Cviceni 5: Linearni podprostory

Monday, November 2, 2020 9:09 AM

V – vekt. prostor $\Rightarrow$

$W \subset V$ – podprostor, když:

1) $\forall a \in W,\ ka \in W$ $(\forall k \in \mathbb{R})$

2) $\forall a, b \in W,\ a + b \in W$

$\boxed{(\Leftrightarrow \forall a, b \in W,\ \lambda, \mu \in \mathbb{R},\ \lambda a + \mu b \in W)}$

W je taky lin. prostor!

Dulež. představa:

Lin. kombinace: $\{a_1, \ldots, a_n\} \subset V$,

$\lambda_1 a_1 + \ldots + \lambda_n a_n \in V$ (konkr. vektor)

Lin. obal: $\{\lambda_1 a_1 + \ldots + \lambda_n a_n\} \subset V$ $\forall \lambda_1, \ldots, \lambda_n$

$[a_1, \ldots, a_n]$

$[a_1, \ldots, a_n]$ $\lambda_1, \ldots, \lambda_n$

Veta: (1) Lin. obal je lin. podpr.

(2) $\forall$ lin. podpr. je lin. obal nekolik vektorů

1) $V = \{\text{polyn}\} = R_3[X]$

$W = \{\text{polyn stupnu} \leq 3\}$

$W = [1, X, X^2, X^3]$

$p = a_0 \cdot 1 + a_1 \cdot X + a_2 \cdot X^2 + a_3 X^3$

2) $V = R^2$, $a = (1,1)$

$[a] = \{k \cdot (1,1)\} = \{y = x\} \subset R^2$

3) $a = (1,1,0)$, $b = (0,1,1)$

$[a, b] = \{\lambda a + \mu b\} \Leftrightarrow \begin{cases} x = \lambda \\ y = \lambda + \mu \\ z = \mu \end{cases}$

$\Rightarrow [a, b] = \{y = x + z\} \subset R^3$

$(0-\lambda+2)\in\mathbb{R}$

4) Leží-li $\bar{a}=(8,7,1)$ v lin. obalu $\bar{a}_1=(4,3,5)$, $a_2=(-3,-2,-3)$, $a_3=(2,1,1)$, $a_4=(-1,-3,-8)$?

Zapišme: $a=x_1a_1+\dots+x_4a_4$

$(V=\mathbb{R}^3)$

$$\begin{cases}4x_1-3x_2+2x_3-x_4=8\\3x_1-2x_2+x_3-3x_4=7\\5x_1-3x_2+x_3-8x_4=1\end{cases}$$

$$\left(\begin{array}{cccc|c}4&-3&2&-1&8\\3&-2&1&-3&7\\5&-3&1&-8&1\end{array}\right)\xrightarrow{(1)}\left(\begin{array}{cccc|c}-1&0&1&7&7\\3&-2&1&-3&7\\5&-3&1&-8&1\end{array}\right)\xrightarrow{(2)}\left(\begin{array}{cccc|c}-1&0&1&7&7\\0&-2&4&18&28\\0&-3&6&27&36\end{array}\right)\xrightarrow{(3)}$$

$$\to\left(\begin{array}{cccc|c}-1&0&1&7&7\\0&-1&2&9&14\\0&1&-2&-9&-12\end{array}\right)\xrightarrow{(4)}\left(\begin{array}{cccc|c}-1&0&1&7&7\\0&-1&2&9&14\\0&0&0&0&2\end{array}\right)$$

Není řešení $\Rightarrow$ $\boxed{\text{Ne}}$

5) $V=\{\text{polyn} \leq 2 \text{ stupňů}\} = \mathbb{R}_2[x]$

$P_1=4+5x+3x^2$

$P_2 = 2 + 3x + 2x^2$

$P_3 = -1 - 2x - 3x^2$

Leží-li $P = 2x + 1$ v lin obalu P_1, P_2, P_3?

Pokud ano, zapište P jak konkr. lin. kombin.

