8.2 c) Show that im Sym consists of symmetric tensors.

Ans: $\rho_\tau \circ \mathrm{Sym} = \mathrm{Sym}$

d) Observe by the universal property of the quotient $S^pV = \otimes^p V/\sim$ that $\mathrm{Sym} : \otimes^p V \to \otimes^p V$ induces a map $\mathrm{sym} : S^pV \to \otimes^p V$. Describe sym.

Ans:

$$\begin{tikzcd} \otimes^p V \arrow[r, "\mathrm{Sym}"] \arrow[d, "q"'] & \otimes^p V \\ S^pV \arrow[ur, dashed, "\exists!\ \mathrm{sym}"'] & \end{tikzcd}$$

$\mathrm{sym} : S^pV \longrightarrow \otimes^p V$

where $\mathrm{sym}(u_1 \vee u_2) = \mathrm{Sym}(u_1 \otimes u_2) = \frac{1}{2}(u_1 \otimes u_2 + u_2 \otimes u_1)$

$\therefore\ \mathrm{sym}(u_1 \vee \cdots \vee u_p) = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma (u_1 \otimes \cdots \otimes u_p)$

e) Prove that there is a bijection

$S^pV \quad \leftrightarrow \quad \{\text{symmetric tensor}\}$

Pf. im sym consists of symmetric tensors
$\Rightarrow S^pV \subseteq \{\text{symmetric tensor}\}$

It suffices to show that sym is an injective linear map.

Injectivity:

Let $a, b \in S^pV$.

Fix a basis $\{w_1, \dots, w_k\}$ for V.

Write $a = \sum_{i_1 \leq i_2 \leq \cdots \leq i_p} a_{i_1 i_2 \cdots i_p} w_{i_1} \vee \cdots \vee w_{i_p}$

$b = \sum_{i_1 \leq i_2 \leq \cdots \leq i_p} b_{i_1 i_2 \cdots i_p} w_{i_1} \vee \cdots \vee w_{i_p}$

Now suppose $\mathrm{sym}(a) = \mathrm{sym}(b)$

then $\frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma \left(\sum_{i_1 \leq i_2 \leq \cdots \leq i_p} a_{i_1 i_2 \cdots i_p} w_{i_1} \vee \cdots \vee w_{i_p} \right)$

$= \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \rho_\sigma \left(\sum_{i_1 \leq i_2 \leq \cdots \leq i_p} b_{i_1 i_2 \cdots i_p} w_{i_1} \vee \cdots \vee w_{i_p} \right)$

$\Rightarrow a_{i_1 i_2 \cdots i_p} = b_{i_1 i_2 \cdots i_p} \Rightarrow a = b$

Linearity is clear.

8.3 Define $\mathrm{Alt} : \otimes^p V \to \otimes^p V$, $\mathrm{Alt}(-) = \frac{1}{p!} \sum_{\sigma \in S_p} \operatorname{sgn}\sigma \cdot \rho_\sigma(-)$

Compute the antisymmetrisation (image under Alt) of

$u_1 + u_2 + 3u_3 \in \otimes^1 V$,

$u_1 \otimes u_2 - u_2 \otimes u_3 + u_3 \otimes u_3 \in \otimes^2 V$

Ans: $\mathrm{Alt}(u_1 + u_2 + 3u_3) = u_1 + u_2 + 3u_3 \quad \because p = 1$

$$\begin{aligned}
&\mathrm{Alt}(u_1 \otimes u_2 - u_2 \otimes u_3 + u_3 \otimes u_3) \\
&= \mathrm{Alt}(u_1 \otimes u_2) - \mathrm{Alt}(u_2 \otimes u_3) + \mathrm{Alt}(u_3 \otimes u_3) \\
&= \tfrac{1}{2}(u_1 \otimes u_2 - u_2 \otimes u_1) - \tfrac{1}{2}(u_2 \otimes u_3 - u_3 \otimes u_2) + \tfrac{1}{2}(u_3 \otimes u_3 - u_3 \otimes u_3) \\
&= \tfrac{1}{2}(u_1 \otimes u_2 - u_2 \otimes u_1 - u_2 \otimes u_3 + u_3 \otimes u_2)
\end{aligned}$$

8.4 Let $\omega \in \mathrm{Lin}_3(V, V, V; V)$ s.t.

$\omega(u, v, w) = \omega(v, u, w)$ — ①

$\omega(u, v, w) = -\omega(u, w, v)$ — ②

Show that $\omega \equiv 0$.

