqrt{2b(T-t)},$$

  this is the scaling law for the blow up speed.

Now a crucial point again is -[103]- that the solutions to (5.8) never belong to L^2 from a logarithmic divergence at infinity:

$$|P_b(y)| \sim \frac{C(P_b)}{|y|^{\frac{N}{2}}} \ \text{ as } \ |y| \to +\infty.$$

This behavior is a consequence of the oscillations induced by the linear group after the turning point $|y| \geq \frac{2}{|b|}$. Nevertheless, in the ball $|y| < \frac{2}{|b|}$, the operator $-\Delta + 1 - \frac{b^2|y|^2}{4}$ is coercive, and no oscillations will take place in this zone.

Because we track a log-log correction to the self similar law as an upper bound on the blow up speed, the profiles $\overline{Q}_b = e^{-i\frac{b}{4}|y|^2}\overline{P}_b$ with $\overline{P}_b$ solving (5.8) are natural candidates as refinements of the Q profile in the geometrical decomposition (4.11). Nevertheless, as they are not in L^2, we need to build a smooth localized version avoiding the non L^2 tale, what according to the above discussion is doable in the coercive zone $|y| < \frac{2}{|b|}$.

Proposition 5.2 (Localized self similar profiles)**.** *There exist universal constants $C > 0$, $\eta^* > 0$ such that the following holds true. For all $0 < \eta < \eta^*$, there exist constants $\nu^*(\eta) > 0$, $b^*(\eta) > 0$ going to zero as $\eta \to 0$ such that for all $|b| < b^*(\eta)$, let*

$$R_b = \frac{2}{|b|}\sqrt{1-\eta}, \quad R_b^- = \sqrt{1-\eta}R_b,$$

$B_{R_b} = \{y \in \mathbb{R}^N, \quad |y| \leq R_b\}$. *Then there exists a unique radial solution* Q_b *to*

$$\begin{cases} \Delta Q_b - Q_b + ib\left(\frac{N}{2}Q_b + y \cdot \nabla Q_b\right) + Q_b|Q_b|^{\frac{4}{N}} = 0, \\ P_b = Q_b e^{i\frac{b|y|^2}{4}} > 0 \quad in \quad B_{R_b}, \\ Q_b(0) \in (Q(0) - \nu^*(\eta), Q(0) + \nu^*(\eta)), \quad Q_b(R_b) = 0. \end{cases}$$

Moreover, let a smooth radially symmetric cut-off function $\phi_b(x) = 0$ *for* $|x| \geq R_b$ *and* $\phi_b(x) = 1$ *for* $|x| \leq R_b^-$, $0 \leq \phi_b(x) \leq 1$ *and set*

$$\tilde{Q}_b(r) = Q_b(r)\phi_b(r),$$

then

$$\tilde{Q}_b \to Q \quad as \quad b \to 0$$

in some very strong sense, and $\tilde{Q}_b$ *satisfies*

$$\Delta \tilde{Q}_b - \tilde{Q}_b + ib(\tilde{Q}_b)_1 + \tilde{Q}_b|\tilde{Q}_b|^{\frac{4}{N}} = -\Psi_b \tag{5.9}$$

with

$$Supp(\Psi) \subset \{R_b^- \leq |y| \leq R_b\} \quad and \quad |\Psi_b|_{C^1} \leq e^{-\frac{C}{|b|}}.$$

Eventually, $\tilde{Q}_b$ *has supercritical mass:*

$$\int |\tilde{Q}_b|^2 = \int Q^2 + c_0 b^2 + o(b^2) \quad as \quad b \to 0 \tag{5.10}$$

for some universal constant $c_0 > 0$.

The meaning of this proposition is that one can build localized profiles $\tilde{Q}_b$ on the ball B_{R_b} which are a smooth function of b and approximate Q in a very strong sense as $b \to 0$, and these profiles satisfy the self similar equation up to an exponentially small term Ψ_b supported around the turning point $\frac{2}{b}$. The proof of this Proposition uses standard variational tools in the setting of non linear elliptic problems. In fact, the implicit function theorem would do the job as well, see [95].

Now one can think of making a formal expansion of $\tilde{Q}_b$ in terms of b, and the first term is non zero:

$$\frac{\partial \tilde{Q}_b}{\partial b}_{|b=0} = -\frac{i}{4}|y|^2 Q.$$

However, the energy of $\tilde{Q}_b$ is degenerated in b at all orders:

$$|E(\tilde{Q}_b)| \leq e^{-\frac{C}{|b|}}, \tag{5.11}$$

for some universal constant $C > 0$.

The existence of a one parameter family of profiles satisfying the self similar equation up to an exponentially small term and having an exponentially small energy is an algebraic property of the structure of (1.1) around Q which is at the heart of the existence of the log-log regime.

5.3. **Modulation theory.** We are now in position to exhibit the sharp decomposition needed for the proof of the log-log upper bound. From Theorem 3.7 and the proximity of $\tilde{Q}_b$ to Q in H^1, the solution $u(t)$ to (1.1) is for all time close to the four dimensional manifold

$$\mathcal{M} = \{e^{i\gamma}\lambda^{\frac{N}{2}}\tilde{Q}_b(\lambda y + x), \quad (\lambda, \gamma, x, b) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}\}.$$

We now sharpen the decomposition according to the following Lemma. In the sequel, we let

$$\varepsilon = \varepsilon_1 + i\varepsilon_2$$

be the real and imaginary parts decomposition.

Lemma 5.3 (Non linear modulation of the solution close to $\mathcal{M}$)**.** *There exist $\mathcal{C}^1$ functions of time $(\lambda, \gamma, x, b) : [0, T) \to (0, +\infty) \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ such that:*

$$\forall t \in [0, T), \quad \varepsilon(t, y) = e^{i\gamma(t)} \lambda^{\frac{N}{2}}(t) u(t, \lambda(t)y + x(t)) - \tilde{Q}_{b(t)}(y) \tag{5.12}$$

satisfies:
(i)

$$\left(\varepsilon_1(t), \Lambda\Sigma_{b(t)}\right) + \left(\varepsilon_2(t), \Lambda\Theta_{b(t)}\right) = 0, \tag{5.13}$$

$$\left(\varepsilon_1(t), y\Sigma_{b(t)}\right) + \left(\varepsilon_2(t), y\Theta_{b(t)}\right) = 0, \tag{5.14}$$

$$-\left(\varepsilon_1(t), \Lambda^2\Theta_{b(t)}\right) + \left(\varepsilon_2(t), \Lambda^2\Sigma_{b(t)}\right) = 0, \tag{5.15}$$

$$-\left(\varepsilon_1(t), \Lambda\Theta_{b(t)}\right) + \left(\varepsilon_2(t), \Lambda\Sigma_{b(t)}\right) = 0, \tag{5.16}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $\tilde{Q}_b = \Sigma_b + i\Theta_b$ in terms of real and imaginary parts;

(ii) $|1 - \lambda(t)\dfrac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}| + |\varepsilon(t)|_{H^1} + |b(t)| \leq \delta(\alpha^*)$ *with* $\delta(\alpha^*) \to 0$ *as* $\alpha^* \to 0$.

