Remark. *All sets are assumed to be topological spaces and all maps are assumed to be continuous unless stated otherwise. The symbol '=' will denote that two topological spaces are homeomorphic. The closed unit interval will be denoted by I or J.*

Exercise 1. *Prove that being homotopic is an equivalence relation (on the set of continuous maps between topological spaces).*

Solution. Let $f, g, k : X \to Y$ be such that $f \sim g$, $g \sim k$, i.e. there exist maps $h, h' : X \times I \to Y$ such that $h(x, 0) = f(x)$, $h(x, 1) = g(x)$, $h'(x, 0) = g(x)$, $h'(x, 1) = k(x)$.

- Reflexivity: the map $H_1 : X \times I \to Y$ defined by $H_1(x, t) := f(x)$ for all $t \in I$ is a homotopy between f and itself.
- Symmetry: the map $H_2 : X \times I \to Y$ defined by $H_2(x, t) := h(x, 1 - t)$ for all $t \in I$ is a homotopy between g and f.
- Transitivity: the map $H_3 : X \times I \to Y$ defined by $H_3(x, t) := \begin{cases} h(x, 2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ h'(x, 2t) & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$ is a homotopy between f and k.

□

Exercise 2. *Let $\simeq$ be an equivalence relation on a topological space X. Prove that the map $f : X/\simeq \to Y$ is continuous iff $f \circ p : X \to Y$ is continuous, where $p : X \to X/\simeq$ is the canonical quotient projection.*

Solution. The direction "⇒" follows from the facts that p is continuous (in fact, the quotient topology is the final topology with respect to p) and the composition of continuous functions is again continuous. For "⇐", let $U \subseteq Y$ be open. Then

$$p^{-1}(f^{-1}(U)) = (f \circ p)^{-1}(U)$$

is open by continuity of $f \circ p$, so $f^{-1}(U)$ must also be open by the definition of quotient topology and we are done. □

Exercise 3. *Show that $D^n/S^{n-1} = S^n$ using the map $f : D^n \to S^n$ given by*

$$f(x_1, \ldots, x_n) = (2\sqrt{1 - \|\boldsymbol{x}\|}\boldsymbol{x}, 2\|\boldsymbol{x}\|^2 - 1).$$

Solution. It's easy to see that f is continuous. Moreover, its restriction to the interior of D^n gives a bijection to $S^n \setminus \{(0, \ldots, 0, 1)\}$ (the inverse function is given by $(\mathbf{y}, z) \mapsto \frac{1}{\sqrt{\frac{1-z}{2}}}\mathbf{y}$) and we have $f(S^{n-1}) = \{(0, \ldots, 0, 1)\}$, so we can define $f' : D^n/S^{n-1} \to S^n$ by $f'([\mathbf{x}]) = f(\mathbf{x})$. Then f' is a bijection, and by the previous exercise it is continuous. Finally, both D^n/S^{n-1} and S^n are compact (Hausdorff) spaces (since both D^n and S^n are closed bounded subsets of $\mathbb{R}^n$ and R^{n+1}, respectively, and $S^{n-1} \subseteq D^n$ is closed), so f' must be a homeomorphism (a general fact for continuous bijections between compact spaces). □

Exercise 4. *Let* $f : X \to Y$ *and* $M_f = X \times I \cup_{j\times 1} Y$*. Moreover, let* $\iota_X : X \to M_f$ *be given by* $x \mapsto (x, 0)$*,* $\iota_Y : Y \to M_f$ *be given by* $y \mapsto [y]$ *and* $r : M_f \to Y$ *be given by* $r(y) = y$*,* $r(x, t) = f(x)$*. Show that*

i) *Y is a deformation retract of* M_f*,*

ii) $r \circ \iota_X = f$*,*

iii) $\iota_Y \circ f \sim \iota_X$*.*

Solution.

i) Geometrically, the deformation retraction is realized by pushing X along I towards Y.

ii) We have $r \circ \iota_X(x) = r(x, 0) = f(x)$ for all $x \in X$.

iii) The required homotopy $h : X \times J \to M_f$ is given by $h(x, s) = [(x, s)]$.

□

Exercise 5. *Show that the pair* (M_f, X) *has the homotopic extension property (HEP), i.e.* ι_X *is a cofibration.*

Solution. Let $g : I \times J \to \{0\} \times J \cup I \times \{0\}$ be any retraction such that $g(0, s) = (0, s)$ and $g(1, s) = (1, 0)$. Then the map $r : M_f \times J \to X \times \{0\} \times J \cup M_f \times \{0\}$ defined by $r(x, t, s) = (x, g(t, s))$ and $r(y, s) = (y, 0)$ is the required retraction. □

Exercise 6. *The smash product between two based spaces is defined by*

$$(C, c_0) \wedge (D, d_0) := (C \times D)/ (C \times \{d_0\} \cup \{c_0\} \times D) .$$

Show that $X/A \wedge Y/B = (X \times Y)/(X \times B \cup A \times Y)$*.*

Solution. Let $p_1 : X \times Y \to X/A \times Y/B$ be given by $p_1(x, y) = ([x], [y])$ and $p_2 : X/A \times Y/B \to X/A \wedge Y/B$ be given by $p_2(([x], [y])) = ([[x]], [[y]])$. Then the composition $p_2 \circ p_1$ is continuous and factors through $(X \times Y)/(X \times B \cup A \times Y)$, which implies that the canonical bijection between $(X \times Y)/(X \times B \cup A \times Y)$ and $X/A \wedge Y/B$ is continuous (using exercise 2). Using the definition of quotient topology several times, it can be shown that this bijection is also open, hence a homeomorphism. □

Exercise 7. *Let* $A = \{\frac{1}{n} \cup \{0\}\} \subseteq \mathbb{R}$*. Show that* (I, A) *does not have the HEP, i.e. the inclusion* $A \hookrightarrow I$ *is not a cofibration.*

Solution. If $A \times J \cup I \times \{0\}$ was a retract of $I \times J$, the retraction would have to preserve connected subsets. But $I \times J$ is locally connected while $A \times J \cup I \times \{0\}$ is not, a contradiction.

□

Exercise 8. *Show that* $(S^m,*) \wedge (S^n,*) \cong (S^{m+n},*)$.

Solution. Using exercises 3 and 6 from the previous tutorial, we have

$$\begin{aligned}(S^m,*) \wedge (S^n,*) &\cong (D^m/S^{m-1}) \wedge (D^n/S^{n-1}) \cong \\ &\cong (D^m \times D^n)/(S^{m-1} \times D^m \cup D^n \times S^{n-1}) \cong \\ &\cong (I^m \times I^n)/(\partial I^m \times I^n \cup \partial I^n \times I^m) = \\ &= I^{m+n}/(\partial(I^{m+n})) \cong D^{m+n}/\partial D^{m+n} \cong S^{m+n}.\end{aligned}$$

□

Exercise 9. *Show that* $\mathbb{C}P^n$ *is a CW-complex.*

Solution. Clearly $\mathbb{C}P^0$ is a point. Next, we have

$$\begin{aligned}\mathbb{C}P^n &= \mathbb{C}^{n+1} \setminus \{0\}/\{v \sim \lambda v, \lambda \in \mathbb{C} \setminus \{0\}\} \cong S^{2n+1} \setminus \{0\}/\{v \sim \lambda v, |\lambda| = 1\} \cong \\ &\cong \{(w, \sqrt{1-|w^2|}) \in \mathbb{C}^{n+1}, w \in D^{2n}\}/\{w \sim \lambda w \text{ for } |w| = 1\} \cong \\ &\cong (D^{2n} \cup S^{2n-1})/\{w \sim \lambda w \text{ for } w \in S^{2n-1}\} = D^{2n} \cup_f \mathbb{C}.\end{aligned}$$

Now taking the canonical projection $S^{2n-1} \to \mathbb{C}P^{n-1} \cong S^{2n-1}/\sim$ as the attaching map f yields a CW-complex with one cell in every even dimension till $2n$ and none in the odd ones. □

Exercise 10. *From the lecture we know that* $A := \{\frac{1}{n}, n \in \mathbb{N}\} \cup \{0\}$ *as a subspace of* $\mathbb{R}$ *is not a CW-complex. Show that* $X := I \times \{0\} \cup A \times I$ *is not a CW-complex either.*

Solution. Suppose that X is a CW-complex. Then it cannot contain cells of dimension ≥ 2, because it becomes disconnected after removing any point. In fact, the space obtained after removing any point $(a, 0)$ with $a \in A$ has more than two connected components (three for $a > 0$, to be exact), so these points cannot lie inside a 1-cell. Therefore these points must form 0-cells, but we already know that A does not have discrete topology, a contradiction. □

Exercise 11. *Show that the Hawaiian earring given by*

$$X = \{(x,y) \in \mathbb{R}^2, (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2} \textit{ for some } n\}$$

is not a CW-complex.

Solution. Suppose that X is a CW-complex. Using similar arguments as in the previous exercise, we can see that $(0, 0)$ must be a 0-cell and that X must have either infinitely many 0-cells, or infinitely many 1-cells. But since X is compact, exercise 5 implies that X can have only finitely many cells, a contradiction. □

Exercise 12. *Prove that every compact set* A *in a CW-complex* X *can have a nonempty intersection with only finitely many cells.*

Solution. X is comprised of cells that are indexed by elemnts of some set J. Let B be a set containing exactly one point from each intersection $A \cap e^\beta, \beta \in J$. We need to show that B is closed and discrete, which will imply that B is compact (since $B \subseteq A$) and discrete, hence finite. We know that a set $C \subseteq X^n$ is closed iff both $C \cap X^{n-1}$ and $C \cap e^n_\alpha$ for each $\alpha \in J$ are closed, because $D^n \cup_f X^{n-1}$ is a pushout. Using induction, this implies that $C \subseteq X$ is closed iff $C \cap e_\alpha$ is closed for each $\alpha \in J$. Since $B \cap e_\alpha$ contains at most one point for any $\alpha \in J$ and X is T_1 (even Hausdorff), this shows that B is closed. Using the same argument, B with any one point removed is closed. Therefore B is also discrete and we are done. □

Exercise 13. *Show that for a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *of abelian groups (or more generally modules over a commutative ring) the following are equivalent:*

(1) There exists $p : B \to A$ *such that* $pf = \mathrm{id}_A$.

(2) There exists $q : C \to B$ *such that* $gq = \mathrm{id}_C$.

(3) There exist $p : B \to A$ *and* $q : C \to B$ *such that* $fp + qg = \mathrm{id}_B$.

(Another equivalent condition is $B \cong A \oplus C$, *with* (p, g) *and* $f + q$ *being the respective inverse isomorphisms.)*

Solution.
(1) $\implies$ (2) and (3):
Since g is surjective, for any $c \in C$ there is some $b \in B$ such that $g(b) = c$. Moreover, for any other $b' \in B$ such that also $g(b') = c$, we have $b - fp(b) = b' - fp(b')$, since $b - b' \in \ker g = \mathrm{im} f$, so that $b - b' = f(a)$ and

$$fp(b - b') = fpf(a) = f(a) = b - b'.$$

This shows that we can correctly define $q(c) := b - fp(b)$ for any such b. Then we have

$$gq(c) = g(b) - gfp(b) = g(b) = c$$

(since $gf = 0$), which shows that $gq = \mathrm{id}_C$, and also $qg(b) = b - fp(b)$, hence $fp + qg = \mathrm{id}_B$.

(3) $\implies$ (1) and (2):
Applying f from the right to the equation $fp + qg = \mathrm{id}_B$ yields $fpf = f$ (since $gf = 0$), which together with the fact that f is injective implies $pf = \mathrm{id}_A$. Similarly, applying g from the left yields $gqg = g$, which together with the fact that g is surjective implies $gq = \mathrm{id}_C$. □

Exercise 14. *Let* $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$ *be a short exact sequence of chain modules. We have defined the connecting homomorphism* $\partial_* : H_n(C) \to H_{n-1}(A)$ *by the formula* $\partial_*[c] = [a]$, *where* $\partial c = 0$, $f(a) = \partial b$ *and* $g(b) = c$. *Show that this definition does not depend on a nor b.*

Solution. We have $\partial a = 0$ iff $f(\partial a) = 0$ (using injectivity of f) iff $0 = \partial f(a) = \partial\partial b$, and the last condition is true.
Now let $b, b' \in B$ be such that $g(b) = g(b') = c$ with $a, a' \in A$ such that $f(a) = b, f(a') = b'$. Then $b - b' \in \ker g = \mathrm{im} f$, so $b - b' = f(\overline{a})$ for some $\overline{a} \in A$. Therefore $f(\partial\overline{a}) = \partial b - \partial b' = f(a - a')$ and the injectivity of f implies $\partial\overline{a} = a - a'$, hence $[0] = [\partial a'] = [a] - [a']$ and we are done. □

Exercise 15. *Show $\partial\partial = 0$. Use formula $\varepsilon^i_{n+1} \circ \varepsilon^j_n = \varepsilon^{j+1}_{n+1} \circ \varepsilon^i_n$, where $i < j$. The definition for $\sigma \in C_n(X)$, $\sigma\colon \Delta^n \to X$, is*

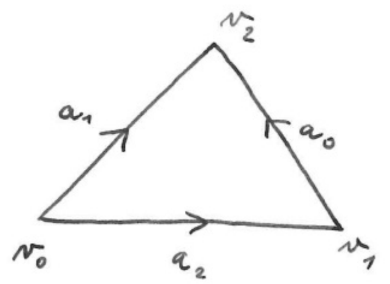
$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \, \sigma \circ \varepsilon^i_n.$$

Solution. Easily workout

$$\begin{aligned}
\partial(\partial\sigma) &= \partial\Big(\sum_{i=0}^{n+1} (-1)^i \, \sigma \circ \varepsilon^i_{n+1}\Big) = \sum_{i=0}^{n+1} (-1)^i \, \partial(\sigma \circ \varepsilon^i_{n+1}) = \\
&= \sum_{i=0}^{n+1} (-1)^i \Big(\sum_{j<i} (-1)^j \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^j_n + \sum_{j>i} (-1)^{j-1} \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^{j-1}_n \Big) = \\
&= \sum_{i=1}^{n+1} \sum_{j<i} (-1)^i (-1)^j \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^j_n + \sum_{i=1}^{n} \sum_{j>i} (-1)^{j+i-1} \sigma \circ \varepsilon^i_{n+1} \circ \varepsilon^{j-1}_n,
\end{aligned}$$

now, with proper reindex and shift, this yields $\varepsilon^i_{n+1} \circ \varepsilon^j_n = \varepsilon^{j+1}_{n+1} \circ \varepsilon^i_n = \varepsilon^j_{n+1} \circ \varepsilon^{i-1}_n$, both sums are of the same elements but with opposite signs. Hence, $\partial\partial = 0$. □

Exercise 16. *Simplicial homology of $\partial\Delta^2$.*



Solution. Chain complex of this simplicial homology is $C_0 = \mathbb{Z}[v_0, v_1, v_2] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, $C_1 = \mathbb{Z}[a_0, a_1, a_2] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. So

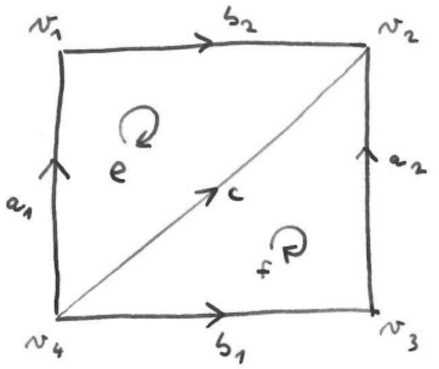
$$0 \to C_1 \xrightarrow{\partial} C_0 \to 0,$$

where we want to determine ∂ and we know $\partial a_0 = v_2 - v_1$, $\partial a_1 = v_2 - v_0$, $\partial a_2 = v_1 - v_0$. Using simple linear algebra, we study generators $\ker \partial$ and $\mathrm{im}\partial$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{array}\right),$$

therefore $\ker \partial$ has a generator $a_0 - a_1 + a_2$ and $\mathrm{im}\partial$ has two generators $-v_1 + v_2$ and $-v_0 + v_2$. We get $H_0 = \dfrac{\mathbb{Z}[v_0, v_1, v_2]}{\mathbb{Z}[-v_1 + v_2, -v_0 + v_2]} = \dfrac{\mathbb{Z}[-v_1 + v_2, v_0 + v_2, v_2]}{\mathbb{Z}[-v_1 + v_2, -v_0 + v_2]} = \mathbb{Z}[v_0] = \mathbb{Z}$ and $H_1 = \ker \partial = \mathbb{Z}[a_0 - a_1 + a_2] = \mathbb{Z}$. □

Exercise 17. *Simplicial complex, model of torus, compute differentials and homology.*



Solution. Again, we get simplicial chain complex C_* formed by free abelian groups generated by equivalence classes of simplicies. Note a_1, a_2 are actually one generator, same for b_1, b_2. All the vertices are also equivalent. We choose the orientation and fix it.

