$$f : \underset{\beta}{\mathbb{V}} \longrightarrow \underset{\alpha}{\mathbb{U}} \qquad f \longmapsto (f)_{\alpha,\beta}$$

$$\big(f(x)\big)_\alpha = (f)_{\alpha,\beta} \cdot (x)_\beta$$

$$\mathrm{id} : \underset{\beta}{\mathbb{V}} \to \underset{\alpha}{\mathbb{V}} \qquad (x)_\alpha = (\mathrm{id})_{\alpha,\beta} (x)_\beta$$

3.1. (iii) $\Rightarrow$ (i)

Dz.

$$\alpha \cdot P = \beta$$

$$x = \beta (x)_\beta = \alpha \cdot \underbrace{P(x)_\beta}_{= (x)_\alpha}$$

$$P_{\alpha,\alpha} = (\mathrm{id})_{\alpha,\alpha} = \left((u_1)_\alpha, \dots, (u_n)_\alpha \right)$$

$$\alpha = (u_1, \dots, u_n)$$

$$I_n = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$P_{\beta,\alpha} = P_{\alpha,\beta}^{-1}$$

$$(x)_\beta = P_{\beta,\alpha}(x)_\alpha \qquad \Big| \; P_{\beta,\alpha}^{-1}$$

$$\underset{\displaystyle \left(P_{\beta,\alpha}\right)^{-1}}{\overset{\displaystyle P_{\alpha,\beta}}{\shortparallel}} (x)_\beta \overset{\text{ug.}}{=} (x)_\alpha$$

$$\alpha P = \beta$$

$$P^{-1}$$

$$\alpha \cdot P^{-1}$$

$$\overset{= I_n}{(e_1, \ldots, e_n)} \cdot P = \beta$$

$$e_j(P) = e_j(\beta) = v_j$$

$$(\alpha \mid \beta) \sim \dots\dots \sim (I_m \mid \alpha^{-1}\beta)$$

$$\underline{\underline{P_{\alpha,\beta}}} = \alpha^{-1}\beta$$

$$P_{\beta_2,\alpha_2} \cdot (\varphi)_{\alpha_2,\alpha_1} \cdot \underbrace{P_{\alpha_1,\beta_1} (x)_{\beta_1}} =$$

$$P_{\beta_2,\alpha_2} \cdot (\varphi)_{\alpha_2,\alpha_1} (x)_{\alpha_1} = P_{\beta_2,\alpha_2} \big(\varphi(x)\big)_{\alpha_2}$$

$$= \big(\varphi(x)\big)_{\beta_2} = (\varphi)_{\beta_2,\beta_1} (x)_{\beta_1}$$

$(ii) \Uparrow (iii)$:

$$B_{\beta_2,\beta_1} = P_{\beta_2,\alpha_2} A_{\alpha_2,\alpha_1} P_{\alpha_1,\beta_1}$$

$(ii) \Rightarrow (iii)$

$$B = P \cdot A \cdot Q$$

$$h(B) = h(P \cdot A \cdot Q) \leq h(A)$$

$$A = P^{-1} B Q^{-1}$$

$$h(A) \leq h(B)$$

$$\Downarrow$$

$$h(A) = h(B)$$

(iii) $\Rightarrow$ (ii):

$$A \sim \begin{pmatrix} \; \\ 0 \end{pmatrix} \sim \begin{pmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ 0 & \end{pmatrix}$$