Let us insist onto the fact that the reason for this precise choice of orthogonality conditions is a fundamental issue which will be addressed in the next section.

Proof of Lemma 5.3. This Lemma follows the standard frame of modulation theory and is obtained from Theorem 3.7 using the implicit function theorem. From Theorem 3.7, there exist parameters $\gamma_0(t) \in \mathbb{R}$ and $x_0(t) \in \mathbb{R}^N$ such that with $\lambda_0(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}}$,

$$\forall t \in [0, T), \quad \left| Q - e^{i\gamma_0(t)} \lambda_0(t)^{\frac{N}{2}} u(\lambda_0(t)x + x_0(t)) \right|_{H^1} < \delta(\alpha^*)$$

with $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$. Now we sharpen this decomposition using the fact that $\tilde{Q}_b \to Q$ in H^1 as $b \to 0$, i.e. we chose $(\lambda(t), \gamma(t), x(t), b(t))$ close to $(\lambda_0(t), \gamma_0(t), x_0(t), 0)$ such that

$$\varepsilon(t, y) = e^{i\gamma(t)} \lambda^{1/2}(t) u(t, \lambda(t)y + x(t)) - \tilde{Q}_{b(t)}(y)$$

is small in H^1 and satisfies suitable orthogonality conditions (5.13), (5.14), (5.15) and (5.16).The existence of such a decomposition is a consequence of the implicit function Theorem. For $\delta > 0$, let $V_\delta = \{v \in H^1(\mathbb{C}); \quad |v - Q|_{H^1} \leq \delta\}$, and for $v \in H^1(\mathbb{C})$, $\lambda_1 > 0$, $\gamma_1 \in \mathbb{R}$, $x_1 \in \mathbb{R}^N$, $b \in \mathbb{R}$ small, define

$$\varepsilon_{\lambda_1, \gamma_1, x_1, b}(y) = e^{i\gamma_1} \lambda_1^{\frac{N}{2}} v(\lambda_1 y + x_1) - \tilde{Q}_b. \tag{5.17}$$

We claim that there exists $\overline{\delta} > 0$ and a unique C^1 map : $V_{\overline{\delta}} \to (1 - \overline{\lambda}, 1 + \overline{\lambda}) \times (-\overline{\gamma}, \overline{\gamma}) \times B(0, \overline{x}) \times (-\overline{b}, \overline{b})$ such that if $v \in V_{\overline{\delta}}$, there is a unique $(\lambda_1, \gamma_1, x_1, b)$ such that $\varepsilon_{\lambda_1, \gamma_1, x_1, b} = (\varepsilon_{\lambda_1, \gamma_1, x_1, b})_1 + i(\varepsilon_{\lambda_1, \gamma_1, x_1, b})_2$ defined as in (5.17) satisfies

$$\rho^1(v) = ((\varepsilon_{\lambda_1, \gamma_1, x_1, b})_1, \Lambda\Sigma_b) + ((\varepsilon_{\lambda_1, \gamma_1, x_1, b})_2, \Lambda\Theta_b) = 0,$$

$$\rho^2(v) = ((\varepsilon_{\lambda_1, \gamma_1, x_1, b})_1, y\Sigma_b) + ((\varepsilon_{\lambda_1, \gamma_1, x_1, b})_2, y\Theta_b) = 0,$$

$$\rho^3(v) = -\left((\varepsilon_{\lambda_1, \gamma_1, x_1, b})_1, \Lambda^2\Theta_b\right) + \left((\varepsilon_{\lambda_1, \gamma_1, x_1, b})_2, \Lambda^2\Sigma_b\right) = 0,$$

$$\rho^4(v) = ((\varepsilon_{\lambda_1, \gamma_1, x_1, b})_1, \Lambda\Theta_b) - ((\varepsilon_{\lambda_1, \gamma_1, x_1, b})_2, \Lambda\Sigma_b) = 0.$$

Moreover, there exists a constant $C_1 > 0$ such that if $v \in V_{\overline{\delta}}$, then $|\varepsilon_{\lambda_1,\gamma_1,x_1}|_{H^1} + |\lambda_1 - 1| + |\gamma_1| + |x_1| + |b| \leq C_1\overline{\delta}$. Indeed, we view the above functionals $\rho^1, \rho^2, \rho^3, \rho^4$ as functions of $(\lambda_1, \gamma_1, x_1, b, v)$. We first compute at $(\lambda_1, \gamma_1, x_1, b, v) = (1, 0, 0, 0, v)$:

$$\frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial x_1} = \nabla v, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial \lambda_1} = \frac{N}{2}v + x\cdot\nabla v, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial \gamma_1} = iv, \quad \frac{\partial \varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial b} = -\left(\frac{\partial \tilde{Q}_b}{\partial b}\right)_{|b=0}.$$

Now recall that $(\tilde{Q}_b)_{|b=0} = Q$ and $\left(\frac{\partial \tilde{Q}_b}{\partial b}\right)_{|b=0} = -i\frac{|y|^2}{4}Q$. Therefore, we obtain at the point $(\lambda_1, \gamma_1, x_1, b, v) = (1, 0, 0, 0, Q)$,

$$\begin{aligned}
&\frac{\partial \rho^1}{\partial \lambda_1} = |\Lambda Q|_2^2, \quad \frac{\partial \rho^1}{\partial \gamma_1} = 0, \quad \frac{\partial \rho^1}{\partial x_1} = 0, \frac{\partial \rho^1}{\partial b} = 0,\\
&\frac{\partial \rho^2}{\partial \lambda_1} = 0, \quad \frac{\partial \rho^2}{\partial \gamma_1} = 0, \quad \frac{\partial \rho^2}{\partial x_1} = -\frac{1}{2}|Q|_2^2, \frac{\partial \rho^2}{\partial b} = 0,\\
&\frac{\partial \rho^3}{\partial \lambda_1} = 0, \quad \frac{\partial \rho^3}{\partial \gamma_1} = -|\Lambda Q|_2^2, \quad \frac{\partial \rho^3}{\partial x_1} = 0, \frac{\partial \rho^3}{\partial b} = 0,\\
&\frac{\partial \rho^4}{\partial \lambda_1} = 0, \quad \frac{\partial \rho^4}{\partial \gamma_1} = 0, \quad \frac{\partial \rho^4}{\partial x_1} = 0, \frac{\partial \rho^4}{\partial b} = \frac{1}{4}|yQ|_2^2.
\end{aligned}$$