Thus we get $C_0 = \mathbb{Z}[v] = \mathbb{Z}$, $C_1 = \mathbb{Z}[a, b, c] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, $C_2 = \mathbb{Z}[e, f] = \mathbb{Z} \oplus \mathbb{Z}$, $C_3 = 0$, and the following holds: $\partial a = 0, \partial b = 0, \partial c = 0$, as well as $\partial e = a + b - c, \partial f = c - a - b, \partial(e + f) = 0$, so we get $\ker \partial = \mathbb{Z}[e + f], \mathrm{im}\partial = \mathbb{Z}[a + b - c]$.

Let T be the torus. Then

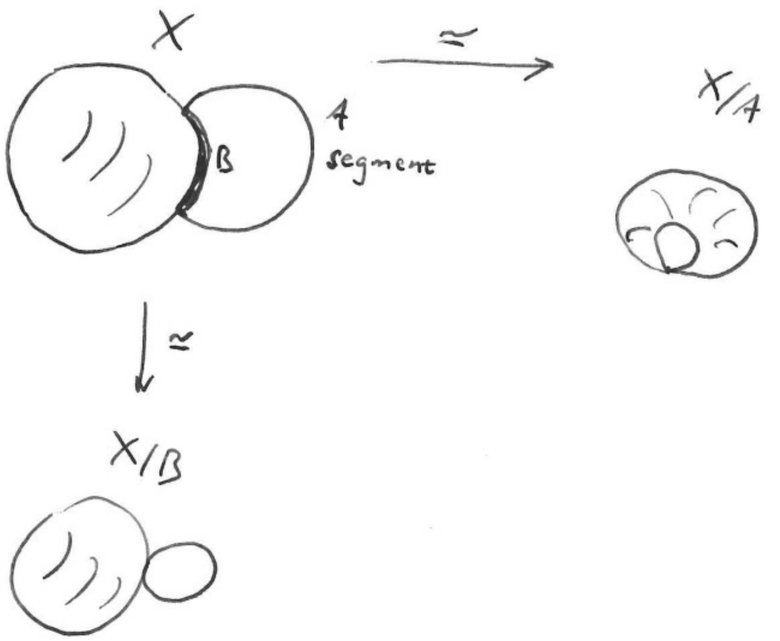
$$\begin{aligned} H_2(T) &= \ker \partial_2 = \mathbb{Z}[e + f] = \mathbb{Z}, \\ H_1(T) &= \mathbb{Z}[a, b, c]/\mathbb{Z}[a + b - c] = \frac{\mathbb{Z}[a, b, a + b - c]}{\mathbb{Z}[a + b - c]} = \mathbb{Z}[a, b] = \mathbb{Z} \oplus \mathbb{Z}, \\ H_0(T) &= \ker \partial_0 = \mathbb{Z}. \end{aligned}$$

□

First criterion of homotopy equivalence (Hatcher)

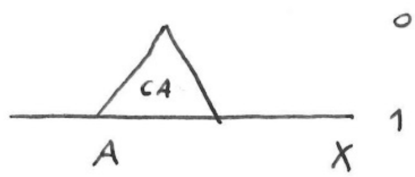
Let (X, A) be a pair that satisfies HEP, i.e. $A \overset{i}{\hookrightarrow} X$ is a cofibration. Let A be contractible in itself. Then $q\colon X \to X/A$ is a homotopy equivalence.

Exercise 18. $S^2 \vee S^1 \simeq S^2/S^0$ *(using First criterion)*



Solution. In the picture (hopefully) above, A is a segment as well as B, so contractible in itself. Clearly $S^2 \vee S^1 = X/B$ and $S^2/S^0 = X/A$ and $X \simeq X/A$ and $X \simeq X/B$ by criterion, therefore $X/A \simeq X/B$ and we are done. □

Exercise 19. *Let $i\colon A \hookrightarrow X$ is a cofibration, show $X/A \simeq X \cup CA = Ci$. (using First criterion)*



Solution. We know $CA \hookrightarrow X \cup CA$ is a cofibration using homework 1, exercise 2, with $Y = CA$. Then by criterion $X \cup CA \simeq X \cup CA/CA$. Also X/A is homeomorphic to $X \cup CA/CA$ (see picture above), which concludes the result.

□

Exercise 20. *Prove the first criterion of homotopy equivalence.*

Solution. We take $h\colon A \times I \to A$, on $A \times \{0\}$ it is identity on A and constant on $A \times \{1\}$.

$$\begin{array}{ccc} X \times I & \xrightarrow{f} & X \\ {\scriptstyle q\times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle q} \\ X/A \times I & \xrightarrow{\bar f} & X/A \end{array}$$

and find $g\colon X/A \to X$. Define $\bar f(x,t) = f(x,t)$, $\bar f([x],t) = [f(x,t)]$. If we define $g\colon X/A \to A, [x] \mapsto f(x,1)$, then it is well defined. Now we want to show, that the compositions are homotopy equivalent to the identities.

$g \circ q \sim \mathrm{id}_X$: $g(q(x)) = g([x]) = f(x,1)$, just the way we defined it, so f is the homotopy, as $f(-,0) = \mathrm{id}_X$ and $f(-,1) = g \circ q$,

$q \circ g \sim \mathrm{id}_{X/A}$: $q(g([x])) = q(f(x,1)) = [f(x,1)] = \bar f([x],1)$ and $\mathrm{id}_{X/A} = \bar f([x],0)$, so in this case the map $\bar f$ is homotopy. □

Exercise 21. *Application of the criterion: two types of suspensions, unreduced and reduced. Unreduced suspension: $SX = X \times I/\sim$, where $(x_1,0) \sim (x_2,0)$, $(x_1,1) \sim (x_2,2)$.*
Reduced suspension: $\Sigma X = SX/\{x_0\} \times I = (X,x_0) \wedge (S^1,s_0)$
(this might be a homework)
The criterion says, that if $\{x_0\} \hookrightarrow X$ is a cofibration, then $SX \simeq \Sigma X$.

$$I \simeq \{(x_0,t), t \in I\} \subseteq SX \longrightarrow SX/\{x_0,t), t \in I\} = \Sigma X$$

Exercise 22. *There is a lemma, that says: Given the following diagram, where rows are long exact sequences and m is iso,*

$$\begin{array}{ccccccccccc} K_n & \xrightarrow{f} & L_n & \xrightarrow{g} & M_n & \xrightarrow{h} & K_{n-1} & \longrightarrow & L_{n-1} & \longrightarrow & M_{n-1} \\ {\scriptstyle k}\downarrow & & {\scriptstyle l}\downarrow & & {\scriptstyle m}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{K}_n & \xrightarrow{\bar f} & \overline{L}_n & \xrightarrow{\bar g} & \overline{M}_n & \xrightarrow{\bar h} & \overline{K}_{n-1} & \longrightarrow & \overline{L}_{n-1} & \longrightarrow & \overline{M}_{n-1} \end{array}$$

we get a long exact sequence

$$K_n \overset{f\oplus k}{\longrightarrow} L_n \oplus \overline{K}_n \overset{l\oplus \bar f}{\longrightarrow} \overline{L}_n \overset{h\circ m^{-1}\circ \bar g}{\longrightarrow} K_{n-1} \longrightarrow \cdots$$

This might be a homework.

Exercise 23. *There is a lemma, that says: Given the following diagram, where rows are long exact sequences and m is an iso*

$$\begin{array}{ccccccccccc} K_n & \xrightarrow{i} & L_n & \xrightarrow{j} & M_n & \xrightarrow{h} & K_{n-1} & \longrightarrow & L_{n-1} & \longrightarrow & M_{n-1} \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle m}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{K}_n & \xrightarrow{\bar i} & \overline{L}_n & \xrightarrow{\bar j} & \overline{M}_n & \longrightarrow & \overline{K}_{n-1} & \longrightarrow & \overline{L}_{n-1} & \longrightarrow & \overline{M}_{n-1} \end{array}$$

we get a long exact sequence

$$K_n \xrightarrow{(i,f)} L_n \oplus \overline{K}_n \xrightarrow{g-\bar{i}} \overline{L}_n \xrightarrow{i \circ iso \circ \bar{j}} K_{n-1} \longrightarrow \cdots$$

We can denote $\partial = h \circ m^{-1} \circ \bar{j}$.
Show exactness in $L_n \oplus \bar{K}_n$ *and also in* $\bar{L}_n$.

Solution. We have $(g - \bar{i}) \circ (i, f) = \bar{i}f - gi = 0$ obviously. For $x \in L_n, y \in \bar{K}_n$ we have $(g - \bar{i})(x, y) = 0$, so $g(x) = \bar{i}(y)$. Now, let x be such that $j(x) = 0$, then there is $z \in K_n$ such that $i(z) = x$. Then, suppose $g(x) = a \in \bar{L}_n$, then by m being iso we know $\bar{j}(a) = 0$, so exists $y \in \bar{K}_n$ such that $\bar{i}(y) = a$. Since $f(z)$ and y have the same image, their difference has a preimage, i.e. exists $b \in \bar{M}_{n+1}$ such that $b \mapsto y - f(z)$. By iso then there exists $c \mapsto z$, or denote $h(c) = z$. Now, all of this is much easier with a picture (that I don't draw). Compute now:
$f(z + c) = f(z) + y - f(z) = y$ and $i(z + h(c)) = i(z) = x$, and we are done.

Exactness in $\bar{L}_n$ is easier. It holds $\partial \circ (g - \bar{i}) = 0$, so take $x \in \ker \partial$ (also, $x \in \bar{L}_n$). Now, $x \mapsto a$, by iso there is b in the upper row that maps to zero. Then there exists y such that $y \mapsto b$. Now we can work with $x - g(y)$. There exists also z such that, obviously, $z \mapsto x - g(y) \mapsto a - a = 0$. Get $x = g(y) + \bar{i}(z) = g(y) - \bar{i}(-z)$, that is we needed to express x as this difference, hence we are done. □

Exercise 24. *There is a long exact sequence of the triple* (X, A, B)*, i.e.* $(B \subseteq A \subseteq X)$*:*

$$\cdots \to H_n(A, B) \xrightarrow{i} H_n(X, B) \xrightarrow{j_X} H_n(X, A) \xrightarrow{D_*} H_{n-1}(A, B) \to \cdots,$$

with $H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{j_A} H_{n-1}(A, B)$. *We get this sequence from a special short exact sequence of chain complexes. Show that it is exact and that the triangle commutes, that is* $D_* = j_A \circ \partial_*$.

Solution. The chain complex of a pair is a quotient. We have

$$0 \longrightarrow \frac{C_*(A)}{C_*(B)} \xrightarrow{i} \frac{C_*(X)}{C_*(B)} \xrightarrow{j} \frac{C_*(X)}{C_*(A)} \to 0.$$

Take $c \in C_*(A)$, then $ji[c] = j[ic] = [jic] = [c]$, but seen as a different class. So, $j \circ i = 0$ and inclusion im $i \subseteq \ker j$ holds. The other inclusion $\ker j \subseteq$ im i is obvious.

Analogous: $0 \to C_*(A) \to C_*(X) \to \dfrac{C_*(X)}{C_*(A)} \to 0$.

Now, show D_* equals the composition:
Note, if you are familiar with the definition, it's clear.
Take a chain complex in $C_*(X)$ with boundary 0, chain in $C_*(A)$, take preimage, boundary in $C_*(B)$, we have $[c] \in C_n(X, B)$ and its image $[c] \in C_n(X, A)$, same representative, but different equivalence. Take $\partial c \in C_{n-1}(X, B)$, it has preimage $[\partial c]_{A,B} \in C_{n-1}(A, B)$. (Drawing diagram and chase helps.) The equality of D_*, that it is composition, holds, basically thanks to j_A being inclusion. □

Exercise 25. *Apply previous exercise to the triple $(D^k, S^{k-1}, *)$, where $*$ is a point.*

Solution.

$$\cdots H_n(S^{k-1}, *) \to H_n(D^k, *) \to H_n(D^k, S^{k-1}) \xrightarrow{\varphi} H_{n-1}(S^{k-1}, *) \to H_{n-1}(D^k, *) \to \cdots,$$

and, note $H_n(*) = \mathbb{Z}$ for $n = 0$ and $H_n(*) = 0$ otherwise.

We also work with reduced homology groups: $\bar{H}_n(X) = H_n(X, x_0)$, $H_*(X) = \bar{H}_*(X) \oplus H_*(*)$. Since D^k is contractible, $H_n(D^k, *) = \bar{H}_n(D^k) = 0$ and $H_{n-1}(D^k, *) = 0$, so we have

$$\cdots H_n(S^{k-1}, *) \to 0 \to H_n(D^k, S^{k-1}) \xrightarrow{\varphi} H_{n-1}(S^{k-1}, *) \to 0 \to \cdots,$$

where φ is *iso*.

Reduced homology for pairs is the same as unreduced:

$$H_n(D^k, S^{k-1}) = \bar{H}_n(D^k, S^{k-1}) \cong \bar{H}_{n-1}(S^{k-1}).$$

With this we know that $H_n(D^k, S^{k-1}) = \bar{H}_n(D^k, S^{k-1}) = \mathbb{Z}$ for $n = k$ and it is 0 for $n \neq k$.