Ans:

$$\begin{aligned}
\omega(u, v, w) &= \omega(v, u, w) && \text{by ①} \\
&= -\omega(v, w, u) && \text{by ②} \\
&= -\omega(w, v, u) && \text{by ①} \\
&= \omega(w, u, v) && \text{by ②} \\
&= \omega(u, w, v) && \text{by ①} \\
&= -\omega(u, v, w) && \text{by ②}
\end{aligned}$$

$\therefore \omega \equiv 0$

Rk. A k-tensor on V is a multilinear map $V \times \cdots \times V \to \mathbb{R}$.
e.g. $w \in \mathrm{Lin}_3(V, V, V; \mathbb{R})$ is a 3-tensor.

A k-tensor is called alternating $\Leftrightarrow$
$$w(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = w(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

In Differential Geometry, a differential k-form is a function w on M (locally $\mathbb{R}^n$), assigning $p \in M$ to an alternating k-tensor
$$w_p : \underbrace{T_pM \times \cdots \times T_pM}_{k} \to \mathbb{R}.$$
T_pM: tangent of M at p

A k-form w has a coordinate expression, e.g. if $k=2$, $n=2$, a 2-form is
$$w = w_{12} \, dx \wedge dy$$
for $w_{12} : M \to \mathbb{R}$

In $\mathbb{R}^n$, for n vectors that span a parallelepiped, the volume of the parallelepiped is given by
$$\det(v_1, \ldots, v_n).$$
Note that det is multilinear and antisymmetric, this motivates us to define a volume element on a manifold by considering an antisymmetric multilinear function $\rightsquigarrow$ w_p alternating k-tensor

In other words, a differential k-form assigns for each point p on a manifold M a volume element, i.e. give a local volume element.

If on a 2-dim manifold M ($n = k = 2$), to compute the area on a region S on M, we compute
$$\int_S w_{12} \, dx \wedge dy.$$

8.5 Define $\times : V \times V \to V$ s.t.

$$\mathrm{Vol}(u, v, w) = \langle u \times v, w \rangle$$

Prove the following

i) $\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0$

ii) $u \times v \neq 0 \Leftrightarrow u$ & v are linearly independent and this implies $(u, v, u \times v)$ is a positive basis

iii) $|u \times v| = |u| \cdot |v| \cdot \sin\alpha$ where α is the angle between u & v

the above properties determined $u \times v$ uniquely in a geometric sense, we can also do it algebraically:

Rk. $u \times v = \langle u \times v, e_1 \rangle e_1 + \langle u \times v, e_2 \rangle e_2 + \langle u \times v, e_3 \rangle e_3$

v) $(u \times v) \times w = \langle u, w \rangle v - \langle v, w \rangle u$

Ans: i) $\langle u \times v, u \rangle = \mathrm{Vol}(u, v, u) = \det(u, v, u) = 0$

$\langle u \times v, v \rangle = \mathrm{Vol}(u, v, v) = 0$

ii) u & v are lin ind $\Rightarrow \exists\, w : (u, v, w)$ is a basis of V

$\Rightarrow \mathrm{Vol}(u, v, w) = \langle u \times v, w \rangle \neq 0$

$\Rightarrow u \times v \neq 0$

u & v are lin dep $\Rightarrow 0 = \mathrm{Vol}(u, v, w) = \langle u \times v, w \rangle \quad \forall w \in V$

$\Rightarrow u \times v = 0$ when fixing $w \neq 0$

Now compute $\mathrm{Vol}(u, v, u \times v)$

$= \langle u \times v, u \times v \rangle$ by def of $\times$

$= |u \times v|^2 > 0$

iii) Consider $|u \times v|^4 = \mathrm{Vol}(u, v, u \times v)^2$

$$= \det(u \;\; v \;\; u \times v) \cdot \det\begin{pmatrix} u \\ v \\ u \times v \end{pmatrix}$$

$$= \det\begin{pmatrix} |u|^2 & \langle u, v \rangle & 0 \\ \langle u, v \rangle & |v|^2 & 0 \\ 0 & 0 & |u \times v|^2 \end{pmatrix}$$

$$= (|u|^2 |v|^2 - \langle u, v \rangle^2) \cdot |u \times v|^2$$

$$= (|u|^2 |v|^2 - (|u| \cdot |v| \cdot \cos\alpha)^2) \cdot |u \times v|^2$$

$$= |u|^2 |v|^2 \sin^2\alpha \cdot |u \times v|^2$$

$\Rightarrow |u \times v| = |u|\,|v| \cdot \sin\alpha$

v) Note that $|e_i \times e_i| = \sin 0 = 0 \Rightarrow e_i \times e_i = 0$

Consider $\mathrm{Vol}(e_1, e_2, e_3) = 1$

$\Rightarrow \langle e_1 \times e_2, e_3 \rangle = 1$

$\Rightarrow e_1 \times e_2 = e_3$

Similarly, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$,

and $e_2 \times e_1 = -e_3$, $e_3 \times e_2 = -e_1$, $e_1 \times e_3 = -e_2$.

