Exercises—Global Analysis

1. Consider the **cylinder** in $\mathbb{R}^3$ given by the equation

$$M := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = R^2\},$$

where $R > 0$. Show that M is a 2-dimensional submanifold in $\mathbb{R}^3$. Moreover, give formula for local parametrizations and local trivializations, and a description of M as a local graph.

2. Consider a **double cone** given by rotating a line through 0 of slope α around the z-axis in $\mathbb{R}^3$. It is given by the equation

$$z^2 = (\tan\alpha)^2(x^2 + y^2).$$

At which points is the double cone a smooth submanifold of $\mathbb{R}^3$? Around the points where it is give a formula for local parametrizations and trivializations, and a description of it as a local graph.

3. Denote by Hom$(\mathbb{R}^n, \mathbb{R}^m)$ the nm-dimensional vector space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Consider the subset Hom$_r(\mathbb{R}^n, \mathbb{R}^m)$ of linear maps in Hom$(\mathbb{R}^n, \mathbb{R}^m)$ of rank r. Show that Hom$_r(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of dimension of $r(n + m - r)$ in Hom$(\mathbb{R}^n, \mathbb{R}^m)$.

Hint: Let $T_0 \in$ Hom$_r(\mathbb{R}^n, \mathbb{R}^m)$ be a linear map of rank r and decompose $\mathbb{R}^n$ and $\mathbb{R}^m$ as follows

$$\mathbb{R}^n = E \oplus E^\perp \quad \text{and} \quad \mathbb{R}^m = F \oplus F^\perp, \tag{0.1}$$

where F equals the image of T_0 and $E^\perp$ the kernel of T_0, and $(\cdot)^\perp$ denotes the orthogonal complement. Note that $\dim E = \dim F = r$. With respect to (0.1) any $T \in$ Hom$(\mathbb{R}^n, \mathbb{R}^m)$ can be viewed as a matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in$ Hom(E, F), $B \in$ Hom$(E^\perp, F)$, $C \in$ Hom$(E, F^\perp)$ and $D \in$ Hom$(E^\perp, F^\perp)$. Show that the set of matrices T with A invertible defines an open neighbourhood of T_0 and characterize the elements in this neighbourhood that have rank r (equivalently, the ones that have an $(n - r)$-dimensional kernel).

4. For $i = 1, ...n$ let $(M_i, \mathcal{A}_i)$ be a smooth manifolds. Suppose $M := M_1 \times ... \times M_n$ is endowed with the product topology. Then show that

$$\mathcal{A} := \{(U_1 \times ... \times U_n, u_1 \times ... \times u_n) : (U_i, u_i) \in \mathcal{A}_i\}$$

defines a smooth atlas on M and that the projections $\text{pr}_i : M \to M_i$ are smooth. Moreover show that, for any smooth manifold N, a map $f : N \to M$ is smooth if and only if $f_i := \text{pr}_i \circ f : N \to M_i$ is smooth for all i, and show that this property characterizes the smooth manifold structure on M uniquely.

5. Suppose $(M_i, \mathcal{A}_i)$ are smooth manifolds for $i \in I$, where I is countable. Consider the disjoint union

$$M := \sqcup_{i \in I} M_i = \cup_{i \in I}\{(x, i) : x \in M_i\}$$

endowed with the disjoint union topology and denote by $\text{inj}_i : M_i \hookrightarrow M$ the canonical injections ($\text{inj}_i(x) = (x, i)$). Show that $\mathcal{A} := \cup_{i \in I} \mathcal{A}_i$ defines a smooth atlas on M and that the injections inj_i are smooth. Moreover, show that for any smooth manifold N, a map $f : M \to N$ is smooth if and only if $f_i := f \circ \text{inj}_i : M_i \to N$ is smooth for all i, and show that this property characterizes the smooth manifold structure on M uniquely.

6. Suppose $U \subset \mathbb{R}^m$ is open and $f : U \to \mathbb{R}^n$ a smooth map such that $D_x f : \mathbb{R}^m \to \mathbb{R}^n$ is of rank r for all $x \in U$. Show that for any $x_0 \in U$ there exists a diffeomorphism ϕ between an open neighbourhood of x_0 and an open neighbourhood of $0 \in \mathbb{R}^m$ and a diffeomorphism ψ between an open neighbourhood of $y_0 = f(x_0)$ and an open neighbourhood of 0 in $\mathbb{R}^n$ such that the locally defined map

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^r \times \mathbb{R}^{m-r} \to \mathbb{R}^r \times \mathbb{R}^{n-r}$$

has the form $(x_1, ..., x_r, ..., x_m) \mapsto (x_1, ..., x_r, 0, ..., 0)$.

Hint: The idea is that f locally around x_0 looks like $D_{x_0} f$, which is a linear map $\mathbb{R}^m \to \mathbb{R}^n$ of rank r, which up to a basis change has the form $(x_1, .., x_m) \mapsto (x_1, ..., x_r, 0, ..., 0)$.

   (a) Set $E_2 := \ker(D_{x_0} f) \subset \mathbb{R}^m$ and $E_1 := E_2^\perp$, and $F_1 := \text{Im}(D_{x_0} f) \subset \mathbb{R}^n$ and $F_2 := F_1^\perp$. Decompose

$$\mathbb{R}^m = E_1 \oplus E_2 \qquad \text{and} \qquad \mathbb{R}^n = F_1 \oplus F_2,$$

   and consider f as a map $f = (f_1, f_2) : E_1 \oplus E_2 \to F_1 \oplus F_2$ defined on $U \subset E_1 \oplus E_2 = \mathbb{R}^m$.