Note also, that $(D^k, S^{k-1}) \cong (\Delta^k, \partial\Delta^k)$, this might be useful later on. □

Exercise 26. *Show that the chain in $C_k(\Delta^k, \partial\Delta^k)$ given by $\mathrm{id}\colon \Delta^k \to \Delta^k$ is the representative of the generator of*

$$H_k(\Delta^k, \partial\Delta^k) \cong \mathbb{Z}.$$

(Use induction and the long exact sequence for triple.)

Solution. First denote $\wedge^{k-1}$ boundary without interior of one face. Then work with the triple $\Delta^k, \partial\Delta^k, \wedge^{k-1}$, so the sequence needed is as follows:

$$0 \to H_k(\Delta^k, \partial\Delta^k) = \mathbb{Z} \to H_{k-1}(\partial\Delta^k, \wedge^{k-1}) \to 0,$$

where we have the zeroes because $\Delta^k, \wedge^{k-1}$ are contractible to points. Use excision theorem, $H_*(X - C, A - C) \cong H_*(X, A)$, where C is the boundary with bottom cut out (imagine upper part of the letter Δ, i.e. triangle). We know that $H_{k-1}(\Delta^{k-1}, \partial\Delta^{k-1}) = \mathbb{Z}$ and this is isomorphic (by excision theorem) to $H_{k-1}(\partial\Delta^k, \wedge^{k-1})$. So, everything is $\mathbb{Z}$. We want to show that images of id_{k-1} and id_k are the same, this is actually the inductive step.

Suppose that the generator in $H_{k-1}(\Delta^{k-1}, \partial\Delta^{k-1})$ is given by $\mathrm{id}_{k-1}\colon \Delta^{k-1} \to \Delta^{k-1}$. Then $\mathrm{id}_k\colon \Delta^k \to \Delta^k$ is cycle again, represents element $[\mathrm{id}_k] \in H_k(\Delta^k, \partial\Delta^k)$.
The Beginning of the Induction *(coming to theaters this summer)*:

$$H_1([-1,1], \{-1,1\}) \xrightarrow{\partial_*} H_0(\{-1,1\}, \{1\}),$$

take $\mathrm{id}\colon [-1,1] \to [-1,1]$ as a chain complex, $1 - (-1) = [-1]$, $[-1]$ generator. □

Exercise 27. *Using the Mayor-Vietoris exact sequence compute the homology groups of the torus. (note: Vietoris died in 2002, aged 110, remarkable)*

Solution. It goes as union, intersection, pair and we will want to determine the union. First draw two disks with holes (these glued together give a torus). Call one interior of the disk A and the other B. We work with $X = A \cup B$, it is not a problem, that we need to work with A, B open, as from the point of view of homology it doesn't matter.

The sequence is

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots,$$

review: $X = A \cup B$ is torus, $A \cap B = S^1 \sqcup S^1$ disjoint union, $H_n(A) = H_n(B) = \mathbb{Z}$ for $n = 0, 1$ and 0 otherwise, $H_n(A \cap B) = H_n(A) \oplus H_n(B) = \mathbb{Z} \oplus \mathbb{Z}$ for $n = 0, 1$ and 0 otherwise. We can therefore continue with this sequence:

$$H_2(A) \oplus H_2(B) \to H_2(X) \to H_1(A \cap B) \xrightarrow{f} H_1(A) \oplus H_1(B) \to H_1(X) \to$$

$$\to H_0(A \cap B) \xrightarrow{g} H_0(A) \oplus H_0(B) \to H_0(X) \to 0,$$

and we can rewrite it as

$$0 \to H_2(X) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \to H_1(X) \to$$

$$\to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

where we use the fact, that torus is connected, so $H_0(X) = \mathbb{Z}$. Now we want to compute $H_2(X)$ and $H_1(X)$.

We know $H_2(X)$ is $\ker f$, $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $(a, b) \mapsto (a + b, a + b)$, then $(a, -a)$ is in the kernel, so $H_2(X) = \mathbb{Z}[a, -a] = \mathbb{Z}$.

For the $H_1(X)$ group use the fact, that $\ker g$ is $\mathbb{Z}$ (it has same idea, basically). Now consider the sequence

$$0 \to \mathbb{Z} \hookrightarrow H_1(X) \to \ker g = \mathbb{Z} \to 0,$$

which splits, so $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$. We are done.

Sphere with two handels might be a homework.

□

Exercise 28. *Prove that S^n has a nonzero vector field if and only if n is odd.*

Solution. First note, that we have $v\colon S^n \to \mathbb{R}^{n+1}$ such that $v(x) \perp x$. Consider the case of S^1 and $(x_0, x_1) \mapsto (x_1, -x_0)$. Take $(x_0, x_1, x_2, x_3, \ldots, x_{2n+1}) \in S^{2n+1} \subseteq \mathbb{R}^{2n}$. Then we get $(x_1, -x_0, x_3, -x_2, \ldots, x_{2n+1}, -x_{2n})$ as image and there is nothing more obvious than that the product is zero, i.e. it's perpendicular.

Note, for $S^1 \subseteq \mathbb{C}$ it is $z \mapsto ez$, where e is the complex unit, usually denoted as i.

Now we want to prove that if S^n has a nonzero vector field, then n is odd. We use the fact that $\deg(\mathrm{id}) = 1$ and $\deg(-\mathrm{id}) = (-1)^{n+1}$. Take $v\colon S^n \to S^n$. If we show $\mathrm{id} \sim -\mathrm{id}$, then $1 = (-1)^{n+1} \Rightarrow n$ is odd. The homotopy is $h(x, t)$ we are looking for is $h(x, t) = cos(t)x + sin(t)v(x)$, where $t \in [0, \pi]$. Also note $\| h(x, t) \| = cos^2 t + sin^2 t = 1$. We are done. □

Exercise 29. *Prove* $\deg(f) = \sum_{i=1}^{k} \deg(f/x_i)$.

Solution. We have $f\colon S^n \to S^n$ and $y \in S^n$ has a neighborhood (nbhd) V. Denote $f^{-1}(y) = \{x_1, \dots, x_k\}$ and U_i nbhd's of x_i, $U_i \cap U_j = \emptyset$. Let degree of f/x_i be n_i. We have $f\colon (U_i, U_i - x_i) \to (V, V-y)$ and $(f/U_i)_*\colon H_n(U_i, U_i - x_i) \cong \mathbb{Z} \to H_n(V, V-y) \cong \mathbb{Z}$, so $z \mapsto n_i z$. Consider the following diagram, its evolution (i.e. drawing order) was revealed in the tutorial class:

$$\begin{array}{ccccc}
 & & \bigoplus H_n(U_i, U_i - x_i) & \xrightarrow{(f/U_i)_*} & H_n(V, V-y) \\
 & \swarrow{\scriptstyle\cong} & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle h} \\
H_n(S^n, S^n - x_i) & & H_n(S^n, S^n - f^{-1}(y)) & \longrightarrow & H_n(S^n, S^n - y) \\
 & \nwarrow{\scriptstyle\cong} & \uparrow{\scriptstyle j_*} & & \uparrow{\scriptstyle j_{y*}} \\
 & & H_n(S^n) & \xrightarrow{f_*} & H_n(S^n)
\end{array}$$

We have j_{y*} iso and h is iso by excision theorem.

So, we have $H_n(S^n, S^n - f(y)) \overset{\cong}{\leftarrow} \bigoplus H_n(U_i, U_i - x_i) \cong H_n(\bigsqcup_1^k (U_i, U_i - x_i))$. Iso between $H_n(S^n, S^n - f(y))$ and $H_n(\bigsqcup_1^k (U_i, U_i - x_i))$ by excision. So k is also iso.

We can take $j_{y*} = h \circ \sum (f/U_i)_* \circ k^{-1} \circ j_*$ and for generator $1 \in H_n(S^n)$ we have $((j_y)_* \circ f_*)(1) = \deg(f) \cdot 1$. We are done.

$$\begin{array}{ccc}
(1, \dots, 1) & \longrightarrow & \sum deg(f/x_i) \\
\uparrow & & \downarrow \\
1 & & \sum \deg(f/x_i)
\end{array}$$

$\square$

Exercise 30. *Compute homology groups of* $\mathbb{RP}^n$ *using CW-structure.*

Solution. Let X be the CW-complex, we have $C_n^{CW} = \bigoplus \mathbb{Z}[e_\alpha^n]$, $H_n(C_*^{CW}, d) = H_n(X)$. Also, $d[e_\alpha^n] = \sum_\beta d_\alpha^\beta [e_\beta^{n-1}]$, where α_α^β is the degree of the map

$$S^{n-1} \xrightarrow{f_\alpha} X^{n-1} \to X^{n-1}/(X^{n-1} - e_\beta^{n-1}) \cong S^{n-1}.$$

We know that $\mathbb{RP}^n = e^0 \cup e^1 \cup \dots \cup e^n$, cell in every dimension. Attaching map is

$$f\colon S^i \to (\mathbb{RP})^{(i)} = \mathbb{RP}^i = S^i/(x \sim -x).$$

Then have

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \dots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ appears $n+1$ times. Now, $S^i \xrightarrow{f} \mathbb{RP}^i \to \mathbb{RP}^i/\mathbb{RP}^{i-1} \cong S^i$, hence we can apply the attaching map. Class $[y]$ has two preimages $y, -y$, there are nbhd's, $deg(f/y) = \pm 1$.

Note that $f/-y = f/y \circ (-\mathrm{id})$. Now we can compute the degree: $\deg(f) = \deg(f/y) + \deg(f/-y) = \pm(1 + (-1)^{i+1})$. It is ± 2 if i is odd and 0 otherwise. So, we can add to the sequence above:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0,$$

and compute for $X = \mathbb{RP}^n$, n even: $H_0(X) = \mathbb{Z}, H_1(X) = \mathbb{Z}_2, H_2(X) = 0, H_i(X) = 0$ for i even, $H_i(X) = \mathbb{Z}_2$ for i odd. The homology is zero for $i > 0$. For n even the only difference is $H_n(X) = \mathbb{Z}$.

(Note: this is because for even they are not orientable, for odd it is oriented space.) □

Exercise 31. *Compute homology groups of oriented two dimensional surfaces. What about nonorientable?*

Solution. Denote M_g surface of genus g (it is the same as sphere with g handles, i.e. M_1 is torus and M_2 is double torus (homework 4). The CW-model is $e^0 \cup e^1_1 \cup \cdots \cup e^1_{2g} \cup e^2$ and we have

$$0 \to \mathbb{Z} \to \bigoplus_1^{2g} \mathbb{Z} \to \mathbb{Z} \to 0.$$

Second differential is zero, $e^0 - e^0 = 0$. The first one is zero as well (glue the model, the arrows go with $+$ and then $-$). Then we get $H_0 = \mathbb{Z}, H_1 = \bigoplus_1^{2g} \mathbb{Z}, H_2 = \mathbb{Z}$.

For nonorientable surfaces, N_g is modelled by one 2-dimensional disc which has boundary composed with g segments every of which repeats twice with the same orientation. So we have one cell in dimensions 2 and 0 and g cells in dimension one. We get (quite similarly)

$$0 \to \mathbb{Z} \to \bigoplus_1^{g} \mathbb{Z} \to \mathbb{Z} \to 0.$$

This equality holds: $d[e^2] = 2[e^1_1] + 2[e^1_2] + \ldots$, so $H_2 = 0, H_0 = \mathbb{Z}$ and the only interesting case is

$$H_1 = \frac{\mathbb{Z}[e^1_1, \ldots, e^1_g]}{\mathbb{Z}[2e^1_1 + \cdots + 2e^1_g]} = \mathbb{Z}_2 + \bigoplus_1^{g-1} \mathbb{Z}.$$

□

Exercise 32. *Have $f\colon S^n \to S^n$ map of degree k. (such map always exists). Let $X = D^{n+1} \cup_f S^n$ and compute homology of X and the projection $p\colon X \to X/S^n$ in homology.*

Solution. Easy, $X = e^0 \cup e^n \cup e^{n+1}$ and $0 \to \mathbb{Z} \xrightarrow{k\cdot} \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0$, also

$$S^n \to X^{(n)} \to X^{(n)}/(X^{(n)} - e^n) = S^n.$$

We get $H_{n+1}(X) = 0, H_n = \mathbb{Z}_k, H_0(X) = \mathbb{Z}$. Note, $X/S^n \cong S^{n+1}$. So, for the p_* we have

$$p_*\colon H_{n+1}(X) = 0 \xrightarrow{0} H_{n+1}(S^{n+1}) = \mathbb{Z}$$

and

$$p_*\colon H_n(X) = \mathbb{Z}_k \xrightarrow{0} H_n(S^{n+1}) = 0$$

and at H_0 it is identity $\mathbb{Z} \to \mathbb{Z}$.

□

Exercise 33. *Prove the following equalities (assuming some conditions):*

$$\bar{H}_*(X \vee Y) = \bar{H}_*(X) \oplus \bar{H}_*(Y)$$
$$\bar{H}_*(\bigvee_{i=1}^{n} X_i) = \bigoplus_{i=1}^{n} \bar{H}_*(X_i)$$
$$\bar{H}_*(\bigvee_{i=1}^{\infty} X_i) = \bigoplus_{i=1}^{\infty} \bar{H}_*(X_i)$$

Solution. Denote z the distinguished point of $X \vee Y$. For the pair $(X \vee Y, X)$ we have the following long exact sequence

$$\cdots \to \bar{H}_{i+1}(X \vee Y, X) \xrightarrow{0} \bar{H}_i(X) \to \bar{H}_i(X \vee Y) \to \bar{H}_i(X \vee Y, X) \xrightarrow{0} \bar{H}_{i-1}(X) \to \cdots .$$

Thus we have short exact sequence, which splits, because we have continuous (cts) map $\mathrm{id} \vee \mathrm{const}_z\colon X \vee Y \to X$ (which maps Y to z). Thus we have $\bar{H}_*(X \vee Y) \cong \bar{H}_*(X) \oplus \bar{H}_*(X \vee Y, X)$. Now it remains to prove $\bar{H}_*(X \vee Y, X) \cong \bar{H}_i(Y)$. If $X \vee Y$ is a CW-complex and X its subcomplex, it is known that $\bar{H}_i(X \vee Y, X) \cong \bar{H}_i(X \vee Y / X) = \bar{H}_i(Y)$. More generally, let U be some (sufficiently small) neighborhood of z in X. From excision theorem we have:

$$\bar{H}_i(X \vee Y, X) \cong \bar{H}_i\big(X \vee Y \smallsetminus (X \smallsetminus U), X \smallsetminus (X \smallsetminus U)\big) = \bar{H}_i(U \vee Y, U).$$

Because U should be[1] contractible, $\bar{H}_i(U \vee Y, U) \cong \bar{H}_i(Y, z) = \bar{H}_i(Y)$.

The second equality we get from the first by induction.