$P = x_1 P_1 + x_2 P_2 + x_3 P_3$

$$\begin{cases} 4x_1 + 2x_2 - x_3 = 1 & (1) \\ 5x_1 + 3x_2 - 2x_3 = 2 & (x) \\ 3x_1 + 2x_2 - 3x_3 = 0 & (x^2) \end{cases}$$

$$\left(\begin{array}{ccc|c} 4 & 2 & -1 & 1 \\ 5 & 3 & -2 & 2 \\ 3 & 2 & -3 & 0 \end{array}\right) \xrightarrow{(1)} \left(\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ 5 & 3 & -2 & 2 \\ 3 & 2 & -3 & 0 \end{array}\right) \xrightarrow{(2)} \left(\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ 0 & -2 & 3 & -3 \\ 0 & -1 & 0 & -3 \end{array}\right) \xrightarrow{(3)} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & -2 & 3 & -3 \end{array}\right) \xrightarrow{(4)}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 3 \end{array}\right) \xrightarrow{(5)} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Řešení $\Rightarrow$

$\exists$ řešení $\Rightarrow$ $\boxed{\text{Ano}}$

$x_1 = -1, x_2 = 3, x_3 = 1$

$$P = -P_1 + 3P_2 + P_3$$

6) $V = \mathbb{R}^4$

$$W = \begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 + 2x_3 - x_4 = 0 \end{cases}$$

Popsat W jak lin. obal

$x_3 = s,\ x_4 = t \Rightarrow$

$$\begin{cases} x_1 + 2x_2 = -t - s \\ x_1 - x_2 = t - 2s \end{cases} \Rightarrow \begin{cases} x_2 = \frac{1}{3}(s - 2t) \\ x_1 = \frac{1}{3}(t - 5s) \end{cases}$$

$$\begin{cases} x_1 = -\frac{5}{3}s + \frac{1}{3}t \\ x_2 = \frac{1}{3}s - \frac{2}{3}t \\ x_3 = s,\ x_4 = t \end{cases}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \cdot \begin{pmatrix} -\frac{5}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{pmatrix}}$$

Fakt $\exists\ \infty$ variantu!

7) $V = \mathbb{R}_n[X]$ $\quad (n \geq 1)$

$W = \{p \in V : p(2X) = 2p(X)\}$

Popsat W lin. obalem

$p = a_0 + a_1X + \dots + a_nX^n$

$p(2X) = a_0 + 2a_1X + 4a_2X^2 + \dots + 2^n a_n X^n$

$p(2X) = 2p(X) \Rightarrow$

$a_0 = 2a_0,\ 2a_1 = 2a_1,\ 4a_2 = 2a_2,\ \dots,\ 2^n a_n = 2a_n$

$\Rightarrow a_0 = a_2 = \dots = a_n = 0 \Rightarrow$

$W = [X]$ $\qquad (p = a_1X)$

8) $p_1 = 1+X,\ p_2 = X^2-1,\ p_3 = 2X^2+X$

f) $p_1 = 1+X,\ p_2 = X^2-1,\ p_3 = 2X^2+X$

$V = \mathbb{R}_2[X],\quad [p_1, p_2, p_3] = ?$

$p = x_1 p_1 + x_2 p_2 + x_3 p_3$

$p = a_0 + a_1 X + a_2 X^2 \Rightarrow$

$$\begin{cases} x_1 - x_2 = a_0 \\ x_1 + x_3 = a_1 \\ x_2 + 2x_3 = a_2 \end{cases} \qquad A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

když $\exists A^{-1} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

$\exists A^{-1}$? $\quad A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\Rightarrow \exists A^{-1} \Rightarrow \boxed{[p_1, p_2, p_3] = \mathbb{R}_2[X]}$

g) když $p_4 = 1+X+X^2$,

$[p_1, p_2, p_3, p_4] = \ldots ?$

$$= \boxed{\mathbb{R}_2[X]}$$