   (b) Show that $\phi : E_1 \oplus E_2 \to F_1 \oplus E_2$ given by

$$\phi(x^1, x^2) = (f_1(x^1, x^2) - f_1(x_0^1, x_0^2), x^2 - x_0^2)$$

   is a local diffeomorphism around $x_0 = (x_0^1, x_0^2)$ whose local inverse will be the required map.

(c) Show that $g := f \circ \phi^{-1} : F_1 \oplus E_2 \to F_1 \oplus F_2$ has the form

$$g(y^1, y^2) = (g_1((y^1, y^2), g_2((y^1, y^2)) = (y^1 + y_0^1, g_2(y^1, 0)).$$

Now ψ is easily seen to be...?

7. Suppose M and N are are manifolds of dimension m respectively n and let $f : M \to N$ be a smooth map of constant rank r. Deduce from (1) that for any fixed $y \in f(M)$ the preimage $f^{-1}(y) \subset M$ is a submanifold of dimension $m - r$ in M.

8. Consider the Grassmannian of r-planes in $\mathbb{R}^n$:

$$\text{Gr}(r, n) := \{E \subset \mathbb{R}^n : E \text{ is a r-dimensional subspace of } \mathbb{R}^n\}.$$

Denote by $\text{St}_r(\mathbb{R}^n)$ the set of r-tuples of linearly independent vectors in $\mathbb{R}^n$. Identifying an element $X \in \text{St}_r(\mathbb{R}^n)$ with a $n \times r$ matrix

$$X = (x^1, ..., x^r) \qquad x^i \in \mathbb{R}^n,$$

shows that $\text{St}_r(\mathbb{R}^n)$ equals the subset of rank r matrices in the vector space $\text{M}_{n\times r}(\mathbb{R})$, which we know from Tutorial 1 is an open subset. Write

$$\pi : \text{St}_r(\mathbb{R}^n) \to \text{Gr}(r, n)$$

for the natural projection given by $\pi(X) = \text{span}(x^1, ..., x^r)$ and equip $\text{Gr}(r, n)$ with the quotient topology with respect to π.

(a) Fix $E \in \text{Gr}(r, n)$ and let $F \subset \mathbb{R}^n$ be a subspace of dimension $n - r$ such that $\mathbb{R}^n = E \oplus F$. Show that

$$U_{(E,F)} = \{W \in \text{Gr}(r, n) : W \cap F = \{0\}\} \subset \text{Gr}(r, n)$$

is an open neighbourhood of E.

(b) Show that any element $W \in U_{(E,F)}$ determines a unique linear map

$$\widetilde{W} : E \to F$$

such that its graph equals W, i.e. $W = \{(x, \widetilde{W}x) : x \in E\}$.

(c) Show that the map $u_{E,F} : U_{(E,F)} \to \text{Hom}(E, F)$ given by $u_{E,F}(W) = \widetilde{W}$ is a homeomorphism.

(d) Show that

$$\mathcal{A} := \{(U_{(E,F)}, u_{(E,F)}) : E, F \subset \mathbb{R}^n \text{ complimentary subspaces of dimension } r \text{ resp. } n-r\}$$

is a smooth atlas for $\text{Gr}(r, n)$.

9. For a topological space M denote by $C^0(M)$ the vector space of continuous real-valued functions $f : M \to \mathbb{R}$. Any continuous map $F : M \to N$ between topological spaces M and N induces a map $F^* : C^0(N) \to C^0(M)$ given by $F^*(f) := f \circ F : M \to \mathbb{R}$.

(a) Show that F^* is linear.

(b) If M and N are (smooth) manifolds, show that $F : M \to N$ is smooth $\iff$ $F^*(C^\infty(N)) \subset C^\infty(M)$.

(c) If F is a homeomorphism between (smooth) manifolds, show that F is a diffeomorphism $\iff$ $F^*(C^\infty(N)) \subset C^\infty(M)$ and $F^* : C^\infty(N) \to C^\infty(M)$ is an isomorphism.

10. We have seen in the first tutorial that $\mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ of dimension $r(n+m-r)$ in. For $X \in \mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ compute the tangent space

$$T_X \mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m) \subset T_X \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m).$$

11. We have seen in the first tutorial that the Grassmannian manifold $\mathrm{Gr}(r, n)$ can be realized as a submanifold of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of dimension $r(n-r)$. For $E \in \mathrm{Gr}(r, n)$ compute the tangent space

$$T_E \mathrm{Gr}(r, n) \subset T_E \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n).$$

12. Consider the general linear group $\mathrm{GL}(n, \mathbb{R})$ and the special linear group $\mathrm{SL}(n, \mathbb{R})$. We have seen that they are submanifolds of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ (even so called Lie groups) and that $T_{\mathrm{Id}}\mathrm{GL}(n, \mathbb{R}) \cong M_n(\mathbb{R}) = \mathbb{R}^{n^2}$.

(a) Compute the tangent space $T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R})$ of $\mathrm{SL}(n, \mathbb{R})$ at the identity Id.

(b) Fix $A \in \mathrm{SL}(n, \mathbb{R})$ and consider the conjugation $\mathrm{conj}_A : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$ by A given by $\mathrm{conj}_A(B) = ABA^{-1}$. Show that conj_A is smooth and compute the derivative $T_{\mathrm{Id}}\mathrm{conj}_A : T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R}) \to T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R})$.