Let us prove the third equality. Denote $Y_n = X_1 \vee X_2 \vee \cdots \vee X_n$ and $Y = \bigvee_{n=1}^{\infty} Y_n$ and denote z the distinguished point of Y and Y_n for every n. We have the following diagram (where each arrow is an inclusion):

$$\begin{array}{ccccc} C_*(Y_1, z) & \longrightarrow & C_*(Y_2, z) & \longrightarrow & \cdots \\ & \searrow & \downarrow & \swarrow & \\ & & C_*(Y, z) & & \end{array}$$

Since Δ^k is compact, every continuous (cts) map $\Delta^k \to Y$ has image in some Y_n, thus it is easy to prove $C_*(Y, z) = \operatorname{colim} C_*(Y_n, z)$, thus

$$\bar{H}_*(Y) = \operatorname{colim} \bar{H}_*(Y_n) = \operatorname{colim} \bigoplus_{i=1}^{n} X_i = \bigoplus_{i=1}^{\infty} X_i.$$

□

[1] It is true at least for X locally contractible. It is not true generally.

Let X be a topological space with finitely generated homological groups and let $H_i(X) = 0$ for each sufficiently large i. Every finitely generated abelian group can be written as $\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k-times} \oplus \operatorname{Tor}$, where Tor denote torsion part of the group. The number k is called the rank *of the group.*

Euler characteristic χ of X is defined by:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} H_i(X)$$

Example. *We know $H_i(S^n) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & \text{otherwise.} \end{cases}$ Thus $\chi(S^n) = 1 - (-1)^n$.*

Exercise 34. *Let (C_*, ∂) be a chain complex with homology $H_*(C_*)$. Prove that $\chi(X) = \chi(C_*)$, where*

$$\chi(C_*) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} C_i.$$

Solution. We have two short exact sequences:

$$0 \to Z_i \hookrightarrow C_i \xrightarrow{\partial} B_{i-1} \to 0$$
$$0 \to B_i \hookrightarrow Z_i \to Z_i/B_i = H_i \to 0,$$

where C_i, cycles Z_i and boundaries B_i are free abelian groups, thus $\operatorname{rank} C_i = \operatorname{rank} Z_i + \operatorname{rank} B_{i-1}$ and $\operatorname{rank} H_i = \operatorname{rank} Z_i - \operatorname{rank} B_i$. Thus we have

$$\begin{aligned} \chi(C_*) &= \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} Z_i + \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} B_{i-1} \\ &= \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} Z_i - \sum_{i=0}^{\infty} (-1)^i \operatorname{rank} B_i = \chi(X). \end{aligned}$$

□

Let X be a topological space with finitely generated homological groups and let $H_i(X) = 0$ for every sufficiently large i. Let $f\colon X \to X$ be a continuous map. Map f induces homomorphism on the chain complex $f_*\colon C_*(X) \to C_*(X)$ and on the homologiy groups $H_*f\colon H_*(X) \to H_*(X)$, where $H_*f(\operatorname{Tor} H_*(X)) \subseteq \operatorname{Tor} H_*(X)$. Thus it induces homomorphism

$$H_*f\colon H_*(X)/\operatorname{Tor} H_*(X) \to H_*(X)/\operatorname{Tor} H_*(X).$$

Since $H_*(X)/\operatorname{Tor} H_*(X) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\operatorname{rank} H_*(X)}$, map H_*f can be written as a matrix, thus we can compute its trace. So we can define the Lefschetz number of a map f:

$$L(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} H_i f.$$

Similarly to the case of the Euler characteristic, it can be proved that[2]

$$\sum_{i=0}^{\infty}(-1)^i \operatorname{tr} H_i f = \sum_{i=0}^{\infty}(-1)^i \operatorname{tr} f_i.$$

Theorem. *If $L(f) \neq 0$, then f has a fixed point.*

Exercise 35. *Use the theorem above to show, that every cts map f on D^n and $\mathbb{R}P^n$ where n is even has a fixed point.*

Solution. We know that that $H_i(D^n) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$ Because $H_0 f\colon H_0(D^n) \cong \mathbb{Z} \to \mathbb{Z} \cong H_0(D^n)$ can be only the identity, we have $L(f) = 1$, thus f has a fixed point.

Since $H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}/2, & i < n, i \text{ odd;} \\ 0, & \text{otherwise,} \end{cases}$ and $\mathbb{Z}/2$ is torsion, we have $L(f) = 1$ as in the previous case. □

Exercise 36. *Let M be a smooth compact manifold. Prove, that there is a nonzero vector field on M if and only if $\chi(M) = 0$.*

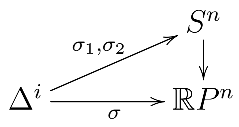
Solution. We will prove only implication $\Rightarrow$. Let v be a nonzero vector field on M. Define a map $X\colon [0,1] \times M \to M$ which satisfies $\dot{X}(t,x) = v(X(t,x))$ for every $x \in M$ and $X(0,x) = x$. There exists t_0 such that $X(t_0,x) \neq x$. Denote $f(x) = X(t_0,x)$, thus f has no fixed point, thus $L(f) = 0$. Because f is homotopic to id and $\operatorname{tr} H_i \mathrm{id} = \operatorname{rank} H_i(M)$, we get from homotopy invariance $0 = L(f) = L(\mathrm{id}) = \chi(M)$. □

Exercise 37. *Use $\mathbb{Z}/2$ coefficients to show, that every cts map $f\colon S^n \to S^n$ satisfying $f(-x) = -f(x)$ has an odd degree.*

Solution. The map f induces a map $g\colon \mathbb{R}P^n \to \mathbb{R}P^n$, since $f(\{x,-x\}) \subseteq \big\{f(x), -f(x)\big\}$. We have the short exact sequence[3]

$$\begin{array}{ccccccccc} & & \sigma & \longmapsto & \sigma_1 + \sigma_2 & \longmapsto & 2\sigma = 0 & & \\ 0 & \to & C_*(\mathbb{R}P^n, \mathbb{Z}/2) & \to & C_*(S^n, \mathbb{Z}/2) & \to & C_*(\mathbb{R}P^n, \mathbb{Z}/2) & \to & 0, \end{array}$$

where $\sigma\colon \Delta^i \to \mathbb{R}P^n$ is an arbitrary element of $C_*(\mathbb{R}P^n)$, σ_1, σ_2 are its preimages of a projection:



[2] $f_i\colon C_i(X) \to C_i(X)$
[3] $2\sigma = 0$ because of the $\mathbb{Z}/2$ coefficient.

From the short exact sequence we get the long exact sequence

$$\begin{CD}
H_i(\mathbb{R}P^n;\mathbb{Z}/2) @>>> H_i(S^n;\mathbb{Z}/2) @>>> H_i(\mathbb{R}P^n;\mathbb{Z}/2) @>>> H_{i-1}(\mathbb{R}P^n;\mathbb{Z}/2) @>>> 0\\
@VV{g_*}V @VV{f_*}V @VV{g_*}V @VV{g_*}V \\
H_i(\mathbb{R}P^n;\mathbb{Z}/2) @>>> H_i(S^n;\mathbb{Z}/2) @>>> H_i(\mathbb{R}P^n;\mathbb{Z}/2) @>>> H_{i-1}(\mathbb{R}P^n;\mathbb{Z}/2) @>>> 0
\end{CD}$$

Because $H_0(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2$ and g_0 on $H_0(\mathbb{R}P^n;\mathbb{Z}/2)$ is an isomorphism, we can show by induction, that $H_i(\mathbb{R}P^n,\mathbb{Z}/2) = \mathbb{Z}/2$ and g_i is an isomorphism for every $i \leq n-1$. An induction step is shown on the following diagram (three isomorphisms imply the fourth):

$$\begin{CD}
0 @>>> \mathbb{Z}/2 @>{\cong}>> \mathbb{Z}/2\\
@. @VV{\cong}V @VV{\cong}V\\
0 @>>> \mathbb{Z}/2 @>>{\cong}> \mathbb{Z}/2
\end{CD}$$

For $i = n$ we have the following situation (the vertical isomorphisms were proved by induction):

$$\begin{CD}
\mathbb{Z}/2 @>{\cong}>> \mathbb{Z}/2 @>{0}>> \mathbb{Z}/2 @>{\cong}>> \mathbb{Z}/2 @>>> 0\\
@VV{\cong}V @VV{?}V @VV{\cong}V @VV{\cong}V\\
\mathbb{Z}/2 @>>{\cong}> \mathbb{Z}/2 @>{0}>> \mathbb{Z}/2 @>>{\cong}> \mathbb{Z}/2 @>>> 0
\end{CD}$$

Thus f_* (the arrow marked by ?) has to be an isomorphism for H_n, thus it maps $[1]_2$ to $[1]_2$, hence f has degree $1 \mod 2$. □

Exercise 38. *Use theorem "$f\colon S^n \to S^n$ s.t. $f(-x) = -f(x) \Rightarrow \deg(f)$ is odd" to prove that for $g\colon S^n \to \mathbb{R}^n$ there exists $x \in S^n\colon g(x) = g(-x)$.*

Solution. We will use proof by contradiction. Suppose that $g(x) - g(-x)$ is always non-zero and define $f(x) = \dfrac{g(x) - g(-x)}{||g(x) - g(-x)||}$. Then obviously $f(-x) = -f(x)$. Since $f\colon S^n \to S^{n-1} \subset S^n$ is odd , the map f has odd degree, but $f(S^n) \subsetneq S^n$ which implies $\deg(f) = 0$ (f is homotopic to a constant map). A contradiction. □

Exercise 39. *Let $\varphi \in C^k(X;R), \psi \in C^l(Y;R)$. Prove $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$. Use $\tau = [e_0, \ldots, e_{k+l+1}] \in C_{k+l+1}(X)$.*

Solution. Easily work out

$$\delta(\varphi \cup \psi)(\tau) = (\varphi \cup \psi)(\delta\tau) = (\varphi \cup \psi)\Big(\sum_{i=0}^{k+l+1} (-1)^i \tau/[e_0, \ldots, \hat{e}_i, \ldots, e_{k+l+1}] \Big) =$$
$$= \sum_{i=0}^{k} (-1)^i \varphi(\tau/[e_0, \ldots, \hat{e}_i, \ldots, e_{k+1}])\psi(\tau/[e_{k+1}, \ldots, e_{k+l+1}]) +$$
$$+ \sum_{i=k+1}^{k+l+1} (-1)^i \varphi(\tau/[e_0, \ldots, e_k])\psi(\tau/[e_k, \ldots, \hat{e}_i, \ldots, e_{k+l+1}]).$$

Now, the right hand side of the formula, the first part gives

$$(\delta\varphi \cup \psi)(\tau) = \delta\varphi(\tau/[e_0, \ldots, e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \ldots, e_{k+l+1}]) =$$
$$= \varphi(\delta\tau/[e_0, \ldots, e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \ldots, e_{k+l+1}]) =$$
$$= \sum_{i=0}^{k+1} (-1)^i \varphi(\delta\tau/[e_0, \ldots, \hat{e}_i, \ldots, e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \ldots, e_{k+l+1}]).$$

The second part is

$$(-1)^k(\varphi \cup \delta\psi)(\tau) = (-1)^k \varphi(\tau/[e_0, \ldots, e_k])\delta\psi(\tau/[e_k, \ldots, e_{k+l+1}]) =$$
$$= \sum_{j=0}^{l+1} (-1)^{j+k} \varphi(\delta\tau/[e_0, \ldots, e_k]) \cdot \psi(\tau/[e_k, \ldots, \hat{e}_{k+j}, \ldots, e_{k+l+1}]).$$

Now, the last summand of the first part plus the first summand of the second part yields

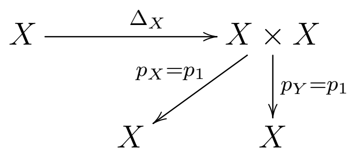
$$(-1)^{k+1}\varphi(\tau/[e_0, \ldots, e_k])\psi(\tau/[e_{k+1}, \ldots, e_{k+l+1}]) +$$
$$+ (-1)^k \varphi(\tau/[e_0, \ldots, e_k])\psi(\tau/[e_{k+1}, \ldots, e_{k+l+1}]) = 0,$$

and we are done, LHS = RHS.

□

Exercise 40. *Prove* $a \cup b = \Delta^*(a \times b)$, *where* $a \in H^k(X), b \in H^l(X), \Delta\colon X \to X \times X, x \mapsto (x, x)$ *and* $\times$ *is cross product defined* $\alpha \times \beta = p_X^*\alpha \cup p_Y^*\beta$ *(*p_X, p_Y *are projections from* $X \times Y$*).*

Solution. By the following diagram



we have that $p_X\Delta$ is id. So, compute

$$\Delta^*(a \times b) = \Delta^*(p_X^* a \cup p_Y^* b) = \Delta^*(p_X^* a) \cup \Delta^*(p_Y^* b) = (p_X\Delta)^* a \cup (p_Y\Delta)^* b,$$

and we are done. The thing is that cup product is natural. □

Exercise 41. *Compute the structure of graded algebra $H^*(S^n \times S^n; \mathbb{Z})$ for n even and n odd. Use the following:*

If $H^n(Y; R)$ is free finitely generated group for all n and $(X, A), Y$ are CW-complexes, then

$$\times \colon H^*(X, A; R) \otimes H^*(Y; R) \to H^*(X \times Y, A \times Y; R)$$

is an isomorphism of graded rings.

Solution. We will omit writing the $\mathbb{Z}$ coefficients.

Now, $H^*(S^n) \otimes H^*(S^n) \to H^*(S^n \times S^n)$ and we know that for spheres $H^0 = \mathbb{Z}$ with generator 1 and $H^n = \mathbb{Z}$, denote generator a. Also, $a \cup a \in H^{2n} = 0, a \cup a = 0$, so we get $\mathbb{Z}[a]/\langle a^2 \rangle$ and $\deg(a) = n$. We can write the same for the second, so denote the other generator b and have $\deg(b) = n$ and we have $\mathbb{Z}[b]/\langle b^2 \rangle$.

Now we compute tensor product $\mathbb{Z}[a]/\langle a^2 \rangle \otimes \mathbb{Z}[b]/\langle b^2 \rangle$, we have four generators: $1_a \otimes 1_b, a \otimes 1_b, 1_a \otimes b, a \otimes b$, we will denote them $1, c, d, c \cdot d$. Compute

$$(a \otimes 1_b) \cdot (1_a \otimes b) = (-1)^{0 \cdot 0} (a \cdot 1_a) \otimes (1_b \cdot b) = a \otimes b,$$

because 0 is an idempotent element, i.e. $0 \cdot 0 = 0$, and $(-1)^n = 1$ for n even, again, as in the first exercise, we use Evenness of Zero. (We refer the reader to "Principia Mathematica" *Whitehead, Russell*,(1910,1912,1913).) Continue with computation

$$(1_a \otimes b) \cdot (a \otimes 1_b) = (-1)^{n \cdot n} (1_a \cdot a) \otimes (b \cdot 1_b) = (-1)^n a \otimes b,$$

so the algebra we get is $H^*(S^n \times S^n) = \mathbb{Z}[c, d]/\langle c^2, d^2, dc - (-1)^n cd \rangle$. For n even we have $dc = cd$. □

Exercise 42. *Prove that there is no multiplication on even dimensional spheres. Multiplication on the sphere S^n is a map $m \colon S^n \times S^n \to S^n$ such that there is an element $1 \in S^n$ satisfying $m(x, 1) = x, m(1, x) = x$.*

Hint: compute $m^* \colon H^*(S^n) \to H^*(S^n \times S^n)$, describe two rings.

