

Remark. All sets are assumed to be topological spaces and all maps are assumed to be continuous unless stated otherwise. The symbol '=' will denote that two topological spaces are homeomorphic. The closed unit interval will be denoted by I or J .

Exercise 1. Prove that being homotopic is an equivalence relation (on the set of continuous maps between topological spaces).

Solution. Let $f, g, k : X \rightarrow Y$ be such that $f \sim g$, $g \sim k$, i.e. there exist maps $h, h' : X \times I \rightarrow Y$ such that $h(x, 0) = f(x)$, $h(x, 1) = g(x)$, $h'(x, 0) = g(x)$, $h'(x, 1) = k(x)$.

- Reflexivity: the map $H_1 : X \times I \rightarrow Y$ defined by $H_1(x, t) := f(x)$ for all $t \in I$ is a homotopy between f and itself.
- Symmetry: the map $H_2 : X \times I \rightarrow Y$ defined by $H_2(x, t) := h(x, 1 - t)$ for all $t \in I$ is a homotopy between g and f .
- Transitivity: the map $H_3 : X \times I \rightarrow Y$ defined by $H_3(x, t) := \begin{cases} h(x, 2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ h'(x, 2t) & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$ is a homotopy between f and k .

□

Exercise 2. Let \simeq be an equivalence relation on a topological space X . Prove that the map $f : X/\simeq \rightarrow Y$ is continuous iff $f \circ p : X \rightarrow Y$ is continuous, where $p : X \rightarrow X/\simeq$ is the canonical quotient projection.

Solution. The direction " \Rightarrow " follows from the facts that p is continuous (in fact, the quotient topology is the final topology with respect to p) and the composition of continuous functions is again continuous. For " \Leftarrow ", let $U \subseteq Y$ be open. Then

$$p^{-1}(f^{-1}(U)) = (f \circ p)^{-1}(U)$$

is open by continuity of $f \circ p$, so $f^{-1}(U)$ must also be open by the definition of quotient topology and we are done. □

Exercise 3. Show that $D^n/S^{n-1} = S^n$ using the map $f : D^n \rightarrow S^n$ given by

$$f(x_1, \dots, x_n) = (2\sqrt{1 - \|\mathbf{x}\|^2}, 2\|\mathbf{x}\|^2 - 1).$$

Solution. It's easy to see that f is continuous. Moreover, its restriction to the interior of D^n gives a bijection to $S^n \setminus \{(0, \dots, 0, 1)\}$ (the inverse function is given by $(\mathbf{y}, z) \mapsto \frac{1}{\sqrt{1-z}}\mathbf{y}$) and we have $f(S^{n-1}) = \{(0, \dots, 0, 1)\}$, so we can define $f' : D^n/S^{n-1} \rightarrow S^n$ by $f'([\mathbf{x}]) = f(\mathbf{x})$. Then f' is a bijection, and by the previous exercise it is continuous. Finally, both D^n/S^{n-1} and S^n are compact (Hausdorff) spaces (since both D^n and S^n are closed bounded subsets of \mathbb{R}^n and \mathbb{R}^{n+1} , respectively, and $S^{n-1} \subseteq D^n$ is closed), so f' must be a homeomorphism (a general fact for continuous bijections between compact spaces). □

Exercise 4. Let $f : X \rightarrow Y$ and $M_f = X \times I \cup_{j \times 1} Y$. Moreover, let $\iota_X : X \rightarrow M_f$ be given by $x \mapsto (x, 0)$, $\iota_Y : Y \rightarrow M_f$ be given by $y \mapsto [y]$ and $r : M_f \rightarrow Y$ be given by $r(y) = y$, $r(x, t) = f(x)$. Show that

- i) Y is a deformation retract of M_f ,
- ii) $r \circ \iota_X = f$,
- iii) $\iota_Y \circ f \sim \iota_X$.

Solution.

- i) Geometrically, the deformation retraction is realized by pushing X along I towards Y .
- ii) We have $r \circ \iota_X(x) = r(x, 0) = f(x)$ for all $x \in X$.
- iii) The required homotopy $h : X \times J \rightarrow M_f$ is given by $h(x, s) = [(x, s)]$.

□

Exercise 5. Show that the pair (M_f, X) has the homotopic extension property (HEP), i.e. ι_X is a cofibration.

Solution. Let $g : I \times J \rightarrow \{0\} \times J \cup I \times \{0\}$ be any retraction such that $g(0, s) = (0, s)$ and $g(1, s) = (1, 0)$. Then the map $r : M_f \times J \rightarrow X \times \{0\} \times J \cup M_f \times \{0\}$ defined by $r(x, t, s) = (x, g(t, s))$ and $r(y, s) = (y, 0)$ is the required retraction. □

Exercise 6. The smash product between two based spaces is defined by

$$(C, c_0) \wedge (D, d_0) := (C \times D) / (C \times \{d_0\} \cup \{c_0\} \times D).$$

Show that $X/A \wedge Y/B = (X \times Y) / (X \times B \cup A \times Y)$.

Solution. Let $p_1 : X \times Y \rightarrow X/A \times Y/B$ be given by $p_1(x, y) = ([x], [y])$ and $p_2 : X/A \times Y/B \rightarrow (X \times Y) / (X \times B \cup A \times Y)$ be given by $p_2([x], [y]) = ([x], [y])$. Then the composition $p_2 \circ p_1$ is continuous and factors through $(X \times Y) / (X \times B \cup A \times Y)$, which implies that the canonical bijection between $(X \times Y) / (X \times B \cup A \times Y)$ and $X/A \wedge Y/B$ is continuous (using exercise 2). Using the definition of quotient topology several times, it can be shown that this bijection is also open, hence a homeomorphism. □

Exercise 7. Let $A = \{\frac{1}{n} \cup \{0\}\} \subseteq \mathbb{R}$. Show that (I, A) does not have the HEP, i.e. the inclusion $A \hookrightarrow I$ is not a cofibration.

Solution. If $A \times J \cup I \times \{0\}$ was a retract of $I \times J$, the retraction would have to preserve connected subsets. But $A \times J \cup I \times \{0\}$ is not locally connected while $I \times J$ is, a contradiction. □