The Jacobian of the above functional is non zero, thus the implicit function Theorem applies and conclusion follows. $\square$

Let us now write down the equation satisfied by ε in rescaled variables. To simplify notations, we note

$$\tilde{Q}_b = \Sigma + \Theta$$

in terms of real and imaginary parts. We have: $\forall s \in \mathbb{R}_+$, $\forall y \in \mathbb{R}^N$,

$$\begin{aligned}
b_s\frac{\partial \Sigma}{\partial b} + \partial_s\varepsilon_1 - M_-(\varepsilon) + b\Lambda\varepsilon_1 &= \left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\Sigma + \tilde{\gamma}_s\Theta + \frac{x_s}{\lambda}\cdot\nabla\Sigma \\
&+ \left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\varepsilon_1 + \tilde{\gamma}_s\varepsilon_2 + \frac{x_s}{\lambda}\cdot\nabla\varepsilon_1 \\
&+ Im(\Psi) - R_2(\varepsilon)
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
b_s\frac{\partial \Theta}{\partial b} + \partial_s\varepsilon_2 + M_+(\varepsilon) + b\Lambda\varepsilon_2 &= \left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\Lambda\Theta - \tilde{\gamma}_s\Sigma + \frac{x_s}{\lambda}\cdot\nabla\Theta \\
&+ \left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\varepsilon_2 - \tilde{\gamma}_s\varepsilon_1 + \frac{x_s}{\lambda}\cdot\nabla\varepsilon_2 \\
&- Re(\Psi) + R_1(\varepsilon),
\end{aligned} \tag{5.19}$$

with $\tilde{\gamma}(s) = -s - \gamma(s)$. The linear operator close to $\tilde{Q}_b$ is now a deformation of the linear operator L close to Q and is $M = (M_+, M_-)$ with

$$M_+(\varepsilon) = -\Delta\varepsilon_1 + \varepsilon_1 - \left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1 - \left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2,$$

$$M_-(\varepsilon) = -\Delta\varepsilon_2 + \varepsilon_2 - \left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2 - \left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1.$$

The formally quadratic in ε interaction terms are:

$$R_1(\varepsilon) = (\varepsilon_1 + \Sigma)|\varepsilon + \tilde{Q}_b|^{\frac{4}{N}} - \Sigma|\tilde{Q}_b|^{\frac{4}{N}} - \left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2} + 1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1 - \left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2,$$

$$R_2(\varepsilon) = (\varepsilon_2+\Theta)|\varepsilon+\tilde{Q}_b|^{\frac{4}{N}} - \Theta|\tilde{Q}_b|^{\frac{4}{N}} - \left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2 - \left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1.$$

Two natural estimates may now be performed:

- First, we may rewrite the conservation laws in the rescaled variables and linearize the obtained identities close to Q. This will give crucial degeneracy estimates on some specific order one in ε scalar products.
- Next, we may inject the orthogonality conditions of Lemma 5.3 into the equations (5.18), (5.19). This will compute the geometrical parameters in their differential form $\frac{\lambda_s}{\lambda}$, $\tilde{\gamma}_s$, $\frac{x_s}{\lambda}$, b_s in terms of ε: these are the so called modulation equations. This step requires estimating the non linear interaction terms. A crucial point here is to use the fact that the ground state Q is exponentially decreasing in space.

The outcome is the following:

Lemma 5.4 (First estimates on the decomposition)**.** *We have for all $s \geq 0$:*
(i) Estimates induced by the conservation of the energy and the momentum:

$$|(\varepsilon_1, Q)| \leq \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}} + C\lambda^2|E_0|, \tag{5.20}$$

$$|(\varepsilon_2, \nabla Q)| \lesssim \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}}. \tag{5.21}$$

(ii) Estimate on the geometrical parameters in differential form:

$$\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s| + |\tilde{\gamma}_s| \lesssim \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}}, \tag{5.22}$$

$$\left|\frac{x_s}{\lambda}\right| \lesssim \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}}, \tag{5.23}$$

where $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$.

Remark 5.5. *The exponentially small term in the degeneracy estimate (5.20) is in fact related to the value of $E(\tilde{Q}_b)$, so we use here in a fundamental way the non linear degeneracy estimate (5.11).*

Comments on Lemma 5.4:

1. $\dot{H}^1$ norm: The norm which appears in the estimates of Lemma 5.4 is essentially a local norm in space. The conservation of the energy indeed relates the $\int |\nabla\varepsilon|^2$ norm with the local norm. These two norms will turn out to play an equivalent role in the analysis. A key is that no global L^2 norm is needed so far.

2. Degeneracy of the translation shift: Comparing estimates (5.22) and (5.23), we see that the term induced by translation invariance is smaller than the ones induced by scaling and phase invariances. This non trivial fact is an outcome of our use of the Galilean transform to ensure the zero momentum condition (5.3).

5.4. **The virial type dispersive estimate.** We now turn to the proof of the dispersive virial type inequality at the heart of the proof of the log-log upper bound. This information will be obtained as a consequence of the virial structure of (1.1) in Σ.

Let us first recall that the virial identity (2.1) corresponds to two identities:

$$\frac{d^2}{dt^2}\int |x|^2|u|^2 = 4\frac{d}{dt}Im(\int x\cdot\nabla u\overline{u}) = 16E_0. \tag{5.24}$$

We want to understand what information can be extracted from this dispersive information in the variables of the geometrical decomposition.

