2. Calculate the eigenvalues λ of A_0:

Case I – all $\lambda \neq 0$:

I.3. Find a_{00} s.t.

$$\begin{vmatrix} a_{00} & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_4 \end{vmatrix} = |A|$$

∵ obtain from trans & rotation

∴ det unchang

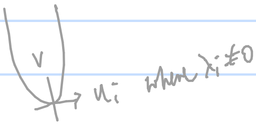
I.4 Then the canonical form is given by

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + a_{00} = 0$$

Case II – some $\lambda = 0$:

a) If $\text{rk}\,A - \text{rk}\,A_0 = 2$:

II.3. Find vertex V of Q by solving $\begin{cases} f(V,V) = 0 \\ f(u_i, V) = 0 \end{cases}$ for $\lambda_i \neq 0$ ($A \mid V \in Q$)



II.4. Find the linear term $c_{0i}\, y_i$ by solving $f(u_i, V) = \frac{c_{0i}}{2}$

II.5. Then the canonical form is given by

i) If there are two λ's $= 0$,

$$\lambda_1^2 y_1^2 + \sum c_{0i} y_i = 0$$

parabolic cylinder 

ii) If only one $\lambda = 0$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \sum c_{0i} y_i = 0$$

if all λ's are of the same sign ⇝ elliptic paraboloid 

if not ⇝ hyperbolic paraboloid 

b) If $\text{rk}\,A - \text{rk}\,A_0 = 1$:

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$$

if all λ's are of the same sign ⇝ elliptic cylinder 

if not ⇝ hyperbolic cylinder 

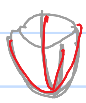
c) If $\text{rk}\,A - \text{rk}\,A_0 = 0$:

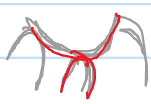
$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 0$$

intersecting planes 

There are other situations, eg. double plane $x^2 = 0$

parallel planes $x^2 = a^2$

elliptic 

hyperbolic 

4.1 Determine the canonical form of the conic section
$x_1^2 - 2x_1x_2 + x_2^2 - 4x_1 - 6x_2 + 3 = 0$

Ans. Homogenize:

$x_1^2 - 2x_1x_2 + x_2^2 - 4x_0x_1 - 6x_0x_2 + 3x_0^2 = 0$

$A = \begin{pmatrix} 3 & -2 & -3 \\ -2 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix}$ $\quad$ $\text{rk}\, A = 3$

$A_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ has eigenvalue $\Rightarrow$ parabola
$\text{rk}\, A_0 = 1$ why?

$\lambda_1 = 2, \quad u_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$; $\lambda_2 = 0, \quad u_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Now solve $\begin{cases} f(v,v) = 0 & — ① \\ f(u_1, v) = 0 & — ② \end{cases}$, $v = (1, x_1, x_2)^T$

②: $\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & -2 & -3 \\ -2 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = 0$

$(1 \quad 2 \quad -2) \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow 1 + 2x_1 - 2x_2 = 0$

①: $(1 \quad x_1 \quad x_2) \begin{pmatrix} 3 & -2 & -3 \\ -2 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = 0$

$(1 \quad x_1 \quad x_2) \begin{pmatrix} 3 - 2x_1 - 3x_2 \\ -2 + x_1 - x_2 \\ -3 - x_1 + x_2 \end{pmatrix} = 0$

$\Rightarrow 3 - 2x_1 - 3x_2 - 2x_1 + x_1^2 - x_1x_2 - 3x_2 - x_1x_2 + x_2^2 = 0$

$\therefore x_1 = \frac{1}{40}, \quad x_2 = \frac{21}{40} \quad \Rightarrow v = (1, \frac{1}{40}, \frac{21}{40})^T$

Now compute $f(u_2, v) = \frac{c_{0i}}{2}$

$= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & -2 & -3 \\ -2 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{40} \\ \frac{21}{40} \end{pmatrix}$

$= \frac{-5}{\sqrt{2}} = \frac{c_{0i}}{2}$

$\Rightarrow c_{0i} = -5\sqrt{2}$

$\therefore$ The canonical form of this parabola
$2x_1^2 - 5\sqrt{2}\, x_2 = 0$

4.2 Determine the canonical form of the conic section

$$5x_1^2 + 8x_2^2 + 5x_3^2 + 4x_1x_2 - 8x_1x_3 + 4x_2x_3 + 6x_1 + 6x_2 + 6x_3 - 27 = 0$$

