Define
$$\delta_U : U \to \mathrm{Hom}(V, U\otimes V)$$
$$u \mapsto \delta_u(u)(-) = u\otimes -$$
i.e., $\delta_U(u)(v) := u\otimes v$, so $\delta_U(u)(-) := u\otimes -$

and
$$\varepsilon_W : \mathrm{Hom}(V, W) \otimes V \to W$$
$$f\otimes v \mapsto f(v)$$
i.e., $\varepsilon_W(f\otimes v) := f(v)$

Set $L := -\otimes V$ and $R := \mathrm{Hom}(V, -)$

Show that
$$\varepsilon_{\Bbbk\otimes V} \cdot L\delta_{\Bbbk} = id_{\Bbbk\otimes V} \quad \text{and}$$
$$R\varepsilon_{\Bbbk} \cdot \delta_{V^*} = id_{V^*}$$

Ans: First of all we understand what L and R do on morphisms.
given $f: A\to B$, $Lf : A\otimes V \to B\otimes V$
$$a\otimes v \mapsto f(a)\otimes v$$
and $Rf : \mathrm{Hom}(V, A) \to \mathrm{Hom}(V, B)$
$$V\xrightarrow{g} A \mapsto V\xrightarrow{g} A \xrightarrow{f} B$$

Now
$$\Bbbk\otimes V \xrightarrow{L\delta_{\Bbbk}} \mathrm{Hom}(V,V)\otimes V \xrightarrow{\varepsilon_{\Bbbk\otimes V}} V \cong \Bbbk\otimes V$$
$$k\otimes v \mapsto (k\cdot -)\otimes v \mapsto k\cdot v \cong k\otimes v$$
and

composition with $\varepsilon_{\Bbbk} : V^*\otimes V \to \Bbbk$
$$V^* \xrightarrow{\delta_{V^*}} \mathrm{Hom}(V, V^*\otimes V) \xrightarrow{R\varepsilon_{\Bbbk}} \mathrm{Hom}(V, \Bbbk) = V^*$$
$$\eta \mapsto \delta_{V^*}(\eta)(-) = \eta\otimes - \mapsto \eta$$

Claim: $R\varepsilon_{\Bbbk}(\eta\otimes -) = \eta$:
$$V \xrightarrow{\eta\otimes -} V^*\otimes V \xrightarrow{\varepsilon_{\Bbbk}} \Bbbk$$
$$v \mapsto \eta\otimes v \mapsto \eta(v)$$

given

$$(\bar{e}_1 \ \bar{e}_2 \ \bar{e}_3) = (e_1 \ e_2 \ e_3) \cdot A$$

if $\begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}$ is the dual basis of $(e_1 \ e_2 \ e_3)$

then $$\begin{pmatrix} \bar{f}^1 \\ \bar{f}^2 \\ \bar{f}^3 \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}$$

$$\begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix} = A \begin{pmatrix} \bar{f}^1 \\ \bar{f}^2 \\ \bar{f}^3 \end{pmatrix}$$

Pf.

$$\bar{e} = e \cdot A$$

$$\bar{f}\bar{e} = E = A^{-1} f e A = A^{-1} f \cdot \bar{e}$$

$$\Rightarrow \bar{f} = A^{-1} f$$

7.10 Let $(\bar{e_1} \ \bar{e_2} \ \bar{e_3}) = (e_1 \ e_2 \ e_3)\begin{pmatrix}1&0&0\\2&1&0\\3&2&1\end{pmatrix}$

For $t = e_3 \otimes e_1 \otimes f^2 \otimes f^1 + e_1 \otimes e_2 \otimes f^3 \otimes f^3$, determine $\underbrace{\bar{t}^{12}_{31}}_{\text{coeff of the term } \bar{e_3}\otimes\bar{e_1}\otimes\bar{f^1}\otimes\bar{f^2} \text{ in } \bar{t}}$

Ans: $(e_1 \ e_2 \ e_3) = (\bar{e_1} \ \bar{e_2} \ \bar{e_3})\begin{pmatrix}1&0&0\\2&1&0\\3&2&1\end{pmatrix}^{-1}$

$= (\bar{e_1} \ \bar{e_2} \ \bar{e_3})\begin{pmatrix}1&0&0\\-2&1&0\\1&-2&1\end{pmatrix}$

$\Rightarrow e_1 = \bar{e_1} - 2\bar{e_2} + \bar{e_3}$, $e_2 = \bar{e_2} - 2\bar{e_3}$, $e_3 = \bar{e_3}$,