To clarify the claim, let us consider an ε solution to the linear homogeneous equation

$$i\partial_s\varepsilon + L\varepsilon = 0 \tag{5.25}$$

where $L = (L_+, L_-)$ is the linearized operator close to Q. A dispersive information on ε may be extracted using a similar virial law like (2.1):

$$\frac{1}{2}\frac{d}{ds}Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon}) = H(\varepsilon,\varepsilon), \tag{5.26}$$

where $H(\varepsilon,\varepsilon) = (\mathcal{L}_1\varepsilon_1,\varepsilon_1) + (\mathcal{L}\varepsilon_2,\varepsilon_2)$ is a Schrödinger type quadratic form decoupled in the real and imaginary parts with explicit Schrödinger operators:

$$\mathcal{L}_1 = -\Delta + \frac{2}{N}\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}-1}y\cdot\nabla Q \quad , \quad \mathcal{L}_2 = -\Delta + \frac{2}{N}Q^{\frac{4}{N}-1}y\cdot\nabla Q.$$

Note that both these operators are of the form $-\Delta+V$ for some smooth well localized time independent potential $V(y)$, and thus from standard spectral theory, they both have a finite number of negative eigenvalues, and then continuous spectrum on $[0,+\infty)$. A simple outcome is then that given an $\varepsilon\in H^1$ which is orthogonal to all the bound states of $\mathcal{L}_1,\mathcal{L}_2$, then $H(\varepsilon,\varepsilon)$ is coercive, that is

$$H(\varepsilon,\varepsilon) \geq \delta_0\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)$$

for some universal constant $\delta_0 > 0$. Now assume that for some reason -it will be in our case a consequence of modulation theory and the conservation laws-, ε is indeed for all times orthogonal to the bound states -and resonances...-, then injecting the coercive control of $H(\varepsilon,\varepsilon)$ into (5.26) yields:

$$\frac{1}{2}\frac{d}{ds}Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon}) \geq \delta_0\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right). \tag{5.27}$$

Integrating this in time yields a standard dispersive information: a space time norm is controlled by a norm in space.

We want to apply this strategy to the full ε equation. There are two main obstructions.

First, it is not reasonable to assume that ε is orthogonal to the exact bound states of H. In particular, due to the right hand side in the ε equation, other second order terms will appear which will need be controlled. We thus have to exhibit a set of orthogonality conditions which ensures both the coercivity of the quadratic form H and the control of these other second order interactions. Note that the number of orthogonality conditions we can ensure on ε is the number of symmetries plus the one from b. A first key is the following Spectral Property which has been proved in dimension $N = 1$ in [71] using the explicit value of Q and checked numerically for $N = 2, 3, 4$.

Proposition 5.6 (Spectral Property). *Let $N = 1, 2, 3, 4$. There exists a universal constant $\delta_0 > 0$ such that $\forall \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$,*

$$\begin{aligned} H(\varepsilon,\varepsilon) &\geq \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \frac{1}{\delta_0} \left\{ (\varepsilon_1, Q)^2 + (\varepsilon_1, \Lambda Q)^2 + (\varepsilon_1, yQ)^2 \right. \\ &+ \left. (\varepsilon_2, \Lambda Q)^2 + (\varepsilon_2, \Lambda^2 Q)^2 + (\varepsilon_2, \nabla Q)^2 \right\}. \end{aligned} \tag{5.28}$$

To prove this property amounts first counting exactly the number of negative eigenvalues of each Schrödinger operator, and then prove that the specific chosen set of orthogonality conditions, which is not exactly the set of the bound states, is enough to ensure the coercivity of the quadratic form. Both these issues appear to be non trivial when Q is not explicit, but obvious to check numerically through the drawing of a small number (less than 10) explicit curves.

Then, the second major obstruction is the fact that the right hand side $Im(\int y \cdot \nabla \varepsilon \bar{\varepsilon})$ in (5.27) is an unbounded function of ε in H^1. This is a priori a major obstruction to the strategy, *but an additional non linear algebra inherited from the virial law (2.1) rules out this difficulty.*

The formal computation is as follows. Given a function $f \in \Sigma$, we let $\Phi(f) = Im(\int y \cdot \nabla f \overline{f})$. According to (5.26), we want to compute $\frac{d}{ds}\Phi(\varepsilon)$. Now from (5.24) and the conservation of the energy:

$$\forall t \in [0,T), \quad \Phi(u(t)) = 4E_0 t + c_0$$

for some constant c_0. The key observation is that the quantity $\Phi(u)$ is scaling, phase and also translation invariant from zero momentum assumption (5.3). Using (5.12), we get:

$$\forall t \in [0,T), \quad \Phi(\varepsilon + \tilde{Q}_b) = 4E_0 t + c_0.$$

We now expand this according to:

$$\Phi(\varepsilon + \tilde{Q}_b) = \Phi(\tilde{Q}_b) - 2(\varepsilon_2, \Lambda\Sigma) + 2(\varepsilon_1, \Lambda\Theta) + \Phi(\varepsilon).$$

A simple algebra yields:

$$\Phi(\tilde{Q}_b) = -\frac{b}{2}|y\tilde{Q}_b|_2^2 \sim -Cb$$

for some universal constant $C > 0$. Next, from the choice of orthogonality condition (5.16),

$$(\varepsilon_2, \Lambda\Sigma) - (\varepsilon_1, \Lambda\Theta) = 0.$$

We thus get using $\frac{dt}{ds} = \lambda^2$:

$$(\Phi(\varepsilon))_s \sim 4\lambda^2 E_0 + Cb_s.$$

In other words, to compute the a priori unbounded quantity $(\Phi(\varepsilon))_s$ *for the full non linear equation* is from the virial law equivalent to computing the time derivative of b_s, what of course makes now perfectly sense in H^1.

The virial dispersive structure on $u(t)$ in Σ thus induces a dispersive structure in $L^2_{loc} \cap \dot{H}^1$ on $\varepsilon(s)$ for the full non linear equation.

The key dispersive virial estimate is now the following.

Proposition 5.7 (Local viriel estimate in ε)**.** *There exist universal constants $\delta_0 > 0$, $C > 0$ such that for all $s \geq 0$, there holds:*

$$b_s \geq \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \lambda^2 E_0 - e^{-\frac{C}{|b|}}. \tag{5.29}$$

Proof of Proposition 5.7. Using the heuristics, we can compute in a suitable way b_s using the orthogonality condition (5.16). The computation -see Lemma 5 in [72]- yields:

$$\begin{aligned} & \frac{1}{4}|yQ|_2^2 b_s = H(\varepsilon,\varepsilon) + 2\lambda^2|E_0| - \frac{x_s}{\lambda} \cdot \{(\varepsilon_2, \nabla\Lambda\Sigma) - (\varepsilon_1, \nabla\Lambda\Theta)\} \quad (5.30) \\ - & \left(\frac{\lambda_s}{\lambda} + b\right) \left\{(\varepsilon_2, \Lambda^2\Sigma) - (\varepsilon_1, \Lambda^2\Theta)\right\} - \tilde{\gamma}_s \left\{(\varepsilon_1, \Lambda\Sigma) + (\varepsilon_2, \Lambda\Theta)\right\} \\ - & (\varepsilon_1, Re\Lambda\Psi)) - (\varepsilon_2, Im(\Lambda\Psi)) + (l.o.t), \end{aligned}$$

where the lower order terms may be estimated from the smallness of ε in H^1:

$$|l.o.t| \leq \delta(\alpha^*) \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right).$$

We now explain how the choice of orthogonality conditions and the conservation laws allow us to deduce (5.29).