(c) Consider the map $\mathrm{Ad} : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{Hom}(T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R}), T_{\mathrm{Id}}\mathrm{SL}(n, \mathbb{R}))$ given by $\mathrm{Ad}(A) := T_{\mathrm{Id}}\mathrm{conj}_A$. Show that Ad is smooth and compute $T_{\mathrm{Id}}\mathrm{Ad}$.

13. Consider $\mathbb{R}^n$ equipped with the standard inner product of signature (p, q) (where $p + q = n$) given by

$$\langle x, y \rangle := \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_i y_i$$

and the group of linear orthogonal transformation of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ given by

$$\mathrm{O}(p, q) := \{A \in \mathrm{GL}(n, \mathbb{R}) : \langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^n\}.$$

(a) Show that

$$\mathrm{O}(p, q) = \{A \in \mathrm{GL}(n, \mathbb{R}) : A^{-1} = I_{p,q} A^t I_{p,q}\},$$

where $I_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$, and that $\mathrm{O}(p, q)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?

(b) Show that $\mathrm{O}(p,q)$ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$ with respect to matrix multiplication μ and that $\mu : \mathrm{O}(p,q) \times \mathrm{O}(p,q) \to \mathrm{O}(p,q)$ is smooth (i.e. that $\mathrm{O}(p,q)$ is a Lie group.)

(c) Compute the tangent space $T_{\mathrm{Id}}\mathrm{O}(p,q)$ of $\mathrm{O}(p,q)$ at the identity Id.

14. Suppose $M = \mathbb{R}^3$ with standard coordinates (x,y,z). Consider the vector field

$$\xi(x,y,z) = 2\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}.$$

How does this vector field look like in terms of the coordinate vector fields associated to the cylindrical coordinates (r,ϕ,z), where $x = r\cos\phi$, $y = r\sin\phi$ and $z = z$? Or with respect to the spherical coordinates (r,ϕ,θ), where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\cos\phi$ and $z = r\cos\theta$?

15. Consider $\mathbb{R}^3$ with coordinates (x,y,z) and the vector fields

$$\xi(x,y,z) = (x^2-1)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}$$

$$\eta(x,y,z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2xz^2\frac{\partial}{\partial z}.$$

Are they tangent to the cylinder $M = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\} \subset \mathbb{R}^3$ with radius 1 (i.e. do they restrict to vector fields on M)?

16. Suppose $M = \mathbb{R}^2$ with coordinates (x,y). Consider the vector fields $\xi(x,y) = y\frac{\partial}{\partial x}$ and $\eta(x,y) = \frac{x^2}{2}\frac{\partial}{\partial y}$ on M. We computed in class their flows and saw that they are complete. Compute $[\xi,\eta]$ and its flow? Is $[\xi,\eta]$ complete?

17. Let M be a (smooth) manifold and $\xi,\eta \in \mathfrak{X}(M)$ two vector fields on M. Show that

(a) $[\xi,\eta] = 0 \iff (\mathrm{Fl}^{\xi}_t)^*\eta = \eta$, whenever defined $\iff \mathrm{Fl}^{\xi}_t \circ \mathrm{Fl}^{\eta}_s = \mathrm{Fl}^{\eta}_s \circ \mathrm{Fl}^{\xi}_t$, whenever defined.

(b) If N is another manifold, $f : M \to N$ a smooth map, and ξ and η are f-related to vector fields $\tilde{\xi}$ resp. $\tilde{\eta}$ on N, then $[\xi,\eta]$ is f-related to $[\tilde{\xi},\tilde{\eta}]$.

18. Consider the general linear group $\mathrm{GL}(n,\mathbb{R})$. For $A \in \mathrm{GL}(n,\mathbb{R})$ denote by

$$\lambda_A : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}) \qquad \lambda_A(B) = AB$$

$$\rho_A : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}) \qquad \rho_A(B) = BA$$

left respectively right multiplication by A, and by $\mu : \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ the multiplication map.

(a) Show that λ_A and ρ_A are diffeomorphisms for any $A \in \mathrm{GL}(n,\mathbb{R})$ and that

$$T_B\lambda_A(B,X) = (AB,AX) \qquad T_B\rho_A(B,X) = (BA,XA),$$

where $(B,X) \in T_B\mathrm{GL}(n,\mathbb{R}) = \{(B,X) : X \in M_n(\mathbb{R})\}$.

(b) Show that

$$T_{(A,B)}\mu((A,B),(X,Y)) = T_B\lambda_A Y + T_A\rho^B X = (AB, AY + XB)$$

where $(A,B) \in \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})$ and $(X,Y) \in M_n(\mathbb{R}) \times M_n(\mathbb{R})$.

(c) For any $X \in M_n(\mathbb{R}) \cong T_{Id}\mathrm{GL}(n,\mathbb{R})$ consider the maps

$$L_X : \mathrm{GL}(n,\mathbb{R}) \to T\mathrm{GL}(n,\mathbb{R}) \qquad L_X(B) = T_{Id}\lambda_B(Id, X) = (B, BX).$$

$$R_X : \mathrm{GL}(n,\mathbb{R}) \to T\mathrm{GL}(n,\mathbb{R}) \qquad R_X(B) = T_{Id}\rho_B(Id, X) = (B, XB).$$

Show that L_X and R_X are smooth vector field and that $\lambda_A^* L_X = L_X$ and $\rho_A^* R_X = R_X$ for any $A \in \mathrm{GL}(n,\mathbb{R})$. What are their flows? Are these vector fields complete?