Solution. We have $H^*(S^n) = \mathbb{Z}[\gamma]/\langle \gamma^2 \rangle$ and $H^*(S^n \times S^n) = \mathbb{Z}[\alpha, \beta]/\langle \alpha^2, \beta^2 \rangle = H^*(S^n) \otimes H^*(S^n)$, because we already know, that $\alpha\beta = \beta\alpha$. Our situation can be described with two diagrams:

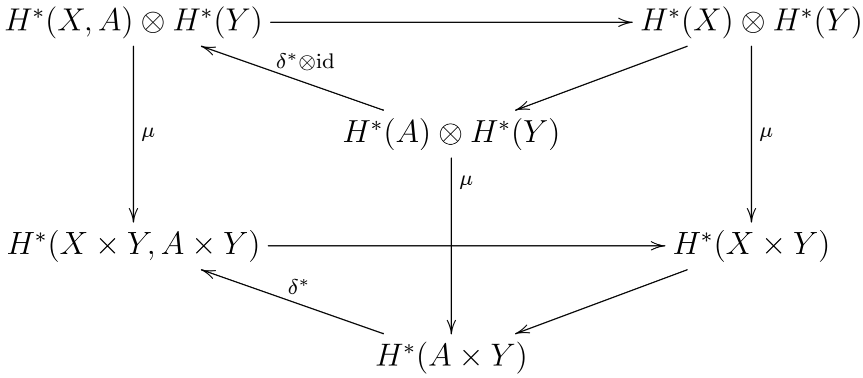
$$\begin{array}{ccc} S^n & \xrightarrow{i_1} & S^n \times S^n \\ & \searrow^{\mathrm{id}} & \downarrow m \\ & & S^n \end{array} \qquad \begin{array}{ccc} H^*(S^n) & \xleftarrow{i_1^*} & H^*(S^n \times S^n) \\ & \nwarrow^{\mathrm{id}} & \uparrow m^* \\ & & H^*(S^n) \end{array}$$

Take $m^*(\gamma) = a\alpha + b\beta$ with $a, b \in \mathbb{Z}$ and prove first, that $a = b = 1$. Use $m \circ i_1 = \mathrm{id}$, so $i_1^*(m^*\gamma) = \gamma$. This gives $i_1^*(a\alpha + b\beta) = \gamma$ and since $i_1^*(a\alpha + b\beta) = a\gamma$, we have $a = 1$, same for b. Final computation yields

$$\begin{aligned} 0 &= m^*(0) = m^*(\gamma^2) = m^*(\gamma \cup \gamma) = m^*\gamma \cup m^*\gamma = \\ &= (\alpha + \beta) \cup (\alpha + \beta) = \alpha^2 + \alpha\beta + \beta\alpha + \beta^2 = 0 + 2\alpha\beta + 0 \neq 0, \end{aligned}$$

and that, my friends, is a contradiction. □

Exercise 43. *(unfinished) Use five lemma to prove that taking any two μ's iso's, the third μ is iso as well. (Proving five lemma might be a homework.) Also, show commutativity of the diagram.*



Solution. We will name the parts of the diagram as follows: back-square, upper-triangle, lower-triangle, left-square, right-square. The triangles come from the long exact sequence of of pairs (X, A) and $(X \times Y, A \times Y)$, the right-square commutativity comes from an inclusion, back-square commutes as well (topology knowledge). The only problematic part is the left-square and it is exercise on computation of connecting homomorphism δ^*. □

Exercise 44 (Five lemma)**.** *Let*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow\cong & & \downarrow\cong & & \downarrow f & & \downarrow\cong & & \downarrow\cong \\ \overline{A} & \longrightarrow & \overline{B} & \longrightarrow & \overline{C} & \longrightarrow & \overline{D} & \longrightarrow & \overline{E} \end{array}$$

be a commutative diagram of modules with exact rows. Show that the middle homomorphism f is an isomorphism.

Solution. We will show the surjectivity of f, injectivity is dual. Let $\overline{c} \in \overline{C}$ be arbitrary and suppose it maps to $\overline{d} \in \overline{D}$. This $\overline{d}$ corresponds to some $d \in D$, and since $\overline{d}$ maps to 0 in $\overline{E}$ by exactness, d also has to map to $0 \in E$ by commutativity. By exactness, there exists $c \in C$ which maps to d. Since $\overline{c}$ and $f(c)$ both map to $\overline{d}$ by commutativity, there is (by exactness) some $\overline{b} \in \overline{B}$ which maps to $f(c) - \overline{c}$. This $\overline{b}$ corresponds to some $b \in B$, which maps to some $c' \in C$. Then by commutativity, $f(c - c') = f(c) - (f(c) - \overline{c}) = \overline{c}$, as desired. (Note that instead of the four vertical maps being isomorphisms, we only needed the surjectivity of $B \to \overline{B}$, $D \to \overline{D}$ and the injectivity of $E \to \overline{E}$). □

Exercise 45. *Show that for a finite CW-complex X and $H^*(Y)$ being finitely generated free group in all dimensions, the cross product*

$$H^*(X) \otimes H^*(Y) \xrightarrow{\mu} H^*(X \times Y)$$

is an isomorphism. (In fact, the same is true for X being an infinite CW-complex.)

Solution. First let $X = pt$ be a point. Then $H^*(pt) = \mathbb{Z}$ with $1 \in H^0(pt)$ and $pt \times Y$ is homeomorphic to Y, hence

$$H^*(pt) \otimes H^*(Y) = \mathbb{Z} \otimes H^*(Y) \cong H^*(Y) \cong H^*(pt \times Y).$$

Now let $X = p_1 \sqcup p_2 \sqcup \cdots \sqcup p_k$ be a finite disjoint union of points (i.e, a discrete set). Then $H^*(X) = \bigotimes_{i=1}^k \mathbb{Z}$, hence

$$\begin{aligned} H^*(X) \otimes H^*(Y) &= \left(\bigoplus_{i=1}^k \mathbb{Z} \right) \otimes H^*(Y) \cong \bigoplus_{i=1}^k (\mathbb{Z} \otimes H^*(Y)) \cong \bigoplus_{i=1}^k H^*(Y) \\ &\cong H^*(\underbrace{Y \sqcup Y \sqcup \cdots \sqcup Y}_{n \text{ times}}) \cong H^*(X \times Y) \end{aligned}$$

(We should also show that the isomorphism is indeed given by μ, but if $e_1, \ldots, e_k \in H^0(X)$ are such that

$$e_i(p_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

it's not hard to see that

$$\mu(e_i \otimes a) = (0, 0, \ldots, \underbrace{a}_{i\text{-th place}}, 0, \ldots, 0)$$

using projections and the definition of the cup product.)

Now we can proceed inductively:

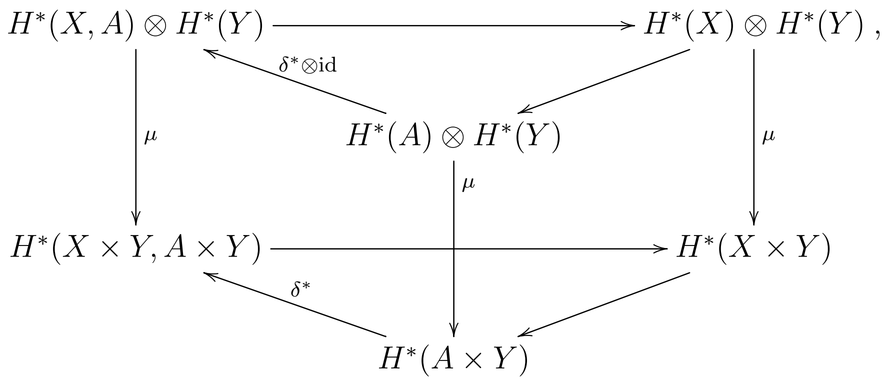
i) We now know that the theorem is true for X of dimension 0.

ii) Since D^n is homotopy equivalent to a point, the theorem is also true for $X = \bigsqcup_{\alpha=1}^k D^n_\alpha$.

iii) Suppose that the theorem holds for finite CW-complexes of dimension $n-1$. Then it is also true for pairs $(\bigsqcup D^n_\alpha, \bigsqcup S^{n-1}_\alpha)$.

iv) Since $H^*(X, A) \cong H^*(X/A)$ for a subcomplex $A \subseteq X$, the theorem also holds for $\bigvee S^n_\alpha \cong \bigsqcup D^n_\alpha / \bigsqcup S^{n-1}_\alpha$.

v) Now let $X = X^{(n)}$ be an n-dimensional CW-complex and consider the diagram



where the lower triangle represents a long exact cohomology sequence by definition, and the same is true for the upper triangle (since free modules are flat). We know that the theorem holds for $X^{(n-1)}$, and also for $X^{(n)}/X^{(n-1)} \cong \bigvee S^{n-1}_\alpha$. Therefore we can use the Five lemma after unfolding the diagram in the appropriate dimensions, from which it follows that the theorem also holds for $X^{(n)}$. This completes the induction.

□

Exercise 46. *Compute the cohomology rings of $\mathbb{C}P^2 \times S^6$ and $\mathbb{C}P^2 \vee S^6$.*

Solution. We have $H^*(\mathbb{C}P^2) = \mathbb{Z}[w]/\langle w^3\rangle$ for $w \in H^2$ and $H^*(S^6) = \mathbb{Z}[a]/\langle a^2\rangle$ for $a \in H^6$, hence

$$H^*(\mathbb{C}P^2 \times S^6) = \mathbb{Z}[w]/\langle w^3\rangle \otimes \mathbb{Z}[a]/\langle a^2\rangle \cong \mathbb{Z}[w,a]/\langle w^3, a^2\rangle.$$

Next, it is true in general that $\overline{H}^*(X \wedge Y) \cong \overline{H}^*(X) \oplus \overline{H}^*(Y)$ is an isomoprhism of graded rings (this can be proven proven straight from the definitions, but it takes some time). Since $\mathbb{C}P^2 \vee S^6$ is connected, we have $H^*(\mathbb{C}P^2 \vee S^6) \cong \overline{H}^*(\mathbb{C}P^2) \oplus \overline{H}^*(S^6) \oplus \mathbb{Z}$. Now $w \cup a \in H^8 = 0$ (more generally, we could use that fact that $(w,0) \cup (0,a) = (0,0)$). Therefore

$$H^*(\mathbb{C}P^2 \vee S^6) \cong \mathbb{Z}[w,a]/\langle w^3, a^2, wa\rangle.$$

□

Exercise 47. *Show that the $\mathbb{C}P^2 \vee S^6$ is not homotopy equivalent to $\mathbb{C}P^3$.*

Solution. It suffices to show that the cohomology rings of these spaces are not isomorphic (note that the additive group structure is not enough to distinguish them). We already know that

$$H^*(\mathbb{C}P^2 \vee S^6) \cong \mathbb{Z}[w,a]/\langle w^3, a^2, wa\rangle$$

and we have $H^*(\mathbb{C}P^3) = \mathbb{Z}[b]/\langle b^4\rangle$ for $b \in H^2$. Any isomorphism would have to map w and b to $\pm$ each other (these are the respective generators in dimension 2), but $w^3 = 0$ while $b^3 \neq 0$, so this is not possible. □

Define the n-th homotopy group of the space X with the base point x_0 as the group of homotopy classes of the cts[4] *maps $(I^n, \partial I^n) \to (X, x_0)$ with the operation given by prescription:*

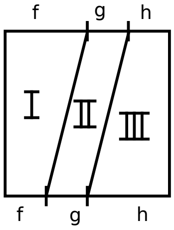
$$(f+g)(t_1,\dots,t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \le t_1 \le \frac{1}{2}, \\ g(2t_1 - 1, t_2, \dots, t_n) & \frac{1}{2} \le t_1 \le 1. \end{cases}$$

Denote it $\pi_n(X, x_0)$.

Exercise 48. *Show the operation on $\pi_n(X, x_0)$ is associative.*

[4] continuous

Solution. We want to show $(f+g)+k \sim f+(g+k)$. We will find prescription for the homotopy by the following diagram:



In the following notation[5], understand $f(t_1)$ as $f(t_1, t_2, \dots, t_n)$ for all $t_2, \dots, t_n \in I$.

$$h(s,t_1) = \begin{cases} f(\frac{4}{1+s}t_1) & (s,t_1) \in \mathrm{I} \quad (t_1 \in [0, \frac{1}{4}+s\frac{1}{4}], s \in [0,1]) \\ g(4t_1 - (1+s)) & (s,t_1) \in \mathrm{II} \quad (t_1 \in [\frac{1}{4}+s\frac{1}{4}, \frac{1}{2}+s\frac{1}{4}], s \in [0,1]) \\ k(\frac{4}{2-s}t_1 - \frac{2+s}{2-s}) & (s,t_1) \in \mathrm{III} \quad (t_1 \in [\frac{1}{2}+s\frac{1}{4}, 1], s \in [0,1]) \end{cases}$$

□

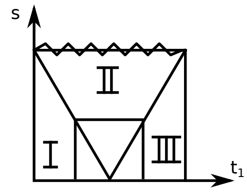
Exercise 49. *Show that the element given by prescription*

$$(-f)(t_1, \dots, t_n) = f(1-t_1, t_2, \dots, t_n)$$

is really the inverse element of f.

Solution. We want to show $f + (-f) \sim const$. The constant will be (function given by) the point $f(2t_1) = f(0)$. Again let us draw a diagram (the square (its boundary) in the middle sign the same value; the wavy line sign one value too).

$$h(s,t_1) = \begin{cases} f(2t_1) & (s,t_1) \in \mathrm{I} \\ f(\frac{1-s}{2}) = (-f)(\frac{1+s}{2}) & (s,t_1) \in \mathrm{II} \\ (-f)(2t_1 - 1) = f(2t_1) & (s,t_1) \in \mathrm{III}, \end{cases}$$



where $\mathrm{II} = \{(s,t_1) \mid s \in [0,1], t_1 \in [\frac{1-s}{2}, \frac{1+s}{2}]\}$. □

Remark. *One can see, that proving by pictures is much more pleasant.*

There is a long exact sequence:

$$\dots \to \pi_{n+1}(X,A,x_0) \xrightarrow{\partial} \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \to \dots$$

Exercise 50. *Show the exactness of this sequence in $\pi_n(X,A,x_0)$ and $\pi_n(A,x_0)$.*

[5] and some other following notations

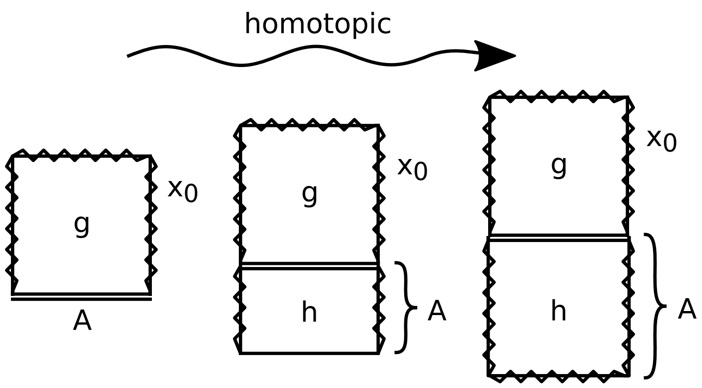
Solution. At first we will show the exactness in $\pi_n(X, A, x_0)$.