step 1 Modulation theory for phase and scaling. The choice of orthogonality conditions (5.15), (5.13) has been made to cancel the two second order in ε scalar products in (5.30):

$$\left(\frac{\lambda_s}{\lambda} + b\right) \left\{(\varepsilon_2, \Lambda^2\Sigma) - (\varepsilon_1, \Lambda^2\Theta)\right\} + \tilde{\gamma}_s \left\{(\varepsilon_1, \Lambda\Sigma) + (\varepsilon_2, \Lambda\Theta)\right\} = 0.$$

step 2 Elliptic estimate on the quadratic form H. We now need to control the negative directions in the quadratic form as given by Proposition 5.6. The directions $(\varepsilon_1, \Lambda Q)$, (ε_1, yQ), $(\varepsilon_2, \Lambda^2 Q)$ and $(\varepsilon_2, \Lambda Q)$ are treated thanks to the choice of orthogonality conditions and the closeness of $\tilde{Q}_b$ to Q for $|b|$ small. For example,

$$\begin{aligned} (\varepsilon_2, \Lambda Q)^2 &= |\{(\varepsilon_2, \Lambda Q - \Lambda\Sigma) + (\varepsilon_1, \Lambda\Theta)\} + (\varepsilon_2, \Lambda\Sigma) - (\varepsilon_1, \Lambda\Theta)|^2 \\ &= |(\varepsilon_2, \Lambda Q - \Lambda\Sigma) + (\varepsilon_1, \Lambda\Theta)|^2 \end{aligned}$$

so that

$$(\varepsilon_2, \Lambda Q)^2 \leq \delta(\alpha^*)(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}).$$

Similarly, we have:

$$(\varepsilon_1, yQ)^2 + (\varepsilon_2, \Lambda^2 Q)^2 + (\varepsilon_1, \Lambda Q)^2 \leq \delta(\alpha^*)(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}). \tag{5.31}$$

The negative direction $(\varepsilon_1, Q)^2$ is treated from the conservation of the energy which implied (5.20). The direction $(\varepsilon_2, \nabla Q)$ is treated from the zero momentum condition which ensured (5.21). Putting this together yields:

$$(\varepsilon_1, Q)^2 + (\varepsilon_2, \nabla Q)^2 \leq \delta(\alpha^*) \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} + \lambda^2 |E_0| \right) + e^{-\frac{C}{|b|}}.$$

step 3 Modulation theory for translation and use of Galilean invariance. The Galilean invariance has been used to ensure the zero momentum condition (5.3) which in turn led together with the choice of orthogonality condition (5.14) to the degeneracy estimate (5.23):

$$|\frac{x_s}{\lambda}| \lesssim \delta(\alpha^*)(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|})^{\frac{1}{2}} + e^{-\frac{C}{|b|}}.$$

Therefore, we estimate the term induced by translation invariance in (5.30) as

$$\left|\frac{x_s}{\lambda}\cdot\{(\varepsilon_2, \nabla\Lambda\Sigma) - (\varepsilon_1, \nabla\Lambda\Theta)\}\right| \lesssim \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + e^{-\frac{C}{|b|}}.$$

step 4 Conclusion. Injecting these estimates into the elliptic estimate (5.28) yields so far:

$$b_s \geq \delta_0\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) - 2\lambda^2 E_0 - e^{-\frac{C}{|b|}} - \frac{1}{\delta_0}(\lambda^2 E_0)^2.$$

We now use in a crucial way *the sign of the energy* $E_0 < 0$ and the smallness $\lambda^2|E_0| \leq \delta(\alpha^*)$ which is a consequence of the conservation of the energy to conclude. □

5.5. **Monotonicity and control of the blow up speed.** The virial dispersive estimate (5.29) means a control of the excess of mass ε by an exponentially small correction in b in time averaging sense. More specifically, this means that in rescaled variables, the solution writes $\tilde{Q}_b + \varepsilon$ where $\tilde{Q}_b$ is the regular deformation of Q and the rest is in a suitable norm exponentially small in b. This is thus an expansion of the solution with respect to an internal parameter in the problem: b.

This virial control is the first dispersive estimate for the infinite dimensional dynamic driving ε. Observe that it means little by itself if nothing is known about $b(t)$. We shall now inject this information into the finite dimensional dynamic driving the geometrical parameters. The outcome will be *a rigidity property for the parameter* $b(t)$ *which will in turn imply the existence of a Lyapounov functional in the problem.* This step will again heavily rely on the conservation of the energy.

We start with exhibiting the rigidity property which proof is a maximum principle type of argument.

Proposition 5.8 (Rigidity property for b)**.** $b(s)$ *vanishes at most once on* $\mathbb{R}_+$.

Note that the existence of a quantity with prescribed sign in the description of the dynamic is unexpected. Indeed, b is no more then the projection of some a priori highly oscillatory function onto a prescribed direction. It is a very specific feature of the blow up dynamic that this projection has a fixed sign.

Proof of Proposition 5.8. Assume that there exists some time $s_1 \geq 0$ such that $b(s_1) = 0$ and $b_s(s_1) \leq 0$, then from (5.29), $\varepsilon(s_1) = 0$. Thus from the conservation of the L^2 norm and $\tilde{Q}_{b(s_1)} = Q$, we conclude $\int |u_0|^2 = \int Q^2$ what contradicts the strictly negative energy assumption. □

The next step is to get the exact sign of b. This is done by injecting the virial dispersive information (5.29) into the modulation equation for the scaling parameter what will yield

$$-\frac{\lambda_s}{\lambda} \sim b. \tag{5.32}$$

The key rigidity property is the following:

Proposition 5.9 (Rigidity of the flow). *There exists a time $s_0 \geq 0$ such that*

$$\forall s > s_0, \quad b(s) > 0.$$

Moreover, the size of the solution is in this regime an almost Lyapounov functional in the sense that:

$$\forall s_2 \geq s_1 \geq s_0, \quad \lambda(s_2) \leq 2\lambda(s_1). \tag{5.33}$$

Proof of Proposition 5.9. **step 1** Equation for the scaling parameter. The modulation equation for the scaling parameter λ inherited from choice of orthogonality condition (5.13) implied control (5.22):

$$\left|\frac{\lambda_s}{\lambda} + b\right| \lesssim \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}},$$

which implies (5.32) in a weak sense. Nevertheless, this estimate is not good enough to possibly use the virial estimate (5.29). We claim using extra degeneracies of the equation that (5.22) can be improved for:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \lesssim \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + e^{-\frac{C}{|b|}} \tag{5.34}$$

step 2 Use of the virial dispersive relation and the rigidity property. We now inject the virial dispersive relation (5.29) into (5.34) to get:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \lesssim b_s + e^{-\frac{C}{|b|}}.$$