Note that

$$u \times v = (u_1 e_1 + u_2 e_2 + u_3 e_3) \times (v_1 e_1 + v_2 e_2 + v_3 e_3)$$

$$= u_1 v_1 (0) + u_1 v_2 e_3 - u_1 v_3 e_2 - u_2 v_1 e_3 + u_2 v_2 (0) + u_2 v_3 e_1 + u_3 v_1 e_2 - u_3 v_2 e_1 + u_3 v_3 (0)$$

$$= (u_2 v_3 - u_3 v_2) e_1 + (u_3 v_1 - u_1 v_3) e_2 + (u_1 v_2 - u_2 v_1) e_3$$

Now $(u \times v) \times w$

$$= ((u_2 v_3 - u_3 v_2) e_1 + (u_3 v_1 - u_1 v_3) e_2 + (u_1 v_2 - u_2 v_1) e_3) \times (w_1 e_1 + w_2 e_2 + w_3 e_3)$$

$$= (u_2 v_3 - u_3 v_2) w_2 e_3 - (u_2 v_3 - u_3 v_2) w_3 e_2 - (u_3 v_1 - u_1 v_3) w_1 e_3 + (u_3 v_1 - u_1 v_3) w_3 e_1 + (u_1 v_2 - u_2 v_1) w_1 e_2 - (u_1 v_2 - u_2 v_1) w_2 e_1$$

$$= [(u_3 v_1 - u_1 v_3) w_3 - (u_1 v_2 - u_2 v_1) w_2] e_1 + [(u_1 v_2 - u_2 v_1) w_2 - (u_2 v_3 - u_3 v_2) w_3] e_2 + [(u_2 v_3 - u_3 v_2) w_2 - (u_3 v_1 - u_1 v_3) w_1] e_3$$

Compute

$\langle u, w \rangle v$

$$= \langle u_1 e_1 + u_2 e_2 + u_3 e_3, w_1 e_1 + w_2 e_2 + w_3 e_3 \rangle \cdot (v_1 e_1 + v_2 e_2 + v_3 e_3)$$

$$= (u_1 w_1 + u_2 w_2 + u_3 w_3) \cdot (v_1 e_1 + v_2 e_2 + v_3 e_3)$$

Similarly, $\langle v, w \rangle u$

$$= (v_1 w_1 + v_2 w_2 + v_3 w_3) \cdot (u_1 e_1 + u_2 e_2 + u_3 e_3)$$

$\therefore \quad (u \times v) \times w = \langle u, w \rangle v - \langle v, w \rangle u$

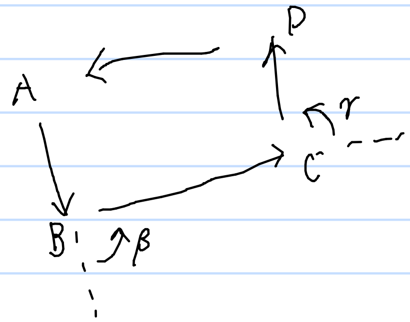
Rk. Not associative

Indeed, $\times$ satisfies the Jacobian Identity:

$$(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 0$$

this means $\mathbb{R}^3$ is a Lie algebra

9.1 i) Decide on the orientation and convexity of the polygon ABCD, where $A = [0,1]$, $B = [3,2]$, $C = [5,0]$, $D = [3,5]$ by computing the oriented volumes and see if they are positive or negative.