(d) Show that $[L_X, R_Y] = 0$ for any $X, Y \in M_n(\mathbb{R})$.

19. Suppose G is a Lie group, i.e. a manifold, which is also a group, where the group multiplication $\mu : G \times G \to G$ is smooth. Denote by $\lambda_g : G \to G$ and $\rho_g : G \to G$ the left resp. right multiplication by $g \in G$, i.e. $\lambda_g(h) = \mu(g,h)$ and $\rho_g(h) = \mu(h,g)$.

(a) Show that the tangent map of μ at $(g,h) \in G \times G$ is given by

$$T_{(g,h)}\mu\,(\xi,\eta) = T_h\lambda_g\eta + T_g\rho_h\xi,$$

where $\xi \in T_gG$ and $\eta \in T_hG$.

(b) Show that the inversion $\iota(g) = g^{-1}$ is smooth and that its tangent map at g is given by

$$T_g\iota = -T_e\rho_{g^{-1}} \circ T_g\lambda_{g^{-1}} = -T_e\lambda_{g^{-1}} \circ T_g\rho_{g^{-1}},$$

where $e \in G$ denotes the neutral element in G. In particular, $T_e\iota = -\mathrm{Id}$.

20. Suppose (G,μ,e) is a Lie group as in the previous example. A vector field $\xi \in \mathfrak{X}(G)$ is called left- resp. right-invariant, if $\lambda_h^*\xi = \xi$ resp. $\rho_h^*\xi = \xi$ for all $h \in G$.

(a) Show that for any $X \in T_eG$, $L_X(g) = T_e\lambda_g X$ and $R_X(g) = T_e\rho_g X$ define a smooth left- resp. right-invariant vector field on G. Moreover, show that $R_X = \iota^*(L_{-X})$.

(b) Show that any left- resp. right-invariant vector field $\xi \in \mathfrak{X}(G)$ is of the form L_X resp. R_X for some $X \in T_eG$.

(c) Show that for any $X \in T_eG$ the vector fields L_X and R_X are complete.

(d) Show that $[L_X, R_Y] = 0$ for any $X, Y \in T_eG$.

21. Suppose α^i_j for $i = 1,...,k$ and $j = 1,...,n$ are smooth real-valued functions defined on some open set $U \subset \mathbb{R}^{n+k}$ satisfying

$$\frac{\partial \alpha^i_j}{\partial x^r} + \sum_{\ell=1}^{k} \alpha^\ell_r \frac{\partial \alpha^i_j}{\partial z^\ell} = \frac{\partial \alpha^i_r}{\partial x^j} + \sum_{\ell=1}^{k} \alpha^\ell_j \frac{\partial \alpha^i_r}{\partial z^\ell},$$

where we write $(x, z) = (x^1, ..., x^n, z^1, ..., z^k)$ for a point in $\mathbb{R}^{n+k}$. Show that for any point $(x_0, z_0) \in U$ there exists an open neighbourhood V of x_0 in $\mathbb{R}^n$ and a unique C^∞-map $f : V \to \mathbb{R}^k$ such that

$$\frac{\partial f^i}{\partial x^j}(x^1, ..., x^n) = \alpha^i_j(x^1, ..., x^n, f^1(x), ..., f^k(x)) \quad \text{and} \quad f(x_0) = z_0.$$

In the class/tutorial we proved this for $k = 1$ and $j = 2$.

22. Which of the following systems of PDEs have solutions $f(x, y)$ (resp. $f(x, y)$ and $g(x, y)$) in an open neighbourhood of the origin for positive values of $f(0, 0)$ (resp. $f(0, 0)$ and $g(0, 0)$)?
    (a) $\frac{\partial f}{\partial x} = f \cos y$ and $\frac{\partial f}{\partial y} = -f \log f \tan y$.
    (b) $\frac{\partial f}{\partial x} = e^{xf}$ and $\frac{\partial f}{\partial y} = xe^{yf}$.
    (c) $\frac{\partial f}{\partial x} = f$ and $\frac{\partial f}{\partial y} = g$; $\frac{\partial g}{\partial x} = g$ and $\frac{\partial g}{\partial y} = f$.

23. Suppose $E \to M$ is a (smooth) vector bundle of rank k over a manifold M. Then E is called *trivializable*, if it isomorphic to the trivial vector bundle $M \times \mathbb{R}^k \to M$.
    (a) Show that $E \to M$ is trivializable $\iff$ $E \to M$ admits a global frame, i.e. there exist (smooth) sections $s_1, ..., s_k$ of E such that $s_1(x)$, ..., $s_k(x)$ span E_x for any $x \in M$.
    (b) Show that the tangent bundle of any Lie group G is trivializable.
    (c) Recall that $\mathbb{R}^n$ has the structure of a (not necessarily associative) normed division algebra over $\mathbb{R}$ for $n = 1, 2, 4, 8$. Use this to show that the tangent bundle of the spheres $S^1 \subset \mathbb{R}^2$, $S^3 \subset \mathbb{R}^4$ and $S^7 \subset \mathbb{R}^8$ is trivializable.

24. Let V be a finite dimensional real vector space and consider the subspace of r-linear alternating maps $\Lambda^r V^* = L^r_{\text{alt}}(V, \mathbb{R})$ of the vector space of r-linear maps $L^r(V, \mathbb{R}) = (V^*)^{\otimes r}$. Show that for $\omega \in L^r(V, \mathbb{R})$ the following are equivalent:
    (a) $\omega \in \Lambda^r V^*$
    (b) For any vectors $v_1, ..., v_r \in V$ one has
    $$\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k)$$
    (c) ω is zero whenever one inserts a vector $v \in V$ twice.
    (d) $\omega(v_1, ..., v_k) = 0$, whenever $v_1, ..., v_k \in V$ are linearly dependent vectors.