$$C = P_1 A \cdot Q_1$$

$$\|$$

$$C = P_2 B Q_2$$

$$B = \underbrace{P_2^{-1} P_1}_{P} \cdot A \underbrace{Q_1 Q_2^{-1}}_{Q}$$

$\varphi(x) = Ax$ $\quad$ $\mathrm{Im}\,A$

$a_j \in \mathrm{Im}\,A$ $\quad$ báze

$\parallel$ $\quad$ B

Ae_j

$m = n = 1000$ $\quad$ $r = 100$

10^6 $\quad$ $2 \cdot 10^5$

$$A = B \cdot C$$

hàng sl. A

$$A = I_n^{-1} \cdot A \cdot I_m \qquad A \sim A$$

$$I_n^{-1} = I_m$$

$$B = P^{-1} \cdot A \cdot P \Rightarrow P \cdot B \cdot P^{-1} = A$$

$$A \sim B \qquad B \sim A$$

$A \sim B$

$B = P^{-1} A \cdot P$ $\qquad$ $(P \cdot Q)^{-1} = Q^{-1} P^{-1}$

$C = Q^{-1} \cdot B \cdot Q$

$B \sim C$

$C = Q^{-1} \cdot B \cdot Q = (Q^{-1} P^{-1}) A (PQ)$

$\underline{A \sim C}$.

$$A \qquad hA = \sum_{i=1}^{n} a_{ii}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \qquad hA = 8$$

1.3.

$$n_i(A) \cdot p_j(B)$$

$$C = A \cdot B$$

$$hC = \sum_{i=1}^{m} n_i(A) \cdot p_i(B)$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} (a_{ik} b_{ki}) = \sum_{k=1}^{n} \sum_{i=1}^{m} b_{ki} a_{ik} = \sum_{k=1}^{n} n_k(B) p_k(A)$$

$$= h(B \cdot A)$$

1.4. STOPA

$$B = P^{-1} A \cdot P$$

$$\mathrm{tr}(B) = \mathrm{tr}(P^{-1} \cdot A \cdot P) = \mathrm{tr}(\underbrace{P \cdot P^{-1}}_{I_n} A) = \underline{\mathrm{tr}(A)}$$

$$(\varphi)_B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$(\varphi(v_1))_B = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{(v_1)_B} = \begin{pmatrix} \lambda_1 \\ \vdots \\ 0 \end{pmatrix}$$

$$\varphi(v_1) = \lambda_1 v_1$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$x = -y$

$\lambda_1 = -1$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$y = x.$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ y \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$y \neq 0 \implies 2 = 0$ SPOR

$y = 0$ CELÁ OSA X

$\varphi : V \to V$

$S \subseteq V \qquad \underline{\underline{\varphi(S) \subseteq S}}$

$\varphi(v) = \lambda v \qquad \varphi([v]) \subseteq [v]$

$$\varphi(u_1) \in [u_1, \dots, u_r]$$

$$\sum_{i=1}^{r} c_i u_i + 0 u_{r+1} + \dots + 0 \cdot u_m .$$

$$\varphi(u_{k+1}) = \sum_{i=1}^{m} c_i u_i \qquad S \oplus T = V$$

$$S = [u_1, \dots, u_k], \quad T = [u_{k+1}, \dots, u_n]$$

$$S \cap T = \{0\}_{\mathcal{O}}$$

$$\varphi(u_{k+1}) = \sum_{i=1}^{k} c_i u_i + \sum_{i=k+1}^{m} c_i u_i$$

$$\varphi(T) \subseteq T$$

$$\varphi(u_{k+1}) = \sum_{j=k+1}^{m} d_j u_j = \sum_{i=1}^{m} c_i u_i$$

$$(\varphi(u_{k+1}))_{\alpha} = \begin{pmatrix} 0 \\ d_{k+1} \\ \vdots \\ d_m \end{pmatrix}$$

$$\mathbb{X}_1 \neq \mathbb{X}_2 \neq \mathbb{X}_3 \quad \neq \mathbb{X}_r$$

$$\mathbb{V}_1 \quad \mathbb{V}_2 \quad \cdots\cdots \quad \mathbb{V}_r$$

$n=1 \quad \mathbb{V}_1 \neq 0$

$n \Rightarrow n+1 \leq r$

$$\sum_{i=1}^{n+1} c_i v_i = 0 \Longrightarrow c_1 = \dots = c_{n+1} = 0$$

$v_1, \dots, v_n$ LN $\qquad c_{n+1} = 1$

$$v_{n+1} = \sum_{i=1}^{n} (-c_i) v_i \qquad \lambda_{n+1} v_{n+1} = \underline{\sum_{i=1}^{n} (-c_i) \lambda_{n+1} v_i}$$

