Two-phase simplex method

1. $\min\{(1 \quad 1)x \mid \begin{pmatrix} 2 & 1 \\ 1 & 7 \end{pmatrix} x \geq \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \; x \geq 0\}$

Ans: Add slack variables to make constraints become equalities:

x_1	x_2	s_1	s_2	$-Z$	RHS
2	1	-1	0	0	4
1	7	0	-1	0	7
1	1	0	0	1	0

Introduce artificial variable t_1, t_2 :

x_1	x_2	s_1	s_2	t_1	t_2	$-Z$	RHS
2	1	-1	0	1	0	0	4
1	7	0	-1	0	1	0	7
-3	-8	1	1	0	0	1	0

Pivot at (1, 1):

x_1		x_2	s_1	s_2	t_1	t_2	$-Z$	RHS	
1	1	$\frac{1}{2}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	0	0	2	/2
0		$\frac{13}{2}$	$\frac{1}{2}$	-1	$\frac{-1}{2}$	1	0	5	$+\frac{1}{2}p$
0		$-\frac{13}{2}$	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	1	0	$+\frac{3}{2}p$

Pivot at (2, 2):

basic	basic					basic		
x_1	x_2	s_1	s_2	t_1	t_2	$-Z$	RHS	
1	0	$-\frac{7}{13}$	$\frac{1}{13}$	$\frac{7}{13}$	$\frac{-1}{13}$	0	$\frac{21}{13}$	
0	1	$\frac{1}{13}$	$\frac{-2}{13}$	$\frac{-1}{13}$	$\frac{2}{13}$	0	$\frac{10}{13}$	
0	0	0	0	1	1	1	11	$+p$

For a basic variable x_i, if the corresponding $c_i \neq 0$, then add multiples of rows of constraint to turn it 0.
3. Repeat the simplex method.

$\Rightarrow t_1 = t_2 = 0$, proceed to Phase II :

x_1	x_2	s_1	s_2	$-Z$	RHS
1	0	$-\frac{7}{13}$	$\frac{1}{13}$	0	$\frac{21}{13}$
0	1	$\frac{1}{13}$	$\frac{-2}{13}$	0	$\frac{10}{13}$
1	1	0	0	1	0

$c_1, \; c_2 \neq 0$.

x_1	x_2	s_1	s_2	$-z$	RHS	
1	0	$-\frac{1}{13}$	$\frac{1}{13}$	0	$\frac{21}{13}$	
0	1	$\frac{1}{13}$	$\frac{-2}{13}$	0	$\frac{10}{13}$	
0	0	$\frac{6}{13}$	$\frac{1}{13}$	1	$-\frac{31}{13}$	$-R_1-R_2$

$$\therefore \quad z = \frac{31}{13}, \qquad x = \left(\frac{21}{13}, \frac{10}{13}\right)$$

Quaternions

Define H to be the real subspace of $Mat_{2\times 2}(\mathbb{C})$ spanned by

$$1 := E$$

$$I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$K := I \cdot J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

2. a) Verify $I^2 = -1$, $J^2 = -1$, $K^2 = -1$, $IJK = -1$.

Ans: $$I^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -E = -1$$

$$J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1$$

$$K^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1$$

$$IJK = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = I^2 = -1$$

b) Fill in the mult table for H.

Ans:

$\cdot$	1	I	J	K
1	1	I	J	K
I	I	-1	K	$-J$
J	J	$-K$	-1	I
K	K	J	$-I$	-1

$$IJ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$KI = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$IK = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$KJ = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$JI = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$JK = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

c) Compute
$$(aI + bJ + cK)(a'I + b'J + c'K)$$
and relate it with inner products and cross products

Ans: $(aI + bJ + cK)(a'I + b'J + c'K)$
$$= -(aa' + bb' + cc') + (bc' - cb')I + (ca' - ac')J + (ab' - ba')K$$
$$= -\langle v, v' \rangle + v \times v'$$

d) Show that the multiplication $\cdot$ above is bilinear and anti-symmetric.

Ans: $v \cdot v' = -\langle v, v' \rangle + v \times v'$
since both the inner prod and cross prod is bilinear, so is $\cdot$.
Since $\langle -, - \rangle$ is symmetric and $- \times -$ is anti-symmetric, so $\cdot$ is anti-symmetric:
$$v' \cdot v = -\langle v', v \rangle + v' \times v$$
$$= -\langle v, v' \rangle - v \times v'$$
$$= -(-\langle v, v' \rangle + v \times v') = -v \cdot v'$$

e) Write $a + u$ for any vector in H, where a is the real part and u is the imaginary part.
Define the conjugate of any $c + c_1 I + c_2 J + c_3 K$:
$$(c + c_1 I + c_2 J + c_3 K)^* := c - c_1 I - c_2 J - c_3 K.$$

Determine the inverse quaternion.

Ans: Consider $(c + c_1 I + c_2 J + c_3 K)(c + c_1 I + c_2 J + c_3 K)^*$
$$= c^2 - (c_1 I + c_2 J + c_3 K)^2$$
$$= c^2 - (c_1^2 + c_2^2 + c_3^2)$$
Let $(c + c_1 I + c_2 J + c_3 K)^{-1} = \frac{1}{c^2 - (c_1^2 + c_2^2 + c_3^2)} \cdot (c + c_1 I + c_2 J + c_3 K)^*$
Check that
$$\frac{1}{c^2 - (c_1^2 + c_2^2 + c_3^2)} \cdot (c + c_1 I + c_2 J + c_3 K)^* \cdot (c + c_1 I + c_2 J + c_3 K)$$
$$= 1$$
and $(c + c_1 I + c_2 J + c_3 K) \cdot \frac{1}{c^2 - (c_1^2 + c_2^2 + c_3^2)} \cdot (c + c_1 I + c_2 J + c_3 K)^*$
$$= 1$$
So the inverse operation is given by $(\ )^{-1}$.