25. Let V be a finite dimensional real vector space. Show that the vector space $\Lambda^* V^* := \bigoplus_{r \geq 0} \Lambda^r V^*$ is an associative, unitial, graded-anticommutative algebra with respect to the wedge product $\wedge$, i.e. show that the following holds:
    (a) $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$ for all $\omega, \eta, \zeta \in \Lambda^* V^*$.

(b) $1 \in \mathbb{R} = \Lambda^0 V^*$ satisfies $1 \wedge \omega = \omega \wedge 1 = 1$ for all $\omega \in \Lambda^* V^*$.

(c) $\Lambda^r V^* \wedge \Lambda^s V^* \subset \Lambda^{r+s} V^*$.

(d) $\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega$ for $\omega \in \Lambda^r V^*$ and $\eta \in \Lambda^s V^*$.

Moreover, show that for any linear map $f : V \to W$ the linear map $f^* : \Lambda^* W^* \to \Lambda^* V^*$ is a morphism of graded unitlal algebras, i.e. $f^* 1 = 1$, $f^*(\Lambda^r W^*) \subset \Lambda^r V^*$ and $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

26. Let V be a finite dimensional real vector space. Show that:

(a) If $\omega_1, ..., \omega_r \in V^*$ and $v_1, ..., v_r \in V$, then

$$\omega_1 \wedge ... \wedge \omega_r(v_1, ..., v_r) = \det((\omega_i(v_j))_{1 \leq i,j \leq r}).$$

In particular, $\omega_1, ..., \omega_r$ are linearly independent $\iff \omega_1 \wedge ... \wedge \omega_r \neq 0$.

(b) If $\{\lambda_1, ..., \lambda_n\}$ is a basis of V^*, then

$$\{\lambda_{i_1} \wedge ... \wedge \lambda_{i_r} : 1 \leq i_1 < ... < i_r \leq n\}$$

is a basis of $\Lambda^r V^*$.

27. Let V be a finite dimensional real vector space. An element $\mu \in L^r(V, \mathbb{R})$ is called *symmetric*, if $\mu(v_1, ..., v_r) = \mu(v_{\sigma(1)}, ..., v_{\sigma(r)})$ for any vectors $v_1, ..., v_r \in V$ and any permutation $\sigma \in S^r$. Denote by $S^r V^* \subset \mu \in L^r(V, \mathbb{R})$ the subspace of symmetric elements in the vector space $L^r(V, \mathbb{R})$.

(a) For $\mu \in L^r(V, \mathbb{R})$ show that

$$\mu \in S^r V^* \iff \mu(v_1, ..., v_i, ..., v_j, ..., v_r) = \mu(v_1, ..., v_j, ..., v_i, ..., v_r),$$

for any vectors $v_1, ..., v_r \in V$.

(b) Consider the map $\mathrm{Sym} : L^r(V, \mathbb{R}) \to L^r(V, \mathbb{R})$ given by

$$\mathrm{Sym}(\mu)(v_1, ..., v_r) = \frac{1}{r!} \sum_{\sigma \in S^r} \mu(v_{\sigma(1)}, ..., v_{\sigma(r)}).$$

Show that $\mathrm{Image}(\mathrm{Sym}) = S^r V^*$ and that $\mu \in S^r V^* \iff \mathrm{Sym}(\mu) = \mu$.

28. Let V be a finite dimensional real vector space and set $S(V^*) := \oplus_{r=0}^{\infty} S^r V^*$ with the convention $S^0 V^* = \mathbb{R}$ and $S^1 V^* = V^*$. For $\mu \in S^r V^*$ and $\nu \in S^t V^*$ define their symmetric product by

$$\mu \odot \nu := \mathrm{Sym}(\mu \otimes \nu) \in S^{r+t} V^*.$$

By blinearity, we extend this to a $\mathbb{R}$-bilinear map $\odot : S(V^*) \times S(V^*) \to S(V^*)$. Show that $S(V^*)$ is an unitial, associative, commutative, graded algebra with respect to the symmetric product $\odot$.

29. Suppose $p : E \to M$ and $q : F \to M$ are vector bundles over M. Show that their direct sum $E \oplus F := \sqcup_{x\in M} E_x \oplus F_x \to M$ and their tensor product $E \otimes F := \sqcup_{x\in M} E_x \otimes F_x \to M$ are again vector bundles over M.

30. Suppose $E \subset TM$ is a smooth distribution of rank k on a manifold M of dimension n and denote by $\Omega(M)$ the vector space of differential forms on M.

    (a) Show that locally around any point $x \in M$ there exists (local) 1-forms $\omega^1, ..., \omega^{n-k}$ such that for any (local) vector field ξ one has: ξ is a (local) section of $E \iff \omega_i(\xi) = 0$ for all $i = 1, ..., n-k$.

    (b) Show that E is involutive $\iff$ whenever $\omega^1, ..., \omega^{n-k}$ are local 1-forms as in (a) then there exists local 1-forms $\mu^{i,j}$ for $i, j = 1, ..., n-k$ such that

    $$d\omega^i = \sum_{j=1}^{n-k} \mu^{i,j} \wedge \omega^j.$$

    (c) Show

    $$\Omega_E(M) := \{\omega \in \Omega(M) : \omega|_E = 0\} \subset \Omega(M)$$

    is an ideal of the algebra $(\Omega(M), \wedge)$. Here, $\omega|_E = 0$ for a ℓ-form ω means that $\omega(\xi_1, ..., \xi_\ell) = 0$ for any sections $\xi_1, ...\xi_\ell$ of E.