$$\varphi(v_{n+1}) = \sum_{i=1}^{n} (-c_i) \varphi(v_i) = \underline{\sum_{i=1}^{n} (-c_i) \lambda_i v_i}$$

$$\sum_{i=1}^{n}(-c_i)\lambda_{n+1}v_i = \sum_{i=1}^{n}(-c_i)\lambda_i v_i =$$

$$\sum_{i=1}^{n}\underbrace{(-c_i)}_{=0}\underbrace{(\lambda_{n+1}-\lambda_i)}_{\neq 0}v_i = 0$$

$$\Rightarrow v_{n+1} = 0$$ <u>SPOR</u>

$\mathbb{X}_1 \oplus \cdots \oplus \mathbb{W}_m$

$\mathbb{W}_1, \ldots, \mathbb{W}_m$ LN

$\varphi([\mathbb{W}_1]) \subseteq [\mathbb{W}_1], \ldots, \varphi([\mathbb{W}_m]) \subseteq [\mathbb{W}_m]$

$$\begin{pmatrix} \square & & & & & 0 \\ & \square & & & & \\ & & \square & & & \\ & & & \square & & \\ & & & & \square & \\ 0 & & & & & \square \end{pmatrix}$$

$$A = \begin{pmatrix} \vdots \\ r_i(A) \\ \vdots \\ r_j(A) \\ \vdots \end{pmatrix}, \quad B = \begin{pmatrix} \vdots \\ r_j(A) \\ \vdots \\ r_j(A) \\ \vdots \end{pmatrix}$$

$\det A$ $\qquad$ $\det B = 0$

$$\det(A+B) = \det(A) + \det B = \det A$$

$$\det A = \begin{pmatrix} \vdots \\ r_i(A) + r_j(A) \\ \vdots \\ r_j(A) \\ \vdots \end{pmatrix}$$

$$-\det A = \begin{pmatrix} \vdots \\ -r_i(A) - r_j(A) \\ \vdots \\ r_j(A) \end{pmatrix}$$

$$\det C :$$

$$C = \begin{pmatrix} \vdots \\ -r_i(A) - r_j(A) \\ \vdots \\ -r_i(A) - r_j(A) \\ \vdots \end{pmatrix}$$

$$+\det A = \det \begin{pmatrix} -r_i(A) - r_j(A) \\ \vdots \\ +r_i(A) \end{pmatrix} \qquad \begin{pmatrix} \vdots \\ r_j(A) \\ \vdots \\ r_i(A) \\ \vdots \end{pmatrix}$$

$$\det A = \det \begin{pmatrix} -r_j(A) \\ \vdots \\ r_i(A) \end{pmatrix} \qquad -\det A = \det \begin{pmatrix} r_j(A) \\ \vdots \\ r_i(A) \end{pmatrix}$$

$$\begin{array}{cc} & \begin{matrix} & & & & j & & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array} = E(i,j)$$

$$\det I_n = 1$$

$$\det E(i,j) = \underline{\underline{-1}}$$

$$\begin{array}{c} \\ i \\ \\ \end{array} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & c & & \\ & 0 & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} = E(c)$$

$$\det I_n = 1$$

$$\det E(c) = c$$

$$\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & c & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} = E(i, j, c)$$

$\underline{c \neq 0}$

$$\det \overset{\det = c}{\begin{pmatrix} 1 & & \\ & \ddots \, c & 0 \\ & c & \ddots \, 1 \end{pmatrix}}$$