We integrate this inequality in time to get: $\forall 0 \leq s_1 \leq s_2$,

$$\left|\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) + \int_{s_1}^{s_2} b(s)ds\right| \leq \frac{1}{4} + \int_{s_1}^{s_2} e^{-\frac{C}{|b(s)|}} ds. \tag{5.35}$$

The key is now to use the rigidity property of Proposition 5.8 to ensure that $b(s)$ has a fixed sign for $s \geq \tilde{s}_0$, and thus: $\forall s \geq \tilde{s}_0$,

$$\left|\int_{s_1}^{s_2} e^{-\frac{C}{|b(s)|}} ds\right| \leq \frac{1}{2}\left|\int_{s_1}^{s_2} b(s)ds\right|. \tag{5.36}$$

step 3 b is positive for s large enough. Assume that $\left|\int_0^{+\infty} b(s)ds\right| < +\infty$, then b has a fixed sign for $s \geq \tilde{s}_0$ and $|b_s| \leq C$, and thus: $b(s) \to 0$ as $s \to +\infty$. Now from (5.35) and (5.36), this implies that $|\log(\lambda(s))| \leq C$ as $s \to +\infty$, and in particular $\lambda(s) \geq \lambda_0 > 0$ for s large enough. Injecting this into virial control (5.29) for s large enough yields:

$$b_s \geq \frac{1}{2}|E_0|\lambda_0^2.$$

Integrating this on large time intervals contradicts the uniform boundedness of b. Here we have used again the assumption $E_0 < 0$. We thus have proved: $\left|\int_0^{+\infty} b(s)ds\right| = +\infty$.

Now assume that $b(s) < 0$ for all $s \geq \tilde{s}_1$, then from (5.35) and (5.36) again, we conclude that $\log(\lambda(s)) \to 0$ as $s \to +\infty$. Now from $\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}$, this yields $|\nabla u(t)|_{L^2} \to 0$ as $t \to T$. But from Gagliardo-Nirenberg inequality and the conservation of the energy and the L^2 mass, this implies $E_0 = 0$, contradicting again the assumption $E_0 < 0$.

step 4 Almost monotonicity of the norm. We now are in position to prove (5.33). Indeed, injecting the sign of b into (5.35) and (5.36) yields in particular: $\forall s_0 \le s_1 \le s_2$,

$$\frac{1}{4} + \frac{1}{2}\int_{s_1}^{s_2} b(s)ds \le -\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \le \frac{1}{4} + 2\int_{s_1}^{s_2} b(s)ds, \tag{5.37}$$

and thus:

$$\forall s_0 \le s_1 \le s_2, \quad -\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \ge \frac{1}{4},$$

what yields (5.33). This concludes the proof of Proposition 5.9.

Note that from the above proof, we have obtained $\int_0^{+\infty} b(s)ds = +\infty$, and thus from (5.37):

$$\lambda(s) \to 0 \quad \text{as} \quad s \to \infty, \tag{5.38}$$

that is finite or infinite time blow up. *On the contrary to the virial argument, the blow up proof is no longer obstructive but completely dynamical, and relies mostly on the rigidity property of Proposition 5.8.* □

Let us now conclude the proof of Theorem 5.1. We need to prove finite time blow up together with the log-log upper bound (5.2) on blow up rate.

Proof of Theorem 5.1. **step 1** Lower bound on $b(s)$. We claim: there exist some universal constant $C > 0$ and some time $s_1 > 0$ such that $\forall s \ge s_1$,

$$Cb(s) \ge \frac{1}{\log|\log(\lambda(s))|}. \tag{5.39}$$

Indeed, first recall (5.29). Now that we know the sign of $b(s)$ for $s \ge s_0$ from Proposition 5.9, and we may thus view (5.29) as a differential inequality for b for $s > s_0$:

$$b_s \ge -e^{-\frac{C}{b}} \ge -b^2 e^{-\frac{C}{2b}} \quad \text{ie} \quad -\frac{b_s}{b^2} e^{\frac{C}{2b}} \le 1.$$

We integrate this inequality from the non vanishing property of b and get for $s \ge \tilde{s}_1$ large enough:

$$e^{\frac{C}{b(s)}} \le s + e^{\frac{C}{b(1)}} \lesssim s \quad \text{ie} \quad b(s) \gtrsim \frac{1}{\log(s)}. \tag{5.40}$$

We now recall (5.37) on the time interval $[\tilde{s}_1, s]$:

$$\frac{1}{2}\int_{\tilde{s}_1}^{s} b \le -\log\left(\frac{\lambda(s)}{\lambda(\tilde{s}_1)}\right) + \frac{1}{4} \le -2\log(\lambda(s))$$

for $s \ge \tilde{s}_2$ large enough from $\lambda(s) \to 0$ as $s \to +\infty$. Inject (5.40) into the above inequality, we get for $s \ge \tilde{s}_3$

$$\frac{s}{\log(s)} \lesssim \int_{\tilde{s}_2}^{s} \frac{d\tau}{\log(\tau)} \le \frac{1}{4}\int_{\tilde{s}_2}^{s} b \le -\log(\lambda(s)) \quad \text{ie} \quad |\log(\lambda(s))| \gtrsim \frac{s}{\log(s)}$$

and thus for s large

$$\log|\log(\lambda(s))| \ge \log(s) - \log(\log(s)) \ge \frac{1}{2}\log(s)$$

and conclusion follows from (5.40). This concludes the proof of (5.39).

step 2 Finite time blow up and control of the blow up speed. We first use the finite or infinite time blow up result (5.38) to consider a sequence of times $t_n \to T \in [0, +\infty]$ defined for n large such that

$$\lambda(t_n) = 2^{-n}.$$

Let $s_n = s(t_n)$ the corresponding sequence and $\bar{t}$ such that $s(\bar{t}) = s_0$ given by Proposition 5.9. Note that we may assume $n \geq \overline{n}$ such that $t_n \geq \bar{t}$. Remark that $0 < t_n < t_{n+1}$ from (5.33), and so $0 < s_n < s_{n+1}$. Moreover, there holds from (5.33)

$$\forall s \in [s_n, s_{n+1}], \quad 2^{-n-1} \leq \lambda(s) \leq 2^{-(n-1)}. \tag{5.41}$$

We now claim that (5.2) follows from a control from above of the size of the intervals $[t_n, t_{n+1}]$ for $n \geq \overline{n}$.
Let $n \geq \overline{n}$. (5.39) implies

$$\int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} \lesssim \int_{s_n}^{s_{n+1}} b(s)ds.$$