Ans : Homogenize:

$$5x_1^2 + 8x_2^2 + 5x_3^2 + 4x_1x_2 - 8x_1x_3 + 4x_2x_3 + 6x_0x_1 + 6x_0x_2 + 6x_0x_3 - 27x_0^2 = 0$$

$$A = \begin{pmatrix} -27 & 3 & 3 & 3 \\ 3 & 5 & 2 & -4 \\ 3 & 2 & 8 & 2 \\ 3 & -4 & 2 & 5 \end{pmatrix} \qquad \operatorname{rk} A = 4$$

$$A_0 = \begin{pmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{pmatrix} \qquad \operatorname{rk} A_0 = 2$$

$$\lambda_1 = 9, \quad u_1 = \left(0, 0, \tfrac{2}{\sqrt5}, \tfrac{1}{\sqrt5}\right)^T$$

$$\lambda_2 = 9, \quad u_2 = \left(0, \tfrac{-5}{\sqrt{45}}, \tfrac{-2}{\sqrt{45}}, \tfrac{4}{\sqrt{45}}\right)^T$$

$$\lambda_3 = 0, \quad u_3 = \left(0, \tfrac{2}{3}, \tfrac{-1}{3}, \tfrac{2}{3}\right)^T$$

$\operatorname{rk} A - \operatorname{rk} A_0 = 2$

$\Rightarrow$ paraboloid

$a_1x_1^2 + a_2x_2^2 - a_3x_3 = 0$

Now solve
$$\begin{cases} f(v,v) = 0 \\ f(u_1, v) = 0 \\ f(u_2, v) = 0 \end{cases}$$

$$\Rightarrow v = \left(1, \tfrac{28}{9}, \tfrac{-37}{18}, \tfrac{28}{9}\right)^T$$

Now compute $f(u_3, v) = \frac{c_{0i}}{2}$

$$f(u_3, v) = \left(0 \ \tfrac{2}{3} \ \tfrac{-1}{3} \ \tfrac{2}{3}\right) \begin{pmatrix} -27 & 3 & 3 & 3 \\ 3 & 5 & 2 & -4 \\ 3 & 2 & 8 & 2 \\ 3 & -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{28}{9} \\ \frac{-37}{18} \\ \frac{28}{9} \end{pmatrix}$$

$$= 3$$

$$\Rightarrow c_{0i} = 6$$

$\therefore$ The canonical form of this elliptic paraboloid is

$$9x_1^2 + 9x_2^2 + 6x_3 = 0$$

Notation: vector - superscript, covector - subscript

4.4

Find a new basis $\alpha = \{e^1, e^2, e^3\}$ for $\mathbb{R}^3$ so that the resulting dual basis α^* for $(\mathbb{R}^3)^*$ is

$$\alpha^* = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad \left.\begin{aligned} f_1(x^1, x^2, x^3) &= 2x^1 - x^2 \\ f_2(x^1, x^2, x^3) &= x^2 - x^3 \\ f_3(x^1, x^2, x^3) &= x^1 + x^2 + x^3 \end{aligned}\right\} \text{ linear functional}$$

unknown, variable

i.e. $f_1 = (2 \;\; -1 \;\; 0)$

$f_2 = (0 \;\; 1 \;\; -1)$

$f_3 = (1 \;\; 1 \;\; 1)$

basis s.t. f is dual basis

Ans: By the def of a dual basis, we must have

$f_1(e^1) = 1, \quad f_2(e^1) = 0, \quad f_3(e^1) = 0$

def: $f_i(e^j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

That means to find e^1 is to solve

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore e^1 = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \\ -\frac{1}{5} \end{pmatrix}$$

Similarly, for e^2, solve

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore e_2 = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ -\frac{3}{5} \end{pmatrix}$$

for e^3, solve

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore e_3 = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}$$

Rk. $(e^1 \;\; e^2 \;\; e^3) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}^{-1}$

Check: $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} (e^1 \; e^2 \; e^3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ☆

4.5 Find a dual basis $\alpha^* = (f_1, f_2, f_3)$ to a basis α of $\mathbb{R}^3$:

$\alpha = (e^1, e^2, e^3)$

$$e^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad e^3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Ans: Recall : $(e^1 \ e^2 \ e^3) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}^{-1}$

$$\Leftrightarrow \quad (e^1 \ e^2 \ e^3)^{-1} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

So we find $\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}^{-1}$

$$= \begin{pmatrix} -\frac{1}{2} & -\frac{5}{4} & 1 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$$\therefore \quad f_1 = \left(-\tfrac{1}{2} \quad -\tfrac{5}{4} \quad 1 \right)$$
$$f_2 = \left(\tfrac{1}{2} \quad -\tfrac{1}{4} \quad 0 \right)$$
$$f_3 = \left(0 \quad \tfrac{1}{2} \quad 0 \right)$$