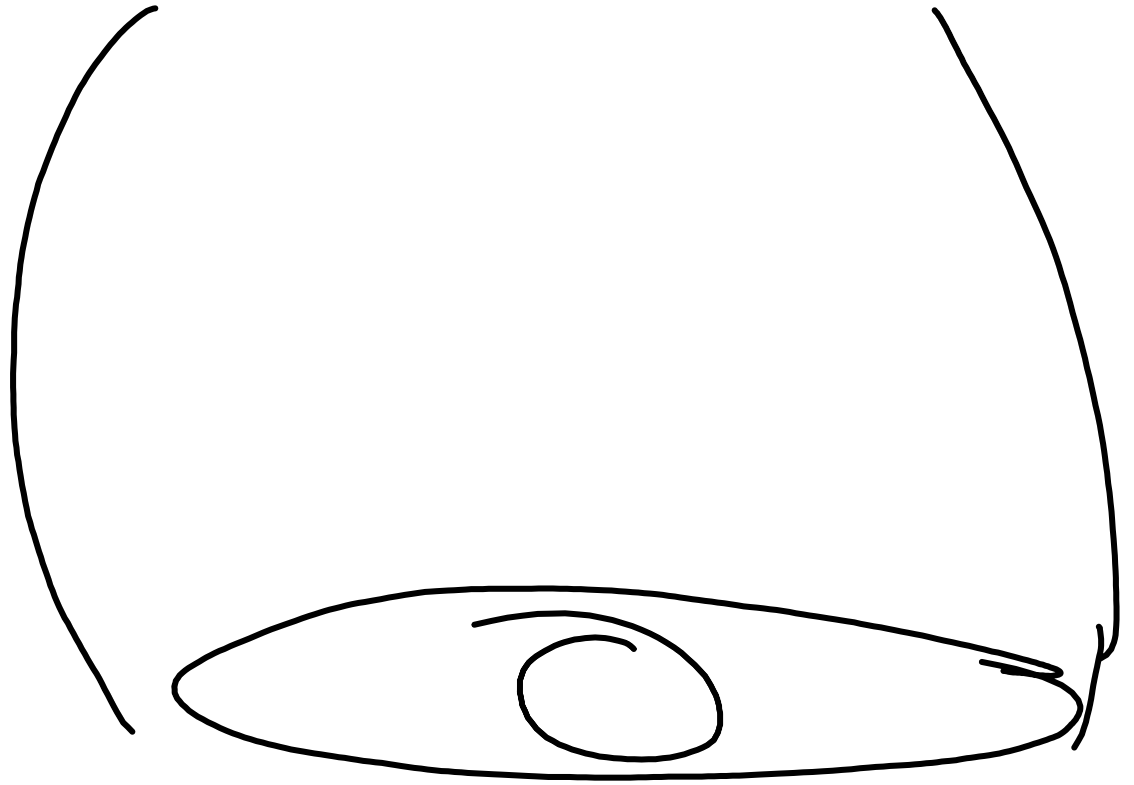
$$=$$

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \rightsquigarrow \det \overset{\det = c}{\begin{pmatrix} 1 & & & \\ & \ddots \, c & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}} + \det \begin{array}{c} \\ i \\ \\ j \end{array} \begin{pmatrix} 1 & & \\ & c & 0 \\ & & \ddots \\ & c & 0 \end{pmatrix}$$

$$\det(E \cdot B) = \det(E) \cdot \det(B)$$

A sing.

$$A \sim C = \begin{pmatrix} \ddots & \\ & \end{pmatrix}$$

$C =$ 

$m \times n$

$\underline{\underline{\det C = 0}}$

$$0\,(0, \ldots, 0) = (0, \ldots, 0)$$

$$0 = 0 \cdot \det C = \det C$$

DZ V. 3.4. A, B regulární

$$A = E_k \cdot \ldots \cdot E_1$$

$$\det(A \cdot B) = \det(E_k \ldots E_1 \cdot B) =$$

$$= \det(E_k) \cdot \det(E_{k-1} \ldots E_1 \cdot B) = \det(E_k) \cdot \det(E_{k-1} \ldots E_1)$$

$$\cdot \det B = \det(A) \cdot \det B$$

$$\begin{pmatrix} a_{11} & & \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