(5.37) with $s_1 = s_n$ and $s_2 = s_{n+1}$ yields:

$$\frac{1}{2}\int_{s_n}^{s_{n+1}} b(s) \leq \frac{1}{4} - |yQ|_{L^2}^2 \log\left(\frac{\lambda(s_{n+1})}{\lambda(s_n)}\right) \lesssim 1.$$

Therefore,

$$\forall n \geq \overline{n}, \quad \int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} \lesssim 1.$$

Now we change variables in the integral at the left of the above inequality according to $\frac{ds}{dt} = \frac{1}{\lambda^2(s)}$ and estimate with (5.41):

$$1 \gtrsim \int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} = \int_{t_n}^{t_{n+1}} \frac{dt}{\lambda^2(t)\log|\log(\lambda(t))|} \geq \frac{1}{10\lambda^2(t_n)\log|\log(\lambda(t_n))|}\int_{t_n}^{t_{n+1}} dt$$

so that

$$t_{n+1} - t_n \lesssim \lambda^2(t_n)\log|\log(\lambda(t_n))|.$$

From $\lambda(t_n) = 2^{-n}$ and summing the above inequality in n, we first get

$$T < +\infty$$

and

$$\begin{aligned} T - t_n &\lesssim \sum_{k\geq n} 2^{-2k}\log(k) = \sum_{n\leq k\leq 2n} 2^{-2k}\log(k) + \sum_{k\geq 2n} 2^{-2k}\log(k) \\ &\lesssim 2^{-2n}\log(n) + 2^{-4n}\log(2n)\sum_{k\geq 0} 2^{-2k}\frac{\log(2n+k)}{\log(2n)} \\ &\lesssim 2^{-2n}\log(n) + 2^{-4n}\log(n) \lesssim 2^{-2n}\log(n) \lesssim \lambda^2(t_n)\log|\log(\lambda(t_n))|. \end{aligned}$$

From the monotonicity of λ (5.33), we extend the above control to the whole sequence $t \geq \bar{t}$. Let $t \geq \bar{t}$, then $t \in [t_n, t_{n+1}]$ for some $n \geq \overline{n}$, and from $\frac{1}{2}\lambda(t_n) \leq \lambda(t) \leq 2\lambda(t_n)$, we conclude

$$\lambda^2(t)\log|\log(\lambda(t))| \gtrsim \lambda^2(t_n)\log|\log(\lambda(t_n))| \gtrsim T - t_n \gtrsim T - t.$$

Now remark that the function $f(x) = x^2\log|\log(x)|$ is non decreasing in a neighborhood at the right of $x = 0$, and moreover

$$f\left(\frac{C}{2}\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right) = \frac{C^2}{4}\frac{(T-t)}{\log|\log(T-t)|}\log\left|\log\left(C\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right)\right| \leq C(T-t)$$

for t close enough to T, so that we get for some universal constant C^*:

$$f(\lambda(t)) \geq f\left(C^*\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right) \quad \text{ie} \quad \lambda(t) \geq C^*\sqrt{\frac{T-t}{\log|\log(T-t)|}}$$

and (5.2) is proved.

□

Appendix

This Appendix is devoted to the proof of the concentration compactness Lemma, i.e. Proposition 1.6. We follow Cazenave [11].

Proof of Proposition 1.6. . Let $u_n \in H^1$ be as in the hypothesis of Proposition 1.6.

step 1 Concentration function. Let the sequence of concentration functions:

$$\rho_n(R) = \sup_{y\in\mathbb{R}^N}\int_{B(y,R)}|u_n(x)|^2dx.$$

The following facts are elementary and left to the reader:

- Monotonicity: $\forall n \geq 0$, $\rho_n(R)$ is a nondecreasing function of R.
- The concentration point is attained:

$$\forall R>0, \quad \forall n\geq 0, \quad \exists y_n(R)\in\mathbb{R}^N \text{ such that } \rho_n(R) = \int_{B(y_n(R),R)}|u_n(x)|^2dx.$$

- Uniform Hölder continuity: $\exists C, \alpha > 0$ independent of n such that

$$\forall R_1, R_2 > 0, \quad \forall n \geq 0, \quad |\rho_n(R_1)-\rho_n(R_2)| \leq C|R_1^N - R_2^N|^\alpha. \tag{5.42}$$

  This last fact is a simple consequence of the H^1 bound (1.13).

step 2 Limit of concentration functions. From (5.42) and Ascoli's theorem, there exists a subsequence $n_k \to +\infty$ and a nondecreasing limit ρ such that

$$\forall R>0, \quad \lim_{k\to+\infty}\rho_{n_k}(R) = \rho(R). \tag{5.43}$$

Let now

$$\mu = \lim_{R\to+\infty}\liminf_{n\to+\infty}\rho_n(R).$$

By definition, there exists $R_k \to +\infty$ such that

$$\lim_{k\to+\infty}\rho_{n_k}(R_k) = \mu.$$

We now claim some stability of the sequence R_k which is a very general and simple fact but crucial for the rest of the argument:

$$\mu = \lim_{k\to+\infty}\rho_{n_k}(R_k) = \lim_{k\to+\infty}\rho_{n_k}(\frac{R_k}{2}) = \lim_{R\to+\infty}\rho(R). \tag{5.44}$$

Proof of (5.44): First oberve from the monotonicity of ρ_{n_k} that

$$\limsup_{k\to+\infty}\rho_{n_k}(\frac{R_k}{2}) \leq \limsup_{k\to+\infty}\rho_{n_k}(R_k) = \mu. \tag{5.45}$$

For the other sense, we argue as follows. For every $R>0$, there holds:

$$\rho(R) = \liminf_{k\to+\infty}\rho_{n_k}(R) \geq \liminf_{n\to+\infty}\rho_n(R)$$

and thus:

$$\lim_{R\to+\infty}\rho(R) \geq \mu. \tag{5.46}$$

Eventually, for any $R>0$, we have $\frac{R_k}{2}\geq R$ for k large enough and thus:

$$\rho_{n_k}(\frac{R_k}{2})\geq\rho_{n_k}(R).$$

Letting $k\to+\infty$ implies:

$$\forall R>0,\quad \lim_{k\to+\infty}\rho_{n_k}(\frac{R_k}{2})\geq\rho(R).$$

Letting now $R>0$ yields:

$$\lim_{k\to+\infty}\rho_{n_k}(\frac{R_k}{2})\geq\lim_{R\to+\infty}\rho(R)\geq\mu$$

where we used (5.46) in the last step. This together with (5.45) concludes the proof of (5.44).
The proof now proceed by making an hypothesis on μ.