$\begin{pmatrix}f^1\\f^2\\f^3\end{pmatrix} = \begin{pmatrix}1&0&0\\2&1&0\\3&2&1\end{pmatrix}\begin{pmatrix}\bar{f^1}\\\bar{f^2}\\\bar{f^3}\end{pmatrix}$

$\Rightarrow f^1 = \bar{f}^1$, $f^2 = 2\bar{f^1} + \bar{f^2}$, $f^3 = 3\bar{f^1} + 2\bar{f^2} + \bar{f^3}$

$\bar{t} = \bar{e_3} \otimes (\bar{e_1} - 2\bar{e_2} + \bar{e_3}) \otimes (2\bar{f^1} + \bar{f^2}) \otimes \bar{f^1}$
$+ (\bar{e_1} - 2\bar{e_2} + \bar{e_3}) \otimes (\bar{e_2} - 2\bar{e_3}) \otimes (3\bar{f^1} + 2\bar{f^2} + \bar{f^3}) \otimes (3\bar{f^1} + 2\bar{f^2} + \bar{f^3})$

$\therefore \bar{t}^{12}_{31} = 0$

7.11 Let $(\bar{e_1} \ \bar{e_2} \ \bar{e_3}) = (e_1 \ e_2 \ e_3)\begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix}$

For t with all $t^{i_1 i_2}_{j_1 j_2 j_3} = 1$, determine $\bar{t}^{12}_{123}$.

Ans: $(e_1 \ e_2 \ e_3) = (\bar{e_1} \ \bar{e_2} \ \bar{e_3})\begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix}^{-1} = (\bar{e_1} \ \bar{e_2} \ \bar{e_3})\begin{pmatrix}1&-2&1\\0&1&-2\\0&0&1\end{pmatrix}$

$\Rightarrow e_1 = \bar{e_1}$, $e_2 = -2\bar{e_1} + \bar{e_2}$, $e_3 = \bar{e_1} - 2\bar{e_2} + \bar{e_3}$

$\begin{pmatrix}f^1\\f^2\\f^3\end{pmatrix} = \begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}\bar{f^1}\\\bar{f^2}\\\bar{f^3}\end{pmatrix}$

$\Rightarrow f^1 = \bar{f^1} + 2\bar{f^2} + 3\bar{f^3}$, $f^2 = \bar{f^2} + \bar{f^3}$, $f^3 = \bar{f^3}$

$t = (e_1 + e_2 + e_3) \otimes (e_1 + e_2 + e_3) \otimes (e_1 + e_2 + e_3)$
$\otimes (f^1 + f^2 + f^3) \otimes (f^1 + f^2 + f^3)$

$\bar{t} = (-\bar{e_2} + \bar{e_3}) \otimes (-\bar{e_2} + \bar{e_3}) \otimes (-\bar{e_2} + \bar{e_3}) \otimes (\bar{f^1} + 3\bar{f^2} + 5\bar{f^3}) \otimes (\bar{f^1} + 3\bar{f^2} + 5\bar{f^3})$

$\therefore t^{12}_{123} = 0$

$S^p U = \otimes^p U / \sim$ where $u_1 \otimes \cdots \otimes u_p \sim u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(p)}$

$\rho_\sigma : \otimes^p U \to \otimes^p U$

$u_1 \otimes \cdots \otimes u_p \mapsto u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(p)}$

$\rho_\sigma \circ \rho_\tau = \underset{\text{doing } \tau \text{ first}}{\rho_{\tau \circ \sigma}}$, $\rho_{id} = id$.

8.2 a) Let char $\mathbb{k} = 0$.
Consider $Sym : \otimes^p U \to \otimes^p U$, $Sym(-) = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma(-)$
Determine the symmetrisation (image under Sym) of

$u_1 + u_2 + 3u_3 \in \otimes^1 U$,

$u_1 \otimes u_2 + u_1 \otimes u_3 \in \otimes^2 U$,

$u_1 \otimes u_2 \otimes u_2 \in \otimes^3 U$.