    (d) An ideal $\mathcal{J}$ of $(\Omega(M), \wedge)$ is called differential ideal, if $d(\mathcal{J}) \subset \mathcal{J}$. Show that $\Omega_E(M)$ is a differential ideal $\iff$ E is involutive.

31. Suppose M is a manifold. Then a graded derivation of the algebra $(\Omega(M), \wedge)$ of degree r is a linear map $D : \Omega(M) \to \Omega(M)$ such that

    - D maps $\Omega^k(M)$ to $\Omega^{k+r}(M)$, and
    - for any $\omega \in \Omega^k(M)$ and any $\eta \in \Omega^\ell(M)$,

    $$D(\omega \wedge \eta) = D(\omega) \wedge \eta + (-1)^{rk} \omega \wedge D(\eta).$$

    In class we have seen that d and $\mathcal{L}_\xi$ for $\xi \in \mathfrak{X}(M)$ are graded derivations of degree 1 respectively 0.

    (a) Show that for two graded derivations D_1 and D_2 of $(\Omega(M), \wedge)$ of degree r_1 respectively r_2,

    $$[D_1, D_2] := D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$$

    is a graded derivation of degree $r_1 + r_2$.

    (b) Suppose D is a graded derivation of $(\Omega(M), \wedge)$. Let $\omega \in \Omega^k(M)$ be a differential form and $U \subset M$ an open subset. Show that $\omega|_U = 0$ implies $D(\omega)|_U = 0$.

    Hint: Think about writing 0 as $f\omega$ for some smooth function f and use the defining properties of a graded derivation.

(c) Suppose D and $\tilde{D}$ are two graded derivations such that $D(f) = \tilde{D}(f)$ and $D(df) = \tilde{D}(df)$ for all $f \in C^\infty(M, \mathbb{R})$. Show that $D = \tilde{D}$.

32. Suppose M is a manifold and $\xi, \eta \in \Gamma(TM)$ vector fields.

    (a) Show that the insertion operator $i_\xi : \Omega^k(M) \to \Omega^{k-1}(M)$ is a graded derivation of degree -1 of $(\Omega(M), \wedge)$.

    (b) Recall from class that $[d, d] = 0$. Verify (the remaining) graded-commutator relations between $d, \mathcal{L}_\xi, i_\eta$:

    (i) $[d, \mathcal{L}_\xi] = 0$.

    (ii) $[d, i_\xi] = d \circ i_\xi + i_\xi \circ d = \mathcal{L}_\xi$.

    (iii) $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]}$.

    (iv) $[\mathcal{L}_\xi, i_\eta] = i_{[\xi,\eta]}$.

    (v) $[i_\xi, i_\eta] = 0$.

    Hint: Use (c) from previous exercise

33. Prove the **Poincaré Lemma**: Suppose $\omega \in \Omega^k(\mathbb{R}^m)$ is a closed k-form, where $k \geq 1$. Show that there exists $\tau \in \Omega^{k-1}(\mathbb{R}^m)$ such that $d\tau = \omega$.

    Hint: Show that for any k-form $\omega = \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ on $\mathbb{R}^m$,

$$P(\omega) = \sum_{\alpha=1}^{k} \sum_{i_1 < \ldots < i_k} (-1)^{\alpha-1} \left[\int_0^1 t^{k-1} \omega_{i_1 \ldots i_k}(tx) dt \right] x^{i_\alpha} dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_\alpha}} \wedge \ldots \wedge dx^{i_k}.$$

    is a $(k-1)$-form on $\mathbb{R}^m$ satisfying

$$\omega = d(P(\omega)) + P(d\omega).$$

    Here, $\widehat{dx^{i_\alpha}}$ means that this term is omitted.

34. Show that for any manifold M its tangent space TM is an orientable manifold.

35. Suppose $M \subset N$ is a submanifold of codimension 1 (i.e. $\dim M = \dim N - 1$) of an oriented manifold N. Suppose there exists a smooth vector field along M that is transverse everywhere to M, that is, a smooth map $\nu : M \to TN$ such that for all $x \in M$ one has

    (i) $\nu(x) \in T_x N$ and

    (ii) $\nu(x)$ and $T_x M$ span $T_x N$.

    Show that M is orientable. Deduce that a hypersurface

$$(M, g) \subset (\mathbb{R}^{m+1}, g) = (\mathbb{R}^{m+1}, g^{\text{euc}})$$

    in Euclidean space is orientable if and only if M admits a globally defined unit normal vector field.

36. Consider $S^m \subset \mathbb{R}^{m+1}$ the unit sphere and the global unit normal vector field $\nu(x) = \sum_{i=1}^{m+1} x^i \frac{\partial}{\partial x^i}$ for S^m. Show that for the nowhere vanishing $m+1$-form

$$\Omega = dx^1 \wedge ... \wedge dx^{m+1}$$

on $\mathbb{R}^{m+1}$,

$$\omega(x) := (i_\nu \Omega)(x) = \Omega(x)(\nu(x), \_\,, ..., \_\,) \text{ for } x \in S^m$$

restricts to a nowhere vanishing m-form on S^m that satisfies

$$A^*\omega = (-1)^{m+1}\omega,$$

where $A : S^m \to S^m$ is the antipodal map $A(x) = -x$.