Set 

as the positive orientation

$\mathrm{Vol}(u,v) = \det(u\ v)$

$= |u||v| \sin\theta$

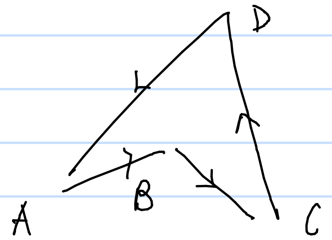
Ans: $\vec{DA} = (-3,-4)$ $\qquad \mathrm{Vol}(\vec{DA}, \vec{AB}) = (-3)(1) - (3)(-4) = 9 > 0$

$\vec{AB} = (3,1)$ $\qquad \mathrm{Vol}(\vec{AB}, \vec{BC}) = (3)(-2) - (2)(1) = -8 < 0$

$\vec{BC} = (2,-2)$ $\qquad \mathrm{Vol}(\vec{BC}, \vec{CD}) = (2)(5) - (-2)(-2) = 6 > 0$

$\vec{CD} = (-2,5)$ $\qquad \mathrm{Vol}(\vec{CD}, \vec{DA}) = (-2)(-4) - (-3)(5) = 23 > 0$

The polygon looks like



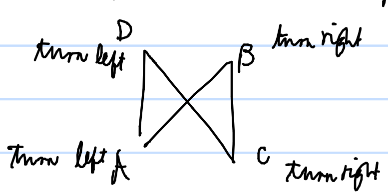
since $\mathrm{Vol}(\vec{DA}, \vec{AB}) > 0$ means 'turning left' and 3 Vols > 0 while only 1 Vol < 0, so the no. of times of left-turns is greater than that of right-turns.

$\therefore$ anticlockwise.

Rk. if all Vol > 0, then convex & positively (anticlockwise) oriented.
all Vol < 0, then convex & negatively (clockwise) oriented.

ii) What happens if 2 of the Vols are positive and 2 are negative?

Ans: no orientation, the image looks like



Thm. The conjugation by q, i.e., $p \mapsto q p q^{-1}$ where $q = e^{\frac{\theta}{2} v} = \cos\frac{\theta}{2} + v \cdot \sin\frac{\theta}{2}$ with $|v| = 1$ is the rotation along v by θ.

Fact. Composition is given as product.

9.2 Describe the composition $S \circ R$ of two rotations S, R via the vector of the axis and the angle,
where R: around $(1,-1,1)$ by $+120^\circ$
S: around $(1,1,1)$ by $+60^\circ$

Ans: $i^2 = -1$, $j^2 = -1$, $k^2 = -1$

$i \times j = k$, $j \times k = i$, $k \times i = j$

$j \times i = -k$, $k \times j = -i$, $i \times k = -j$

$+120^\circ = \frac{2\pi}{3}$, $+60^\circ = \frac{\pi}{3}$

$$R = e^{\frac{2\pi}{3} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{3}} (1,-1,1)} = e^{\frac{\pi}{3} \frac{1}{\sqrt{3}} (1,-1,1)}$$
$$S = e^{\frac{\pi}{3} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{3}} (1,1,1)} = e^{\frac{\pi}{6} \frac{1}{\sqrt{3}} (1,1,1)}$$

$(i+j+k)(i-j+k)$
$= (-1 - k - j - k + 1 + i + j + i - 1)$
$= (2i - 2k - 1)$

$$S \circ R = e^{\frac{\pi}{6}\frac{1}{\sqrt{3}}(1,1,1)} \cdot e^{\frac{\pi}{3}\frac{1}{\sqrt{3}}(1,-1,1)}$$
$$= \left(\cos\frac{\pi}{6} + \frac{1}{\sqrt{3}}(1,1,1)\sin\frac{\pi}{6}\right) \cdot \left(\cos\frac{\pi}{3} + \frac{1}{\sqrt{3}}(1,-1,1)\sin\frac{\pi}{3}\right)$$
$$= \left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}(1,1,1) \cdot \frac{1}{2}\right)\left(\frac{1}{2} + \frac{1}{\sqrt{3}}(1,-1,1)\frac{\sqrt{3}}{2}\right)$$
$$= \frac{1}{4\sqrt{3}}\left(3 + (1,1,1)\right) \cdot \left(1 + (1,-1,1)\right)$$
$$= \frac{1}{4\sqrt{3}}\left(3 + 3(1,-1,1) + (1,1,1) - 1 + (2,0,-2)\right)$$
$$= \frac{1}{2\sqrt{3}}\left(1 + (3,-1,1)\right)$$

Find $q = \cos\frac{\phi}{2} + v \cdot \sin\frac{\phi}{2} = \frac{1}{2\sqrt{3}}\left(1 + (3,-1,1)\right)$

Set
$$\cos\frac{\phi}{2} = \frac{1}{2\sqrt{3}}, \quad \sin\frac{\phi}{2} = \frac{1}{2\sqrt{3}} \cdot \frac{1}{11}$$

$\therefore$ $S \circ R$ is the rotation around $(3,-1,1)$ by $2\arccos\frac{1}{2\sqrt{3}}$