Ans: $Sym(u_1 + u_2 + 3u_3) = \frac{1}{1!}(\rho_{id}(u_1 + u_2 + 3u_3)) = u_1 + u_2 + 3u_3$
as $\Sigma_1 = \{id\}$.

$$\begin{aligned} Sym(u_1 \otimes u_2 + u_1 \otimes u_3) &= Sym(u_1 \otimes u_2) + Sym(u_1 \otimes u_3) \\ &= \tfrac{1}{2!}\left(\rho_{id}(u_1 \otimes u_2) + \rho_{(12)}(u_1 \otimes u_2)\right) \\ &\quad + \tfrac{1}{2!}\left(\rho_{id}(u_1 \otimes u_3) + \rho_{(13)}(u_1 \otimes u_3)\right) \\ &= \tfrac{1}{2}(u_1 \otimes u_2 + u_2 \otimes u_1 + u_1 \otimes u_3 + u_3 \otimes u_1) \end{aligned}$$

$$\begin{aligned} Sym(u_1 \otimes u_2 \otimes u_2) &= \tfrac{1}{3!}\big(\rho_{id}(u_1 \otimes u_2 \otimes u_2) + \rho_{(12)}(u_1 \otimes u_2 \otimes u_2) \\ &\quad + \rho_{(13)}(u_1 \otimes u_2 \otimes u_2) + \rho_{(23)}(u_1 \otimes u_2 \otimes u_2) \\ &\quad + \rho_{(123)}(u_1 \otimes u_2 \otimes u_2) + \rho_{(132)}(u_1 \otimes u_2 \otimes u_2)\big) \\ &= \tfrac{1}{6}(u_1 \otimes u_2 \otimes u_2 + u_2 \otimes u_1 \otimes u_2 \\ &\quad + u_2 \otimes u_2 \otimes u_1 + u_1 \otimes u_2 \otimes u_2 \\ &\quad + u_2 \otimes u_1 \otimes u_2 + u_2 \otimes u_2 \otimes u_1) \\ &= \tfrac{1}{3}(u_1 \otimes u_2 \otimes u_2 + u_2 \otimes u_1 \otimes u_2 + u_2 \otimes u_2 \otimes u_1) \end{aligned}$$

b) Prove that $\rho_\tau \circ Sym = Sym$ and $Sym \circ \rho_\tau = Sym$

Pf. $$\begin{aligned} \rho_\tau\left(\tfrac{1}{p!} \textstyle\sum_{\sigma \in \Sigma_p} \rho_\sigma(t)\right) &= \tfrac{1}{p!} \textstyle\sum_{\sigma \in \Sigma_p} \rho_\tau \rho_\sigma(t) = \tfrac{1}{p!} \textstyle\sum_{\sigma \in \Sigma_p} \rho_{\sigma \circ \tau}(t) \\ &= \tfrac{1}{p!} \textstyle\sum_{\sigma \in \Sigma_p} \rho_\sigma(t) \\ &= Sym(t) \end{aligned}$$

$$\begin{aligned} Sym(\rho_\tau(t)) &= \tfrac{1}{p!} \textstyle\sum_{\sigma \in \Sigma_p} \rho_\sigma(\rho_\tau(t)) \\ &= \tfrac{1}{p!} \textstyle\sum_{\sigma \in \Sigma_p} \rho_{\tau \circ \sigma}(t) \\ &= Sym(t) \end{aligned}$$

c) Show that $\operatorname{im} \mathrm{Sym}$ consists of symmetric tensors.

Ans: $\rho_\tau \circ \mathrm{Sym} = \mathrm{Sym}$

d) Observe by the universal property of the quotient $S^p V = \otimes^p V/\sim$ that $\mathrm{Sym} : \otimes^p V \to \otimes^p V$ induces a map $\mathrm{sym} : S^p V \to \otimes^p V$. Describe sym.

Ans:
$$\begin{tikzcd} \otimes^p V \arrow[r, "\mathrm{Sym}"] \arrow[d, "q"'] & \otimes^p V \\ S^p V \arrow[ur, dashed, "\exists!\, \mathrm{sym}"'] & \end{tikzcd}$$

$\mathrm{sym} : S^p V \longrightarrow \otimes^p V$

where $\mathrm{sym}(u_1 \vee u_2) = \mathrm{Sym}(u_1 \otimes u_2) = \frac{1}{2}(u_1 \otimes u_2 + u_2 \otimes u_1)$

$\therefore \mathrm{sym}(u_1 \vee \cdots \vee u_p) = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma (u_1 \otimes \cdots \otimes u_p)$

e) Prove that there is a bijection

$$S^p V \leftrightsquigarrow \{\text{symmetric tensor}\}$$

Pf. $\operatorname{im} \mathrm{sym}$ consists of symmetric tensors

$\Rightarrow S^p V \subseteq \{\text{symmetric tensor}\}$

It suffices to show that sym is an injective linear map.

Injectivity:

Let $a, b \in S^p V$.

Fix a basis $\{w_1, \cdots, w_k\}$ for V.