37. Show that n-dimensional projective space $\mathbb{R}P^m$ is orientable $\iff$ m is odd.

Hint: For $,\Longrightarrow'$ consider the natural projection $\pi : S^m \to \mathbb{R}P^m$, given by $\pi(x) = [x]$, and use the previous exercise. For $,\Longleftarrow'$ construct an oriented atlas.

38. Suppose M and N are connected, compact, oriented manifolds of the same dimension m. Let $f_0, f_1 : M \to N$ be smooth maps that are homotopic to each other, i.e. there exists a smooth map $F : M \times [0,1] \to N$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Show that for any $\omega \in \Omega^m(N)$ one has

$$\int_M f_0^*\omega = \int_M f_1^*\omega.$$

Hint: $M \times [0,1]$ is an oriented manifold with boundary $\partial M = -(M \times \{0\}) \cup M \times \{1\}$, where the minus indicates that the orientation on $M \times \{0\}$ is reversed. Use Stokes' Theorem.

39. Use the previous exercise to show that, if the antipodal map $A : S^m \to S^m$ on the sphere S^m is homotopic to the identity Id_{S^m} on S^m (i.e. there exists a map $F : S^m \times [0,1] \to S^m$ such that $F(x,0) = x$ and $F(x,1) = -x$), then m is odd.

40. Show that on a sphere S^{2m} of even dimension any smooth vector field $\xi \in \mathfrak{X}(S^{2m})$ has a zero.
Hint: Show that if $\xi \in \mathfrak{X}(S^{2m})$ is nowhere vanishing, then there exists a homotopy between the antipodal map and the identity.

41. Suppose $(M,g) \subset (\mathbb{R}^3, g) = (\mathbb{R}^3, g^{\mathrm{euc}}) = (\mathbb{R}^3, < -,->)$ is a surface in Euclidean space. Let $u : U \to u(U)$ be a local chart for M with corresponding local parametrization

$$v = u^{-1} : u(U) \to U.$$

With respect to the frame $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ of $T\mathbb{R}^2$, we can write v^*g and v^*II as matrices

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix},$$

where

$$E = g(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}) \circ v \quad F = g(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}) \circ v \quad G = g(\frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2}) \circ v,$$

and

$$\tilde{E} = II(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}) \circ v \quad \tilde{F} = II(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}) \circ v \quad \tilde{G} = II(\frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2}) \circ v.$$

Compute in terms of $E, F, G, \tilde{E}, \tilde{F}$ and $\tilde{G}$, the Weingarten map $L \circ v$, the Gauß curvature $K \circ v$, the mean curvature $H \circ v$, and the principal curvatures $\kappa_1 \circ v$ and $\kappa_2 \circ v$.

42. Let us write (x^1, x^2, x^3) for the coordinates in $\mathbb{R}^3$. Take a circle of radius $r > 0$ in the (x^1, x^3)-plane and rotate it around a circle of radius $R > r$ in the (x^1, x^2)-plane. The result is a 2-dimensional torus M in $\mathbb{R}^3$. If $I \subset \mathbb{R}$ is an open interval of length $< 2\pi$ the map $v : I \times I \to \mathbb{R}^3$ given by

$$v(\phi, \theta) = ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)$$

defines a local parametrization of M. With respect to v, compute, using the previous exercise, the metric g on M induced by the Euclidean metric on $\mathbb{R}^3$, the 2nd fundamental form, the Gauß and the mean curvature, the principal curvatures and the principal curvature directions of the surface (M, g) in $\mathbb{R}^3$.

Hint: Note that $\nu(\phi, \theta) = (\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta)$ defines a local unit normal vector field for M.

43. Suppose $(M, g) \subset (\mathbb{R}^{m+1}, g) = (\mathbb{R}^{m+1}, g^{\text{euc}})$ is a connected oriented hypersurface in Euclidean space. Show that all points in M are umbilic if and only if M is part of an affine hyperplane or a sphere.

Hint: For $,\Longrightarrow'$ show the following:

- Fix a global unit normal vector field $\nu : M \to \mathbb{R}^{m+1}$. Then, by assumption, for any $x \in M$ there exists $\lambda(x) \in \mathbb{R}$ such that

$$L_x = \lambda(x)\mathrm{Id}_{T_x M}.$$

Since $\lambda = \frac{g(L(\xi),\xi)}{g(\xi,\xi)}$ for any local vector field ξ on M, $\lambda : M \to \mathbb{R}$ is smooth. Show that λ is constant, by, for instance, picking a chart and computing the left-hand-side of $[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] \cdot \nu = 0$.

- If $\lambda = 0$, show that any curve in M is contained in an affine hyperplane with (constant) normal vector ν.
- If $\lambda \neq 0$, show that $f : M \to \mathbb{R}^{m+1}$, given by $f(x) = x - \frac{1}{\lambda}\nu(x)$, is constant.

44. Suppose ∇ is an affine connection on a manifold M.

(a) Show that its curvature, given by,

$$R(\xi,\eta)(\zeta) = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta,$$

for vector fields $\xi, \eta, \zeta \in \mathfrak{X}(M)$ defines a $\binom{1}{3}$-tensor on M.

(b) Show that, if ∇ is torsion-free, the Bianchi identity holds:

$$R(\xi,\eta)(\zeta) + R(\eta,\zeta)(\xi) + R(\zeta,\xi)(\eta) = 0,$$

for any $\xi, \eta, \zeta \in \mathfrak{X}(M)$.