Let us show the inclusion "$\mathrm{im} j_* \subseteq \ker \partial$". Take an arbitrary $f \in \pi_n(X, x_0)$, thus $f\colon (I^n, \partial I^n) \to (X, x_0)$. From definition $j_*(f) = j \circ f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, where $J^{n-1} = I^n - I^{n-1}$, and $\partial([f]) = [f|_{I^{n-1}}] = const$, since f is constant on whole $\partial I^n \supseteq I^{n-1}$.

"$\mathrm{im} j_* \supseteq \ker \partial$": Take an arbitrary $g \in \ker \partial \subseteq \pi_n(X, A, x_0)$, thus $g\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$. Since $g \in \ker \partial$, there is the homotopy $h\colon (I^{n-1}, \partial I^{n-1}) \times I \longrightarrow (A, x_0)$ such that $h(x, 0) = g|_{I^{n-1}}(x)$ and $h(x, 1) = const$. Because $h(x, t) \in A$ and $h(x', t) = x_0$ for all $x \in I^{n-1}$, $x \in \partial I^{n-1}$ and $t \in [0, 1]$, we can take $f \in \pi_n(X, x_0)$ defined by

$$f(x, t) = \begin{cases} g(x, 2t) & \text{for } t \in [0, \frac{1}{2}] \\ h(x, 2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

It is not hard to prove that $j_*(f)$ is homotopic to g, see picture below.



Now, let us show the exactness in $\pi_n(A, x_0)$.

"$\mathrm{im}\partial \supseteq \ker i_*$" Let $f \in \ker i_* \subseteq \pi_n(A, x_0)$ be an arbitrary. Because $i_* f \sim const$, we have homotopy $h\colon (I^n, \partial I^n) \times I \to (X, x_0)$ such that $h(x, 0) = f(x)$ and $h(x, 1) = x_0$. It holds $h \in \pi_{n+1}(X, A, x_0)$, since $h(x, 0) \in A$, $h(x, 1) = x_0$ and $h(x', t) = x_0$ for all $x \in I^n$ and $x' \in \partial I^n$.

"$\mathrm{im}\partial \subseteq \ker i_*$" Let $h \in \pi_{n+1}(X, A, x_0)$ be an arbitrary. Denote $h|_{I^n} = f$. Then h gives the homotopy $i_* f \sim const$ in (X, x_0), since $h(x, 0) = f(x)$, $h(x, 1) = x_0$ and $h(x', t) = x_0$ for all $x' \in \partial I^n$ and $x \in I^n$. □

A map $p\colon E \to B$ is called a fibration if it has the homotopy lifting property for all $(D^n, \emptyset)$:

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ D^n \times I & \longrightarrow & B \end{array}$$

If p is a fibration then it has the homotopy lifting property also for all pairs (X, A) of

CW-complexes:

$$\begin{array}{ccc} X \times \{0\} \cup A \times I & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ X \times I & \longrightarrow & B \end{array}$$

Recall that $p\colon E \to B$ is a fiber bundle with fibre F if there are open subsets U_α such that $B = \bigcup_\alpha U_\alpha$ and the following diagram commutes for all U_α:

$$\begin{array}{ccc} U_\alpha \times F & \xrightarrow{\text{homeo.}} & p^{-1}(U_\alpha) \\ & \searrow^{pr_1} & \downarrow p \\ & & U_\alpha \end{array}$$

Exercise 51. *Show that every fibre bundle is a fibration.*

Solution. At first consider a trivial fibre bundle $E = B \times F$. Take an arbitrary commutative diagram of the form:

$$\begin{array}{ccc} D^n \times \{0\} & \xrightarrow{f} & B \times F \\ \downarrow & & \downarrow pr \\ D^n \times I & \xrightarrow{h} & B \end{array}$$

Then $h(-,0) = f$ and we can define $H\colon D^n \times I \to B \times F$ by $H(x,t) = \big(h(x,t), g(x)\big)$. One can see that the diagram commutes with H too.

Now, let $p\colon E \to B$ be an arbitrary fibre bundle with fiber F and $B = \bigcup_\alpha U_\alpha$. We can take I^n instead of D^n and consider a diagram:

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{f} & E \\ \downarrow & & \downarrow p \\ I^n \times I & \xrightarrow{h} & B \end{array}$$

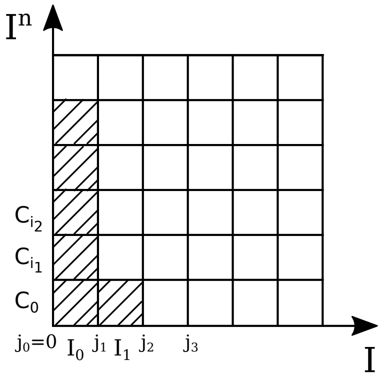
Because I^n is compact, we can divide $I^n \times I$ to finitely many subcubes $C_i \times I_k$ where $I_i = [j_k, j_{k+1}]$ such that $h(C_i \times I_k) \subseteq U_\alpha$ for some α. Since $U_\alpha \times F \to B$ makes a trivial bundle, we can use the same approach as above for each subcube. Since we know $H|_{C_i \times \{0\}} = f|_{C_i \times \{0\}}$, we can find the lift H for all cubes in the first "column" (see the picture below) in the same way as for the trivial case:

$$\begin{array}{ccc} C_i \times \{0\} & \xrightarrow{f} & U_\alpha \times F \\ \downarrow & \overset{H}{\nearrow} & \downarrow pr \\ C_i \times I_0 & \xrightarrow{h} & U_\alpha \end{array}$$

Since we know $H|_{C_i\times\{j_1\}}$ now, we can continue with the second "column":

$$\begin{array}{ccc} C_i \times \{j_1\} & \xrightarrow{f} & U_\alpha \times F \\ \downarrow & \overset{H}{\nearrow} & \downarrow pr \\ C_i \times I_1 & \xrightarrow{h} & U_\alpha \end{array}$$

Thus, we can proceed through all columns in this way until we will get H on the whole $I^n \times I$. The illustration of this situation[6]:



$\square$

Exercise 52. *Show the structure of the fibre bundle $S^n \xrightarrow{p} \mathbb{R}\mathrm{P}^n$.*

Solution. The fibre is $S^0 = \{-1, 1\}$, since $x, -x \longmapsto [x]$. Now, we want to find a neighbourhood U of $[x]$ such that $p^{-1}(U) = U \times S^0$. Set $U = \{[x + v] \mid v \in [x]^\perp\}$ then we have homeomorphism $\varphi\colon U \times S^0 \longrightarrow p^{-1}(U) \subseteq S^n$ given by $\varphi([x + v], 1) = \frac{x+v}{\|x+v\|}$ and $\varphi([x + v], -1) = \frac{-x-v}{\|x+v\|}$. We can cover the whole $\mathbb{R}\mathrm{P}^n$ by the open subsets $U_i = \{(x_0 : x_1 : \cdots : x_n) \mid x_i \neq 0\}$. $\square$

Exercise 53. *Show the structure of the fibre bundle $S^{2n+1} \xrightarrow{p} \mathbb{C}\mathrm{P}^n$ with the fibre S^1.*

Solution. Let us look on the special case $S^1 \hookrightarrow S^3 \to \mathbb{C}\mathrm{P}^1 \cong S^2$ called "Hopf fibration". Realise that we can consider $S^3 \subseteq \mathbb{C}^2$, so we can (locally) define the projection $S^3 \to \mathbb{C}\mathrm{P}^1$ by $(z_1, z_2) \mapsto \frac{z_1}{z_2}$.

In the general case, realise that we can consider $S^{2n+1} \subseteq \mathbb{C}^{n+1}$. Take $U_0 = \{[z_0 : z_1 : \cdots : z_n] \mid z_0 \neq 0\} \subseteq \mathbb{C}\mathrm{P}^n$. We can consider $U_0 = \{[1 : z_1 : \cdots : z_n]\}$. Then the map $U_0 \times S^1 \to S^{2n+1}$ is given by

$$[(1 : z_1 : \cdots : z_n), e^{it}] \longmapsto \frac{(e^{it}, e^{it} z_1, \ldots, e^{it} z_n)}{\|(e^{it}, e^{it} z_1, \ldots, e^{it} z_n)\|}$$

We can do the same for other U_i from the covering of $\mathbb{R}\mathrm{P}^n$. $\square$

[6] it is drawn as planar, but it should be n-dimensional

Exercise 54. *Long exact sequence of the fibration (Hopf) $S^1 \to S^3 \to \mathbb{C}P^1 = S^2$.*

Solution. This is an important example of a fibration, it deserves our attention. First we write long exact sequence

$$\pi_3(S^1) \xrightarrow{i_*} \pi_3(S^3) \xrightarrow{j_*} \pi_3(S^2) \xrightarrow{\partial} \pi_2(S^1) \to \pi_2(S^3) \to \pi_2(S^2) \to$$
$$\to \pi_1(S^1) \to \pi_1(S^3) \to \pi_1(S^2) \to \cdots$$

and also for $\mathbb{Z} \to \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$, that is $\pi_n(\mathbb{Z}, 0) \to \pi_n(\mathbb{R}) \to \pi_n(S^1) \to \pi_{n-1}(\mathbb{Z})$ for $n > 0$. Since $\mathbb{Z}$ is discrete, $(S^n, s_0) \to (\mathbb{Z}, 0), s_0$ goes to a base point 0, therefore the map is constant . Hence we get $\pi_n(\mathbb{Z}, 0) = 0$ for $n \geq 1$. Also, $\pi_n(\mathbb{R})$ is zero as well, because $\mathbb{R}$ is homotopy equivalent to point. We get that this whole sequence are zeroes for $n \geq 2$. What we are left with is

$$0 \to \pi_1(S^1) \to \pi_0(\mathbb{Z}) \to \pi_0(\mathbb{R}) = 0,$$

so $\pi_1(S^1) \cong \mathbb{Z}$. Continue now with the updated long exact sequence:

$$0 \to \pi_3(S^3) \to \pi_3(S^2) \to 0 \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(S^3) \cdots,$$

and $\pi_1(S^3) = 0$, because $\pi_k(S^n) = 0$ for $k < n$ considering the map $S^k \to S^n$ that can be deformed into cellular map and so it is not surjective. So we get

$$\pi_3(S^3) \cong \pi_3(S^2), \quad \pi_2(S^2) \cong \pi_1(S^1) = \mathbb{Z}.$$

It implies that $\pi_2(S^2) \cong \mathbb{Z}$.

Let us remark that if $\pi_3(S^3) = \mathbb{Z}$, then $\pi_3(S^2) = \mathbb{Z}$. Later we will prove that

$$\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z}.$$

□

Remark. *(Story time) Eduard Čech was the first who defined higher homotopy groups (1932, Höherdimensionale Homotopiegruppen) but the community od these groups didn't support the study as they were considered not interesting. Were they mistaken? The rest of this remark is left as an exercise for the reader.*

Remark. *(For geometers) For G Lie group and H its subgroup we have a fibre bundle $H \hookrightarrow G \to G/H$.*

As an example consider otronormal group $O(n)$, we have inclusions $O(1) \subseteq O(2) \subseteq \cdots O(n)$. Then

$$O(n-k) \to O(n) \to O(n)/O(n-k) = V_{n,k}$$

is a fibre bundle, we call $V_{n,k}$ Stiefel manifold, k-tuples of orthonormal vectors in $\mathbb{R}^n$.

Also, we can take $O(k) \to O(n)/O(n-k) \to O(n)/(O(k) \times O(n-k))$, Grassmannian manifold (k-dimensional subspaces in $\mathbb{R}^n$):

$$O(k) \to V_{n,k} \to G_{n,k},$$

is also a fibre bundle.

Exercise 55. *Long exact sequence of the fibration $F \to E \to B$ ends with*

$$\pi_1(B) \to \pi_0(F, x_0) \to \pi_0(E, x_0) \to \pi_0(B).$$

Show exactness in $\pi_0(E)$.

Solution. We showed $\pi_n(B) \cong \pi_n(E, F)$ for $n \geq 0$ which gave us the exactness of the fibration sequence from the exactness of the sequence of the pair (E, F) till $\pi_0(E, x_0)$.

Denote $S^0 = \{-1, 1\}$ and consider the composition

$$(\{-1, 1\}, -1) \to (F, x_0) = p^{-1}(b_0) \to (E, x_0) \to (B, p(x_0) = b_0),$$

where -1 goes to x_0 and b_0 and 1 goes to F but that is $p^{-1}(b_0)$ so we get the constant map.

For the other part of the exactness we will use homotopy lifting property.

Consider $f : (\{-1, 1\}, -1) \to (E, x_0)$ such that pf is homotopic to the constant map into b_0. It means that $pf(1)$ is connected with b_0 by a curve. Have a diagram

$$\begin{array}{ccc} (1) & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ (1) \times I & \longrightarrow & B \end{array}$$

and remark that $f(1)$ is connected by a curve with $x \in E$ such that $p(x) = b_0$. So $x \in F$ and f is homotopic to the map $g : (\{-1, 1\}, -1) \to (F, x_0)$ which maps -1 into x_0 and 1 into x.

□

Exercise 56. *Covering: $G \to X \to X/G$, with action of G on X properly discontinuous, where X is path connected.*

Take

$$\pi_1(G, 1) \to \pi_1(X) \to \pi_1(X/G) \to \pi_0(G) \to \pi_0(X),$$

that is

$$0 \to \pi_1(X) \to \pi_1(X/G) \xrightarrow{\partial} \pi_0(G) \to 0.$$

Show that ∂ is a group homomorphism.

Solution. First note that $\pi_0(G)$ is G taken as a set. We recall that for $F \to E \to B$ we had

$$\begin{array}{ccccc} \pi_n(E) & \xrightarrow{j_*} & \pi_n(E, F, x_0) & \xrightarrow{\bar{\partial}} & \pi_{n-1}(F) \\ & & \cong \big\downarrow p_* & \nearrow \partial & \\ & & \pi_n(B, b_0) & & \end{array}$$

We showed the $\cong$ finding a map going from $\pi_n(B, b_0) \to \pi_n(E, F, x_0)$. Also,

$$f \colon (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0),$$

$\bar{\partial}[F] = [f/\partial I^n]$. The prescription for $\partial\colon \pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$: take $f\colon (I^n, \partial I^n) \to (B, b_0)$ and make a lift $F\colon (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$, $\partial[f] = F/I^{n-1}\colon (I^{n-1}, \partial I^{n-1} \to (F, x_0)$. The diagram is:

$$\begin{array}{ccc} I^{n-1} \times \{1\} \cup \partial I^{n-1} \times I & \xrightarrow{const} & E \quad x_0 \\ \big\downarrow & \overset{F}{\nearrow} & \big\downarrow p \\ I^{n-1} \times I & \longrightarrow & B \quad b_0 \end{array}$$

and $pF(I^{n-1} \times \{0\}) = b_0$, so $F(I^{n-1} \times \{0\}) \subseteq p^{-1}(b_0) = F$.