9.3 Describe the composition $S \circ R$ of two rotations S, R via the vector of the axis and the angle,

where R: around $(1, 0, 1)$ by $+90^\circ$

S: around $(1, 2, 1)$ by $+120^\circ$

Ans:

$$R = e^{\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}(1,0,1)}$$
$$S = e^{\frac{\pi}{3}\cdot\frac{1}{\sqrt{6}}(1,2,1)}$$

$$S \circ R = e^{\frac{\pi}{3}\cdot\frac{1}{\sqrt{6}}(1,2,1)}\, e^{\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}(1,0,1)}$$
$$= \left(\cos\tfrac{\pi}{3} + \tfrac{1}{\sqrt{6}}(1,2,1)\cdot\sin\tfrac{\pi}{3}\right)\left(\cos\tfrac{\pi}{4} + \tfrac{1}{\sqrt{2}}(1,0,1)\cdot\sin\tfrac{\pi}{4}\right)$$
$$= \left(\tfrac{1}{2} + \tfrac{1}{\sqrt{6}}(1,2,1)\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{\sqrt{2}}{2} + \tfrac{1}{\sqrt{2}}(1,0,1)\tfrac{\sqrt{2}}{2}\right)$$
$$= \tfrac{1}{2\sqrt{2}}\left(\sqrt{2} + (1,2,1)\right)\cdot\tfrac{1}{2}\left(\sqrt{2} + (1,0,1)\right)$$
$$= \tfrac{1}{4\sqrt{2}}\left(2 + \sqrt{2}(1,0,1) + \sqrt{2}(1,2,1) - 2 + (2,0,-2)\right)$$
$$= \tfrac{1}{2\sqrt{2}}\left(0 + (1+\sqrt{2}, \sqrt{2}, -1+\sqrt{2})\right)$$

Set
$$\cos\tfrac{\phi}{2} = 0 \Rightarrow \phi = \pi$$

$\therefore$ $S \circ R$ is the rotation around $(1+\sqrt{2}, \sqrt{2}, -1+\sqrt{2})$ by π

8.6 i) Prove that if $(e_1, \ldots, e_n)$ is a basis of a complex vector space V then $(e_1, ie_1, \ldots, e_n, ie_n)$ is a basis of V^R (viewing V as a real vec space).

ii) Prove that if $(\bar{e_1}, \ldots, \bar{e_n})$ is any other basis for V, then the resulting real bases have the same orientation,
i.e., the determinant of the transformation matrix is positive

Ans: i) In a complex vec sp V, an arbitrary vector is expressed as
$(a_1 + ib_1) e_1 + \cdots + (a_n + ib_n) e_n$
which is just $a_1 e_1 + \cdots + a_n e_n + b_1 ie_1 + \cdots + b_n ie_n$.

ii) Suppose both $(e_1, \ldots, e_n)$ & $(\bar{e_1}, \ldots, \bar{e_n})$ are bases for a complex vector space V.
That means $(\bar{e_1}, \bar{e_2}, \ldots, \bar{e_n})$ can be obtained from $(e_1, \ldots, e_n)$ by the following 3 operations:

multiplying by $\begin{pmatrix} 1 & 0 - z \cdots 0 \\ 0 & \ddots & \vdots \\ \vdots & & 0 \\ 0 & \cdots 0 & 1 \end{pmatrix} = P_1$ (add multiple of one basis vector to another)

or by $\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \ddots 1 \end{pmatrix} = P_2$ (swapping 2 basis vectors)

or by $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & z & \\ & & & \ddots 1 \end{pmatrix} = P_3$ (multiple 1 of the vectors by non-zero mult)

We first consider multiplication by a complex no. z in terms of matrix multiplication in $\mathbb{R}^2$, as $\mathbb{C} \cong \mathbb{R}^2$.
Write $z = a + ib$
then $(a+ib)(c+id) = ac - bd + i(ad + bc)$
so the operation is expressed as $\mathbb{R}^2 \ni (c, d) \mapsto (ac - bd, ad + bc)$
which is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with det $a^2 + b^2 > 0$

So P_1, P_2, P_3 can be expressed in real basis as

$P_1 = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \begin{smallmatrix} a & -b \\ b & a \end{smallmatrix} \\ & \ddots & 0 \\ 0 & & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \end{pmatrix}$, $P_2 = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & 0 \\ & \ddots \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \\ 0 & & \ddots \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$, $P_3 = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ddots & & 0 \\ & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} & \\ 0 & & \ddots \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$

and they all have positive determinant.