45. Suppose $E \to M$ is a vector bundle over a manifold M equipped with a linear connection ∇, that is, a $\mathbb{R}$-bilinear map

$$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(E) &\to \Gamma(E) \\ (\xi, s) &\mapsto \nabla_\xi s \end{aligned}$$

such that for $\xi \in \Gamma(TM)$, $s \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$ one has

- $\nabla_{f\xi} s = f \nabla_\xi s$
- $\nabla_\xi f s = f \nabla_\xi s + (\xi \cdot f)s.$

(a) Show that $\nabla : \Gamma(TM) \times \Gamma(E^*) \to \Gamma(E^*)$ (typically also denoted by ∇) given by

$$(\nabla_\xi \mu)(s) = \xi \cdot \mu(s) - \mu(\nabla_\xi s), \text{ for } \mu \in \Gamma(E^*), \xi, \in \Gamma(TM), s \in \Gamma(E)$$

defines a linear connection on the dual vector bundle $E^* \to M$.

(b) Suppose $\tilde{E} \to M$ is another vector bundle equipped with a linear connection $\tilde{\nabla}$. Show the vector bundle $E \otimes \tilde{E} \to M$ admits a linear connection characterized by

$$\nabla_\xi (s \otimes \tilde{s}) = \nabla_\xi s \otimes \tilde{s} + s \otimes \tilde{\nabla}_\xi \tilde{s}$$

for $\xi \in \Gamma(TM)$, $s \in \Gamma(E)$ and $\tilde{s} \in \Gamma(\tilde{E})$.

46. Suppose ∇ is an affine connection on a manifold M. Then the previous exercise shows that ∇ induces a linear connection $\nabla : \Gamma(TM) \times \mathcal{T}^p_q(M) \to \mathcal{T}^p_q(M)$ on all tensor bundles. Show that it also induces a linear connection on the bundles $\Lambda^k T^* M$ for $k = 1, \ldots \dim(M)$ characterized by

$$\nabla_\xi (\omega \wedge \mu) = \nabla_\xi \omega \wedge \mu + \omega \wedge \nabla_\xi \mu$$

for $\omega \in \Gamma(\Lambda^k T^* M)$ and $\mu \in \Gamma(\Lambda^\ell T^* M)$ and give a formula.

47. Suppose (M, g) is a Riemannian manifold.

(a) For vector fields $\xi, \eta \in \mathfrak{X}(M)$, let $\nabla_\xi \eta \in \mathfrak{X}(M)$ be the unique vector field such that

$$g(\nabla_\xi \eta, \zeta) = \frac{1}{2}\Big(\xi \cdot g(\eta,\zeta) + \eta \cdot g(\zeta,\xi) - \zeta \cdot g(\xi,\eta) + g([\xi,\eta],\zeta) - g([\xi,\zeta],\eta) - g([\eta,\zeta],\xi)\Big)$$

for all $\zeta \in \mathfrak{X}(M)$. Show that ∇ defines a torsion-free affine connection satisfying

$$\xi \cdot g(\eta,\zeta) = g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta)$$

for $\xi, \eta, \zeta \in \mathfrak{X}(M)$.

(b) The connection ∇ in (a) is called the Levi-Civita connection of (M, g). Show that its curvature satisfies:

- $g(R(\xi,\eta)(\zeta),\mu) = -g(R(\xi,\eta)(\mu),\zeta)$,
- $g(R(\xi,\eta)(\zeta),\mu) = g(R(\zeta,\mu)(\xi),\eta)$,

for $\xi, \eta, \zeta, \mu \in \mathfrak{X}(M)$.

(c) Suppose (U, u) is a chart for M and let R be the Riemann curvature, i.e. the curvature of the Levi-Civita connection of (M, g). Compute

$$R(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j})(\frac{\partial}{\partial u^k})$$

in terms of the Christoffel symbols.

48. Suppose $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is the upper-half plane and equip it with the Riemannian metric

$$g = \frac{1}{y^2} dx \otimes dx + \frac{1}{y^2} dy \otimes dy.$$

(a) Compute the Christoffel symbols of g.

(b) Compute the geodesics of g.

(c) Compute the Riemann curvature.

49. Identifying $H = \{(x, y) \in \mathbb{R}^2 : y > 0\} = \{z = x + iy \in \mathbb{C} : y > 0\}$ in the previous example, we may write g as

$$g = \frac{1}{\text{Im}(z)^2}\text{Re}(dz \otimes d\bar{z}) = \frac{4}{|z - \bar{z}|^2}\text{Re}(dz \otimes d\bar{z}),$$

where $dz = dx + idy$, $d\bar{z} = dx - idy$, Im and Re denote imaginary and real part, and $|\_|$ is the absolute value of complex numbers.

Consider $\text{SL}(2, \mathbb{R}) = \{A \in \text{GL}(2, \mathbb{R}) : \det A = 1\}$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ and $z \in \mathbb{C}$ let

$$f_A(z) = \frac{az + b}{cz + d}.$$

(a) Show that f_A is a diffeomorphism from H to itself for any $A \in \text{SL}(2, \mathbb{R})$ and that $f_{AB} = f_A \circ f_B$.

(b) Show that f_A is an isometry of H.

(c) Show that for any two points $z, z' \in H$ there exists $A \in \mathrm{SL}(2, \mathbb{R})$ such that $f_A(z) = z'$.

(d) Characterize the elements $A \in \mathrm{SL}(2, \mathbb{R})$ such that $f_A(i) = i$.