Now, $\pi_1(X/G) \xrightarrow{\partial} \pi_0(G)$, $f\colon (I, \partial I) \to (X/G, [x_0])$. Take two closed curves ω, τ at $[x_0]$ and lift them to curves $\bar{\omega}, \bar{\tau}$ in X starting in x_0. We denote the end point of $\bar{\omega}$ by $g_1 x_0$ and the end point of $\bar{\tau}$ by $g_2 x_0$. Then the operation $\cdot$ (that is not a composition $\circ$) is:

$$\begin{aligned} \overline{\omega \cdot \tau} &= \bar{\omega} \cdot g_1 \bar{\tau} \\ \overline{\omega \cdot \tau}(0) &= x_0 \\ \overline{\omega \cdot \tau}(1/2) &= g_1 x_0 \\ \overline{\omega \cdot \tau}(1) &= g_2(g_1 x_0) \end{aligned}$$

And it's a homomorphism.

□

Exercise 57. *Van Kampen theorem - Applications.*

Klein bottle K. Model as a square with identified sides as seen in Fig 1,2,3.

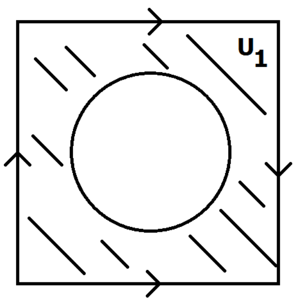


Fig 1

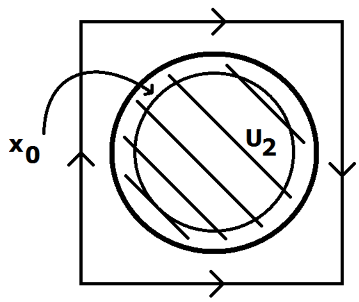


Fig 2

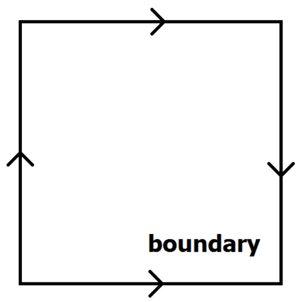


Fig 3

We denote open sets U_1, U_2 (disc) and point x_0 as in figure 1 and 2. (some of the notation in the solution is established in the theorem)

Solution. Set U_1 is homotopy equivalent to the boundary (Fig 3), and this boundary is in fact a wedge of two circles, so $U_1 \simeq S^1 \vee S^1$.

We can compute: $\pi_1(U_2, x_0) = \{1\}$ by contractibility, $\pi_1(U_1, x_0) = \pi_1(S^1 \vee S^1, x_0) =$ free group on two generators α, β as was already shown in lecture.

Then $\pi_1(U_1, x_0) * \pi_1(U_2, x_0) =$ free group on two generators α, β, also $\pi_1(U_1 \cap U_2, x_0) = \mathbb{Z}$.

Now, the intersection $U_1 \cap U_2 \simeq (S^1, x_0)$ and we take generator ω, $i_{2,1*}(\omega^{-1}) = 1$ in $\pi_1(U_2, x_0)$, $i_{1,2*}(\omega) = \alpha\beta\alpha^{-1}\beta$.

So, kernel of φ (the map from the theorem) is generated by element $\alpha\beta\alpha^{-1}\beta$
$\pi_1(K)$ is the group with two generators α, β and one relation $\alpha\beta\alpha^{-1}\beta = 1$ □

Exercise 58. *Recall definitions of n-connectness and n-equivalence. Prove the following lemma: Inclusion $A \hookrightarrow X$ is n-equivalence if and only if (X, A) is n-connected.*

Solution. "$\Leftarrow$" Take long exact sequence:

$$\to \pi_n(A, x_0) \xrightarrow{f_1} \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \xrightarrow{f_2} \pi_{n-1}(X) \to \pi_{n-1}(X, A) \to$$

and use the assumption that $\pi_i(X, A, x_0) = 0$ for $i \leq n$. Then we get that f_1 is epimorphism and f_2 is isomorphism.

"$\Rightarrow$" Reasoning is the same as in the other direction, the only thing we need to realize is $\pi_0(A, x_0) \xrightarrow{\cong} \pi_0(X, x_0)$. □

Exercise 59. *Show $\pi_k(S^\infty) = 0$ for all k, where S^∞ is $colim\, S^n$.*

Solution. We have $S^1 \subset S^2 \subset S^3 \subset \cdots \subset S^n \subset \cdots S^\infty$. Take element in $\pi_k(S^\infty)$, that is $f\colon S^k \to S^\infty$. We know that $f(S^k)$ is compact in S^∞. Consider other CW-complex structure than $e^0 \cup e^k$ for S^k that is $S^k = \bigcup_{i=0}^k e_1^i \cup e_2^i$ (two hemispeheres). Then the following holds: $S^\infty = \bigcup_{i=0}^\infty e_1^i \cup e_2^i$. So $f(S^k) \subseteq (S^\infty)^{(N)} = S^N$ for some N where $(S^\infty)^{(N)}$ is N-skeleton of S^∞. Now,

$$f\colon S^k \to S^N \to S^{N+1} \hookrightarrow S^\infty,$$

so the composition $S^k \to S^{N+1}$ is a map that is not onto and therefore f is homotopic to constant map. (map into a disc is homotopic to constant map, disc is contractible) Then we have $[f] = 0$. Thus we have proved $\pi_k(S^\infty) = 0$. □

Exercise 60. *Compute homotopy groups of $\mathbb{R}P^\infty$.*

Solution. Suprisingly use previous exercise: We can view $\mathbb{R}P^\infty$ as lines going through origin in S^∞, or...just take $S^\infty/_{\mathbb{Z}/2}$, where the action is $x \mapsto -x$. So we work with the following fibration (we don't write the distinguished points as they are not needed) $\mathbb{Z}/2 \to S^\infty \to \mathbb{R}P^\infty$ and the long exact sequence

$$\pi_n(\mathbb{Z}/2) \to \pi_n(S^\infty) \to \pi_n(\mathbb{R}P^\infty) \xrightarrow{\partial} \pi_{n-1}(\mathbb{Z}/2) \to \pi_{n-1}(S^\infty),$$

where for all $n \geq 2$ we have all zeroes, for $n = 1$ consider $0 \to \pi_1(\mathbb{R}P^\infty) \xrightarrow{\partial} \pi_0(\mathbb{Z}/2) \to \pi_0(S^\infty)$. Since $\pi_0(S^\infty) = 0$ and $\pi_0(\mathbb{Z}/2) = \mathbb{Z}/2$, we get that the homomorphism ∂ (it is homomorphism, really, we did it in previous tutorial, but it's still a homomorphism independetly on whether we did it or not) is an isomorphism of groups. By connectness we also know the π_0 group. So the final results are:

$$\pi_n(\mathbb{R}P^\infty) = 0 \text{ for } n \geq 2,\ \pi_1(\mathbb{R}P^\infty) = \mathbb{Z}/2,\ \pi_0(\mathbb{R}P^\infty) = 0.$$

□

Exercise 61. *Show that the spaces $S^2 \times \mathbb{R}P^\infty$ and $\mathbb{R}P^2$ have the same homotopy groups but they are not homotopy equivalent.*

Solution. Here, also use pre²vious exercise. It is known that $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$. With this we can compute

$$\pi_0(S^2 \times \mathbb{R}P^\infty) = 0,\ \pi_1(S^2 \times \mathbb{R}P^\infty) = \pi_1(S^2) \times \pi_1(\mathbb{R}P^\infty) = \{0\} \times \mathbb{Z}/2 \cong \mathbb{Z}/2,$$

$$\text{for } n \geq 2\ \pi_n(S^2 \times \mathbb{R}P^\infty) = \pi_n(S^2) \times \{0\} \cong \pi_n(S^2.$$

Now consider $\mathbb{R}P^2$ as $S^2/_{\mathbb{Z}/2}$, work with the fibration (scheme as follows)

$$\begin{array}{lccccc}
 & \mathbb{Z}/2 \longrightarrow & S^2 \longrightarrow & \mathbb{R}P^2 & & \\
 & \pi_n(\mathbb{Z}/2) \longrightarrow & \pi_n(S^2) \longrightarrow & \pi_n(\mathbb{R}P^2) \longrightarrow & \pi_{n-1}(\mathbb{Z}/2) & \\
n \geq 2 & 0 & \pi_n(S^2) & \cong \pi_n(\mathbb{R}P^2) & 0 & \\
n = 1 & & 0 & \pi_1(\mathbb{R}P^2) & \cong \mathbb{Z}/2 & 0
\end{array}$$

and $\pi_0(\mathbb{R}P^2) = 0$. Thus we showed that these two spaces have the same homotopy groups. Marvelous. How to show, that they are not homotopy equivalent? Use cohomology group! That's right. It is well known (or we should already know) that

$$H^*(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2[\alpha]/\langle \alpha^3 \rangle, \alpha \in H^1 \text{ and}$$

$$H^*(S^2 \times \mathbb{R}P^\infty; \mathbb{Z}/2) = H^*(S^2; \mathbb{Z}/2) \otimes H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[\beta]/\langle \beta^2 \rangle \otimes \mathbb{Z}/2[\gamma], \beta \in H^2.$$

The former space obviously has no non-zero elements of order 4, while the latter has a non-zero element of order 4. This is impossible for homotopy equivalent spaces. We are done.

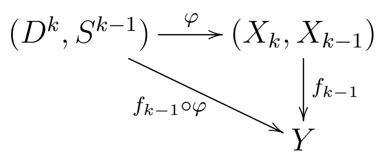
□

Exercise 62. *Extension lemma: Let (X, A) be a pair of CW-complexes, Y a space with $\pi_{n-1}(Y, y_0) = 0$ whenever there is a cell of dimension n in $X - A$. Then every map $f\colon A \to Y$ can be extended to a map $F\colon X \to Y$.*

Solution. Set $X_{-1} = A, X_0 = X^{(0)} \cup A, X_k = X^{(k)} \cup A$, and $f = f_{-1}\colon X_{-1} \to Y, f_0\colon X_0 \to Y$ which extends f_{-1}, $f_0(x_0)$ to any point in Y.

We have $f_{k-1}\colon X_{k-1} \to Y$ and want to define $f_k\colon X_k \to Y$. Also, $\pi_{k-1}(Y, \bullet) = 0$.

Consider following diagram:



The map $f_{k-1} \circ \varphi$ is in $\pi_{k-1}(Y, \bullet)$ so it is homotopic to constant map, so we define f_k on D^k as a constant map. Now,

$$\begin{array}{ccc} D^k \times \{1\} \cup S^{k-1} \times I & \longrightarrow & Y \\ \big\downarrow & \overset{H}{\nearrow} & \\ D^k \times I & & \end{array}$$

and go back to the first diagram:

$$\begin{array}{ccc} (D^k, S^{k-1}) & \xrightarrow{\varphi} & (X_k, X_{k-1}) \\ & \underset{g}{\searrow} & \big\downarrow f_k \\ & & Y \end{array}$$

We extend f_{k-1} to $f_k \colon X_k \to Y$ and proceed to infinity *(and beyond)*, as we always do. $\square$

Exercise 63. *Compare $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$, when distinguished points (are / are not) path connected. Use proof with pillows.*

Solution. First the case where $X = S^1 \sqcup S^2$ and $x_0 \in S^1$ and $x_1 \in S^2$. Then $\pi_1(X, x_0) = \pi_1(S^1) = \mathbb{Z}$ but $\pi_1(X, x_1) = \pi_1(S^2) = 0$. If distinguished points are not path connected, homotopy groups can be different, so consider now ω a curve connecting x_0 and x_1, $\omega \colon I \to X$, $\omega(0) = x_0$ and $\omega(1) = x_1$. We have $\pi_n(X, x_0) \to \pi_n(X, x_1)$ $f \colon I^n \to X$, $\partial I^n \mapsto x_0$, $g \colon I^n \to X$, $\partial I^n \mapsto x_1$, as seen in Figure 1, $x_0 - x_1$ segments. Proofs with pillows!

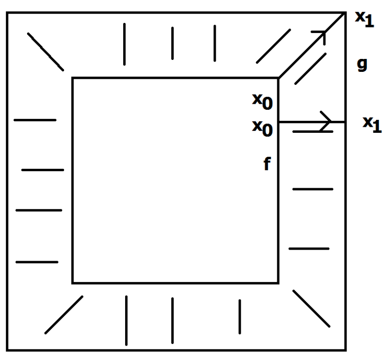


Figure 1

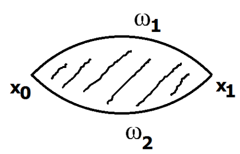


Figure 2

Denote action $f \mapsto \omega \cdot f$, then $f_1 \sim f_2 \Rightarrow \omega \cdot f_1 \sim \omega \cdot f_2$. We are not satisfying algebraist at the moment, only geometers. Let us try to do something about that.

Figure 2 shows other pillow and that $\omega_1 \sim \omega_2 \Rightarrow \omega_1 \cdot f \sim \omega_2 \cdot f$, we can imagine the segment as in Figure 2, two curves.

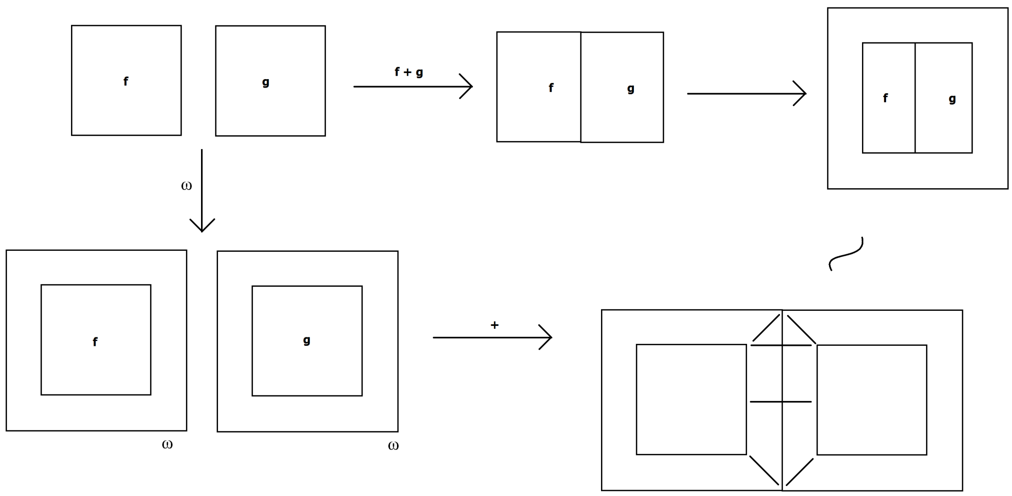


Figure 3

Map given by ω is a bijection $\omega\colon x_0 \to x_1$, $\omega_2(\omega_1 f) \sim (\omega_1\omega_2)f$, $\omega^{-1}(\omega f) \sim (\omega\omega^{-1})f \simeq f$, so the map $\pi_n(X, x_0) \to \pi_n(X, x_1)$ is bijection. Figure 3 tries to explain homomorphism.