Step 3: $\mu=0$ is vanishing. Assume $\mu=0$. Then from (5.44), $\lim_{R\to+\infty}\rho(R)=0$. But ρ is nondecreasing positive so: $\forall R>0$, $\rho(R)=0$. In particular, $\rho(1)=0$ and thus

$$\lim_{k\to+\infty}\rho_{n_k}(1)=\lim_{k\to+\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|u_{n_k}|^2=0. \tag{5.47}$$

We claim that this together with the H^1 bound on u_{n_k} implies (1.15). There is a slight difficulty here which is that we need to pass from a local information -vanishing on every ball- to a global information -vanishing of the global L^q norm-. This relies on a refinement of the Gagliardo Nirenberg interpolation inequality. Indeed, we claim that

$$\forall u\in H^1,\quad \int|u|^{2+\frac{4}{N}}\leq C|u|^2_{H^1}|u|^{\frac{4}{N}}_{L^2} \tag{5.48}$$

can be refined for:

$$\forall u\in H^1,\quad \int|u|^{2+\frac{4}{N}}\leq C|u|^2_{H^1}\left[\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|u|^2\right]^{\frac{2}{N}}. \tag{5.49}$$

This together with (5.47) implies

$$u_{n_k}\to 0\ \text{ in }\ L^{2+\frac{4}{N}}\ \text{ as }\ k\to+\infty$$

and (1.15) follows by interpolation using the global H^1 bound.
Proof of (5.49): Let a partition of $\mathbb{R}^d$ with disjoint rectangles Q_j of side $\frac{1}{2}$. Assume $N\geq 3$ and write Hölder noticing:

$$\frac{1}{2+\frac{4}{N}}=\frac{\alpha}{2}+\frac{1-\alpha}{\frac{2N}{N-2}}\quad\text{with}\quad\alpha=\frac{2}{N+2}$$

so that

$$|u|_{L^{2+\frac{4}{N}}(Q_i)}\leq|u|^{\alpha}_{L^2(Q_j)}|u|^{1-\alpha}_{L^{2^*}(Q_j)}$$

and hence using Sobolev in Q_j:

$$|u|^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}}\leq C|u|^{\frac{4}{d}}_{L^2(Q_j)}|u|^2_{H^1(Q_j)}$$

where the Sobolev constant does not dÕpend on j thanks to the translation invariance of Lebesgue's mesure. We may now sum on the disjoint cubes:

$$\begin{aligned}\int |u|^{2+\frac{4}{N}}dx &= \Sigma_{j\geq 1}\int_{Q_j}|u|^{2+\frac{4}{N}}dx \lesssim \left[\sup_{j\geq 1}|u|^2_{L^2(Q_j)}\right]^{\frac{2}{d}}\Sigma_{j\geq 1}|u|^2_{H^1(Q_j)}\\ &= \left[\sup_{j\geq 1}|u|^2_{L^2(Q_j)}\right]^{\frac{2}{N}}|u|^2_{H^1}\end{aligned}$$

and (5.49) is proved. The cases $N = 1, 2$ is similar and left to the reader.

Step 4: $\mu = M$ is compactness. Let n_k be the sequence satisfying (5.43). For $R > 0$, let $y_k(R)$ such that

$$\rho_{n_k}(R) = \int_{B(y_k(R),R)}|u_{n_k}(x)|^2dx. \tag{5.50}$$

Pick $\varepsilon > 0$. Then from (5.44), there exist $R_0, R(\varepsilon)$ such that

$$\rho(R_0) > \frac{M}{2},\quad \rho(R(\varepsilon)) > M - \varepsilon.$$

Hence there exists $k_0(\varepsilon)$ such that $\forall k \geq k_0(\varepsilon)$,

$$\rho_{n_k}(R_0) = \int_{B(y_k(R_0),R_0)}|u_{n_k}|^2 > \frac{M}{2},\quad \rho_{n_k}(R(\varepsilon)) = \int_{B(y_k(R(\varepsilon)),R(\varepsilon))}|u_{n_k}|^2 > M - \varepsilon.$$

But the total L^2 mass being M, this implies that the balls $B(y_k(R_0), R_0)$ and $B(y_k(R(\varepsilon)), R(\varepsilon))$ cannot be disjoint. Hence -draw a picture- we can find $R_1(\varepsilon)$ such that:

$$\forall\varepsilon > 0,\quad \forall k \geq k_0(\varepsilon),\quad \int_{B(y_k(R_0),R_1(\varepsilon))}|u_{n_k}|^2 \geq M - \varepsilon.$$

By possibly raising the value of $R_1(\varepsilon)$ for the values $k \in [1, k_0(\varepsilon)]$, this implies that the sequence $v_k = u_{n_k}(\cdot + y_k(R_0))$ is L^2 compact:

$$\forall\varepsilon > 0,\quad \exists R_2(\varepsilon) > 0 \text{ such that } \forall k \geq 1,\quad \int_{|y|\geq R_2(\varepsilon)}|v_k(y)|^2dy < \varepsilon.$$

The compactness of the embedding $H^1 \hookrightarrow L^2(B(0, R(\varepsilon)))$ then implies that v_k a Cauchy sequence in L^2, and the H^1 boundedness now implies (1.14) by interpolation.

Step 5: $0 < \mu < M$ is dichotomy. Let again (n_k, R_k) satisfying (5.43), (5.44). Then we can write:

$$u_{n_k} = v_k + w_k + z_k$$

with

$$v_k = u_{n_k}\mathbf{1}_{|y-y_k(\frac{R_k}{2})|\leq\frac{R_k}{2}},\quad w_k = u_{n_k}\mathbf{1}_{|y-y_k(\frac{R_k}{2})|\geq R_k},\quad z_k = u_{n_k}\mathbf{1}_{\frac{R_k}{2}<|y-y_k(\frac{R_k}{2})|<R_k}.$$

The key is to observe from (5.50) and (5.44) that:

$$\begin{aligned}\int|z_k|^2 &= \int_{B(y_k(\frac{R_k}{2}),R_k)}|u_{n_k}|^2 - \int_{B(y_k(\frac{R_k}{2}),\frac{R_k}{2})}|u_{n_k}|^2\\ &\leq \rho_{n_k}(R_k) - \int_{B(y_k(\frac{R_k}{2}),\frac{R_k}{2})}|u_{n_k}|^2 = \rho_{n_k}(R_k) - \rho_{n_k}(\frac{R_k}{2})\\ &\to 0 \text{ as } k \to +\infty.\end{aligned}$$

The claim dichotomy now follows by taking smooth cut off in the localization. The L^p norm of z_k will go to zero using the vanishing of the L^2 norm and the global H^1 bound, and the error introduced by localization will be treated using $R_k \to +\infty$. This is left to the reader.
This concludes the proof of Proposition 1.6. □

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