Write $a = \sum_{i_1 \leq i_2 \leq \cdots \leq i_p} a_{i_1 i_2 \cdots i_p} w_{i_1} \vee \cdots \vee w_{i_p}$

$b = \sum_{i_1 \leq i_2 \leq \cdots \leq i_p} b_{i_1 i_2 \cdots i_p} w_{i_1} \vee \cdots \vee w_{i_p}$

Now suppose $\mathrm{sym}(a) = \mathrm{sym}(b)$

then $\frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma \left(\sum_{i_1 \leq i_2 \leq \cdots \leq i_p} a_{i_1 i_2 \cdots i_p} w_{i_1} \vee \cdots \vee w_{i_p} \right)$

$= \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma \left(\sum_{i_1 \leq i_2 \leq \cdots \leq i_p} b_{i_1 i_2 \cdots i_p} w_{i_1} \vee \cdots \vee w_{i_p} \right)$

$\Rightarrow a_{i_1 i_2 \cdots i_p} = b_{i_1 i_2 \cdots i_p} \Rightarrow a = b$

Linearity is clear.

8.3 Define $\mathrm{Alt}: \otimes^p V \to \otimes^p V$, $\mathrm{Alt}(-) = \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sgn}\,\sigma \cdot \rho_\sigma(-)$

Compute the antisymmetrization (image under Alt) of

$$u_1 + u_2 + 3u_3 \in \otimes^1 V,$$
$$u_1 \otimes u_2 - u_2 \otimes u_3 + u_3 \otimes u_3 \in \otimes^2 V$$

Ans: $\mathrm{Alt}(u_1 + u_2 + 3u_3) = u_1 + u_2 + 3u_3 \quad \because p = 1$

$\mathrm{Alt}(u_1 \otimes u_2 - u_2 \otimes u_3 + u_3 \otimes u_3)$

$= \mathrm{Alt}(u_1 \otimes u_2) - \mathrm{Alt}(u_2 \otimes u_3) + \mathrm{Alt}(u_3 \otimes u_3)$

$= \frac{1}{2}(u_1 \otimes u_2 - u_2 \otimes u_1) - \frac{1}{2}(u_2 \otimes u_3 - u_3 \otimes u_2) + \frac{1}{2}(u_3 \otimes u_3 - u_3 \otimes u_3)$

$= \frac{1}{2}(u_1 \otimes u_2 - u_2 \otimes u_1 - u_2 \otimes u_3 + u_3 \otimes u_2)$

8.4 Let $w \in \mathrm{Lin}_3(V, V, V; V)$ s.t.

$$w(u, v, w) = w(v, u, w) \quad \text{— ①}$$
$$w(u, v, w) = -w(u, w, v) \quad \text{— ②}$$

Show that $w \equiv 0$.

Ans:
$$\begin{aligned}
w(u, v, w) &= w(v, u, w) && \text{by ①} \\
&= -w(v, w, u) && \text{by ②} \\
&= -w(w, v, u) && \text{by ①} \\
&= w(w, u, v) && \text{by ②} \\
&= w(u, w, v) && \text{by ①} \\
&= -w(u, v, w) && \text{by ②}
\end{aligned}$$

$\therefore \quad w \equiv 0$

Rk. A k-tensor on V is a multilinear map $V \times \cdots \times V \to \mathbb{R}$.
e.g. $w \in \mathrm{Lin}_3(V, V, V; \mathbb{R})$ is a 3-tensor.

A k-tensor is called alternating $\Leftrightarrow$

$$w(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = w(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

In Differential Geometry, a differential k-form is a function w on M (locally $\mathbb{R}^n$), assigning $p \in M$ to an alternating k-tensor

$$w_p : \underbrace{T_pM \times \cdots \times T_pM}_{k} \to \mathbb{R}.$$

T_pM: tangent of M at p

A k-form w has a coordinate expression, e.g. if $k=2$, $n=2$, a 2-form is

$$w = w_{12} \, dx \wedge dy$$

for $w_{12} : M \to \mathbb{R}$

In $\mathbb{R}^n$, for n vectors that span a parallelepiped, the volume of the parallelepiped is given by

$$\det(v_1, \dots, v_n).$$

Note that det is multilinear and antisymmetric, this motivates us to define a volume element on a manifold by considering an antisymmetric multilinear function
$\rightsquigarrow$ w_p alternating k-tensor

In other words, a differential k-form assigns for each point p on a manifold M a volume element, i.e. give a local volume element.

If on a 2-dim manifold M ($n = k = 2$), to compute the area on a region S on M, we compute

$$\int_S w_{12} \, dx \wedge dy.$$