We get that x_0, x_1 are in the same path component and if $\omega\colon x_0 \to x_1$ is a curve, then $\pi_n(X, x_0) \to \pi_n(X, x_1)$ is an isomorphism. In particular, if X is simply connected, then every curve gives the same isomorphism. □

Exercise 64. *Using homotopy groups show that $\mathbb{R}P^k$ is not a retract of $\mathbb{R}P^n$, $n > k \geq 1$.*

Solution. Retract: The composition $\mathbb{R}P^k \overset{i}{\hookrightarrow} \mathbb{R}P^n \overset{r}{\to} \mathbb{R}P^k$ is the identity, so we have the following diagram and want to use it to obtain a contradiction.

$$\begin{array}{ccc} \pi_*(\mathbb{R}P^k) & \xrightarrow{\pi_*(i)} & \pi_*(\mathbb{R}P^n) \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \pi_*(r)} \\ & & \pi_*(\mathbb{R}P^k) \end{array}$$

From the long exact sequence of the fibration, we know that $0 = \pi_1(S^k) \to \pi_1(\mathbb{R}P^k) \overset{\cong}{\to} \pi_0(S^0) \to 0$ and $\pi_i(S^k) = \pi_i(\mathbb{R}P^k)$ for $k \geq 2$. Then considering diagram

$$\begin{array}{ccc} \pi_k(S^k) = \mathbb{Z} & \longrightarrow & \pi_k(S^n) = 0 \\ & \searrow & \downarrow \\ & & \pi_k(S^k) = \mathbb{Z} \end{array}$$

we see that we are factoring identity through zero group and that is Mission Impossible: Factor Zero (in theaters never). Continue with $k = 1$. We use the knowledge of $\mathbb{R}P^1 \cong S^1$. Then our diagram is a triangle with $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}$ with identity $\mathbb{Z} \to \mathbb{Z}$, factoring identity through finite group is Mission Impossible: Group protocol (in theaters maybe one day, one can only hope) ...and we are done. □

Exercise 65. *Consider the map $q\colon S^1 \times S^1 \times S^1 \to S^3$ defined as a map $S^1 \times S^1 \times S^1 \to D^3/S^2$ where D^3 is a small disk in the triple torus which is the identity in the interior of D^3 and constant on its complement. Further, consider the Hopf map $p\colon S^3 \to S^2 = \mathbb{C}P^1$ (described in Hopf fibration $S^1 \to S^3 \to S^2$). Compute q_* and $(pq)_*$ in homotopy groups. Show that pq is not homotopic to a constant map.*

Solution. Take following diagram and treat it like your own:

$$\begin{array}{ccccc} & & S^1 & & \text{(fiber)} \\ & & \downarrow & & \\ S^1 \times S^1 \times S^1 & \xrightarrow{q} & S^3 & & \subseteq \mathbb{C}^2 \\ & & \downarrow{\scriptstyle p} & & \\ & & S^2 & & = \mathbb{C}P^1 \end{array}$$

where $p(z_1, z_2) = \dfrac{z_1}{z_2}$. We know that only nontrivial homotopy group of the triple torus is $\pi_1(S^1 \times S^1 \times S^1)$, but $\pi_1 S^3$ is is trivial, so q_* is zero in homotopy groups and the composition as well. By the following diagram we have H, thanks to homotopy lifting property, of course. (denote $(S^1)^3$ the triple torus $S^1 \times S^1 \times S^1$)

$$\begin{array}{ccc} & \nearrow & S^1 = p^{-1}(s_0) \\ & & \downarrow \\ (S^1)^3 & \xrightarrow{q} & S^3 \\ \downarrow & \overset{H}{\nearrow} & \downarrow \\ (S^1)^3 \times I & \xrightarrow[h]{} & (S^2, s_0) \end{array}$$

where $h(0, -)$ is constant map, $h(1, -) = pq$, $p(H(0)) = s_0$,$\operatorname{Im} H(0) \subseteq p^{-1}(s_0) = S^1$ and $H(1) = q \sim H(0)$. However, $H(0)_*$ and q_* differ in the third homology groups:

$$\begin{array}{ccc} & \overset{H(0)_*}{\nearrow} & H_3(S^1) = 0 \\ & & \downarrow \\ H_3((S^1)^3) & \xrightarrow[q_* = id]{} & H_3(S^3) = \mathbb{Z} \end{array}$$

We are trying to factor through zero, a contradiction. □

Exercise 66. *(detail for the lecture 9. 5. 2017) If (X, A) is relative CW-complex such that there are no cells in dimension $\leq n$ in $X \setminus A$, then (X, A) is n-connected.*

Solution. Recall the definition of n-connectness of a pair. For $[f] \in \pi_i(X, A, x_0)$, $i \leq n$, use cell approximation of f: There is a cell map $q\colon (D^i, S^{i-1}, s_0) \to (X, A, x_0)$, such that

$q \sim f$ relatively S^{i-1} and $q(D^i) \subseteq X^{(i)} = A$ since $X^{(-1)} = X^{(i)} = \cdots = X^{(n)} = A$. Note the following very useful criterion:

$[f] = 0$ in $\pi_i(X, A, x_0) \iff f \sim q$ relatively S^{i-1}, $g(D^i) = A$.

Thus $[f] = 0$ in our case, and we are done. □

Exercise 67. *Let $[X, Y]$ denote a set of homotopy classes of maps from X to Y. If (X, x_0) is a CW-complex and Y is path connected, then $[X, Y] \cong [(X, x_0), (Y, y_0)]$.*

Solution. Surely, $[(X, x_0), (Y, y_0)] \subseteq [X, Y]$ and denote the class in the left set as (g) and $[g]$ the class in the $[X, Y]$. The map $(g) \mapsto [g]$ is well defined and injective. To prove that it is also surjective take $[f] \in [X, Y]$. Using HEP for $f : X \times 0 \to Y$ and a curve $\omega : x_0 \times I \to Y$ which connects $f(x_0)$ with y_0 we get $g : X \times 1 \to Y$ such that $f \sim g$ and $g(x_0) = y_0$, then $[f] = [g]$ and $(g) \in [(X, x_0), (Y, y_0)]$. We are done. □

Exercise 68. *(application) We know that deg(f) is an invariant of $[S^n, S^n] = \pi_n(S^n)$. Study $[S^{2n-1}, S^n] \cong \pi_{2n-1}(S^n)$ and describe its co called Hopf invariant $H(f)$.*

Solution. Have $f \colon \partial D^{2n} = S^{2n-1} \to S^n$ and $S^n \cup_f D^{2n}$. For $f \sim g$ we have $S^n \cup_f D^{2n} \simeq S^n \cup_g D^{2n}$, moreover $S^n \cup_f D^{2n} = C_f$ (the cylinder of f). For $n \geq 2$ we have $C_f = e^0 \cup e^n \cup e^{2n}$. Using cohomology: $H^*(C_f) = \mathbb{Z}$ for $* \in \{0, n, 2n\}$ and 0 elsewhere. Take $\alpha \in H^n(C_f)$ generator, we have cup product. Then $\alpha \cup \alpha \in H^{2n}(C_f)$ and for $\beta \in H^{2n}(C_f)$ we have $\alpha \cup \alpha = H(f)\beta$, where $H(f)$ is the Hopf invariant. □

Exercise 69. *(continuation of previous exercise) For n odd, what can we say in this case about Hopf inviariant? And for n even? Thanks.*

Solution. Knowing $\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$ we see that $\alpha \cup \alpha = 0$. So for n odd Hopf invariant is zero.

For n even consider the Hopf fibration $S^1 \to S^3 \to S^2 = \mathbb{C}P^1$. For $\mathbb{C}P^2 = D^4 \cup_f \mathbb{C}P^1$ (recall how $\mathbb{C}P^n$ is built up from $\mathbb{C}P^{n-1}$) we have $C_f = \mathbb{C}P^2$ and $H^*(\mathbb{C}P^2) = \mathbb{Z}[\alpha]/\langle \alpha^3 \rangle$, with $\alpha \in H^2$. The generator of H^4 is α^2. We get that $H(f) = 1$. □

Exercise 70. *Let (X, A) be a pair and show that the following diagram commutes:*

$$\begin{array}{ccc} \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_n(A, x_0) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_n(X, A) & \xrightarrow[\partial_*]{} & H_{n-1}(A) \end{array}$$

where ∂ is boundary homomorphism, h is Hurewicz homomorphism and ∂_* is the connecting homomorphism.

Solution. Take $[f] \in \pi_n(X, A, x_0)$, that is $f \colon (D^n, D^{n-1}, s_0) \to (X, A, x_0)$. Then $\partial[f] = [f/S^{n-1}]$ and $h\partial[f] = h[f/S^{n-1}] = (f/S^{n-1})_*(b)$, where b is generator in $H_{n-1}(S^{n-1})$, as we recall in the following: $g \colon S^{n-1} \to A, g_* \colon H_{n-1}(S^{n-1}) \to H_{n-1}(A)$ and $h[g] = g_*(b) \in$

$H_{n-1}(A)$. Let $a \in H_n(D^n, S^{n-1})$ be a generator, that is $\partial_* a = b$. We proceed with the following diagram:

$$\begin{array}{ccc} H_n(D^n, S^{n-1}) & \xrightarrow[\cong]{\partial_*} & H_{n-1}(S^{n-1}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (f/S^{n-1})_*} \\ H_n(X, A) & \xrightarrow[\partial_*]{} & H_{n-1}(A) \end{array}$$

So, now $\partial_* h[f] = \partial(f_* a) = (f/S^{n-1})_*(\partial_* a)$. With $\partial_* a = b$ we conclude that the diagram commutes. □

Exercise 71. *Homology version of Whitehead theorem. (recall both versions) We show that $f_* \colon \pi_n(X) \to \pi_n(Y)$ is an iso for all n and then use the previous version of the theorem.*

Solution. First, for $f \colon X \hookrightarrow Y$ and inclusion. For pair (X, A) we have the following long exact sequence (with Hurewicz):

$$\begin{array}{ccccccc} \pi_{n+1}(Y, X) & \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(Y, X) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ H_{n+1}(Y, X) & \xrightarrow[0]{} & H_n(Y) & \xrightarrow[\cong]{} & H_n(X) & \xrightarrow[0]{} & H_n(Y, X) \end{array}$$

The thing is that h is not always *iso*, we have $\xrightarrow{0} \pi_1(Y, X) \xrightarrow{0}$, for Y, X simply connected. Now, for $\pi_2 \xrightarrow{h} H_2$ is iso by Hurewicz theorem, then $H_2 = 0$ so $\pi_2 = 0$ and with the same argument we can proceed inductively.

Now, for general map $f \colon X \to Y$ we use the argument with mapping cylinder M_f. Do you remember this one from the first lecture? This construction you thought would not be interesting? Well, hold your hats, its usage is advantageous here!

$$\begin{array}{ccccc} & & X & & \\ & \swarrow{\scriptstyle f} & \downarrow{\scriptstyle i_X} & \searrow{\scriptstyle f} & \\ Y & \xrightarrow[i_Y]{} & M_f & \xleftarrow[r]{} & Y \end{array}$$

We know that $f_* \colon H_n(X) \to H_n(Y)$ is an iso and (as i_Y is homotopy equivalence) $i_{Y*} \colon H_n(Y) \to H_n(M_f)$ is an iso. Thus, we conclude that $i_{X*} \colon H_n(X) \to H_n(M_f)$ is an iso. That is, i_* is homotopy equivalence and therefore f is also homotopy equivalence. (Let us remark that for this exercise you should revisit Whitehead theorem, at.pdf study text is also recommended) □

Exercise 72. *(application of Whitehead theorem) Show that $(n-1)$-connected compact manifold of dim n is homotopy equivalent to S^n $(n \geq 2)$.*

Solution. Take manifold M as two parts: disk and its complement. Map the disk to $S^n = D^n/S^{n-1}$, that is identity on the disk and complement goes to one point. We have $f\colon M \to S^n$ and for $i < n$ by Hurewicz theorem (M is $(n-1)$-connected) following iso $H_i(M) \cong H_i(S^n) = 0$. The same iso for $i > n$ (both H_i are zero).

We conclude our application with this spectacular diagram where all arrows are iso, reasoning goes from the bottom arrows by excision, then with definiton of fundamental class we get the top arrow (our main focus) an iso. (we denote interior of B as B^O)

$$\begin{array}{ccc}
H_n(M) & \longrightarrow & H_n(S^n) \\
\downarrow & & \downarrow \\
H_n(M, M-(B^n)^O) & \longrightarrow & H_n(S^n, S^n-(B^n)^O) \\
\downarrow & & \downarrow \\
H_n(disk, \partial disk) & \xrightarrow[id]{} & H_n(disk, \partial disk)
\end{array}$$

□

Remark. *(Story time) We have that manifold 3-dim (compact) which os pmůy simply connected is homotopy equivalent to S^3. There is another result: M 3-dim manifold simply connected is homeomorphic to S^3. This other result (it might seem we are close to proving it) is actually famous Poincaré conjecture, one of Millenium Prize Problems and it was already solved by Grigori Perelman in 2002. Interesting story and interesting mathematician for sure. It is well known (as we like to say) that Perelman declined Fields medal (among other prizes).*

Exercise 73. *Find a map f with Hopf invariant $H(f) = 2$.*

Solution. We study a space X with a basepoint e. Denote construction $J_2(X) = X \times X/\sim$, where $(x,e) \sim (e,x)$. Apply this idea to S^n. We get a projection $p\colon S^n \times S^n \to J_2(S^n)$. On the left we have one 0-cell, two n-cells and one 2n-cell, while on the right we have one of each. We get that $J_2(S^n)$ has to be a space of the form C_f, so $H^n(J_2) = \mathbb{Z}$ given by a and $H^{2n}(J_2) = \mathbb{Z}$ given by b and $H^n(S^n \times S^n) = \mathbb{Z} \oplus \mathbb{Z}$ (generators a_1, a_2) and $H^{2n}(S^n \times S^n) = \mathbb{Z}$ (with b_0). Now, $p^*\colon H^i(J_2) \to H^i(S^n \times S^n)$ and $p^*(a) = a_1 + a_2, p^*(b) = b_0$.

$$\begin{aligned}
a^2 &= H(f)b \\
p^*(a^2) &= H(f)p^*(b) \\
(a_1 + a_2)^2 &= H(f)b_0 \\
(a_1^2 + a_1a_2 + a_2a_1 + a_2^2) &= H(f)a_1a_2 \\
2a_1a_2 &= H(f)a_1a_2 \\
H(f) &= 2
\end{aligned}$$

because $b_0 = a_1a_2$ and by evenness of the dimension $a_1a_2 = a_2a_1$. □