(1) $a_{ii} = 0 \Rightarrow$ nr RST [illegible]

(2) $\forall i \quad a_{ii} \neq 0$

$$\sim \begin{pmatrix} a_{11} & & & 0 \\ & \ddots & & \\ & & a_{n-1,n-1} & \\ & & & a_{nn} \end{pmatrix} \sim \dots \sim \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

$$\begin{array}{cc} & \begin{matrix} & & & i & & j & & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 0 & & 1 & & \\ & 0 & & 1 & & & \\ & & 1 & & 0 & & \\ & & & 0 & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} & & & j & & \end{matrix} \\ \begin{matrix} \\ \\ j \\ \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & 0 & & \\ & \ddots & & & & \\ & & & & & \\ & & & c & & \\ & & & 0 & 1 & \\ & & & & & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} & i & & j & \end{matrix} \\ \begin{matrix} \\ i \\ \\ j \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & 0 \\ 0 & 1 & & c & 0 \\ & & \ddots & & \\ & 0 & & 1 & \\ 0 & & & \ddots & 1 \end{pmatrix} \end{array}$$

V.3.6. Nechť A je reg.

$\det E_j = \det E_j^T$

$$A = E_k \cdot \ldots \cdot E_1$$

$$\det A = \det E_k \cdot \ldots \cdot \det E_1$$

$$A^T = E_1^T \cdot \ldots \cdot E_k^T$$

$$\det A^T = \det E_1^T \cdot \ldots \cdot \det E_k^T$$

$\det D = 0 \Rightarrow \underline{\det A = 0}$

$\det D \neq 0 \sim I_n$

$\det D^{-1} = \dfrac{1}{\det D}$

$B = A^{-1}$

$\det A \cdot B = \det A \cdot \det B$

$1 = \det A \, \det A^{-1}$

Pa

$\det B = 0$, $\det D \neq 0 \Longrightarrow \underline{\underline{\det A = 0}}$

$\det B \neq 0$, $\det D \neq 0$

$(\det B)^{-1}$

$$\det \begin{pmatrix} I_m & 0 \\ 0 & I_{n-m} \end{pmatrix} = 1$$

$\det A = \det B \cdot \det D$

Cramer:

$$A^{\text{reg.}}\, \mathbb{X} = \mathbb{b} \qquad \mathbb{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$n \times n$

$$x_j = \frac{\det\left(\ldots \overset{j}{\mathbb{b}} \ldots\right)}{\det(A)} \qquad \mathbb{b} = \sum x_j\, c_j(A)$$

$$\det\left(\ \cdots \ \middle|\ \sum_{i=1}^{n} x_i\, a_i(A)\ \middle|\ \cdots\ \right) = \sum_{i=1}^{n} x_i \det\left(\ \cdots\ \middle|\ a_i(A)\ \middle|\ \cdots\ \right)$$

$$= x_j \det A.$$

$$A' = \begin{pmatrix} \lambda_1(A) \\ c \cdot \lambda_i(A) \\ \lambda_n(A) \end{pmatrix} \qquad A'' = \begin{pmatrix} \lambda_1(A) \\ \\ x + y \\ \vdots \\ \lambda_n(A) \end{pmatrix}$$

$$\det A' = c \det A \qquad \det A'' = \det \begin{pmatrix} \lambda_1(A) \\ x \\ \lambda_n(A) \end{pmatrix} + \det \begin{pmatrix} \lambda_1(A) \\ \vdots \\ y \\ \vdots \\ \lambda_n(A) \end{pmatrix}$$