Exercise 1. *Show that* $(S^m, *) \wedge (S^n, *) \cong (S^{m+n}, *)$.

Solution. Using exercises 3 and 6 from the previous tutorial, we have

$$\begin{aligned}(S^m, *) \wedge (S^n, *) &\cong (D^m/S^{m-1}) \wedge (D^n/S^{n-1}) \cong \\ &\cong (D^m \times D^n)/(S^{m-1} \times D^m \cup D^n \times S^{n-1}) \cong \\ &\cong (I^m \times I^n)/(\partial I^m \times I^n \cup \partial I^n \times I^m) = \\ &= I^{m+n}/(\partial(I^{m+n})) \cong D^{m+n}/\partial D^{m+n} \cong S^{m+n}.\end{aligned}$$

□

Exercise 2. *Show that* $\mathbb{C}P^n$ *is a CW-complex.*

Solution. Clearly $\mathbb{C}P^0$ is a point. Next, we have

$$\begin{aligned}\mathbb{C}P^n &= \mathbb{C}^{n+1} \setminus \{0\}/\{v \sim \lambda v, \lambda \in \mathbb{C} \setminus \{0\}\} \cong S^{2n+1} \setminus \{0\}/\{v \sim \lambda v, |\lambda| = 1\} \cong \\ &\cong \{(w, \sqrt{1 - |w^2|}) \in \mathbb{C}^{n+1}, w \in D^{2n}\}/\{w \sim \lambda w \text{ for } |w| = 1\} \cong \\ &\cong (D^{2n} \cup S^{2n-1})/\{w \sim \lambda w \text{ for } w \in S^{2n-1}\} = D^{2n} \cup_f \mathbb{C}.\end{aligned}$$

Now taking the canonical projection $S^{2n-1} \to \mathbb{C}P^{n-1} \cong S^{2n-1}/\sim$ as the attaching map f yields a CW-complex with one cell in every even dimension till $2n$ and none in the odd ones. □

Exercise 3. *From the lecture we know that* $A := \{\frac{1}{n}, n \in \mathrm{N}\} \cup \{0\}$ *as a subspace of* $\mathbb{R}$ *is not a CW-complex. Show that* $X := I \times \{0\} \cup A \times I$ *is not a CW-complex either.*

Solution. Suppose that X is a CW-complex. Then it cannot contain cells of dimension ≥ 2, because it becomes disconnected after removing any point. In fact, the space obtained after removing any point $(a, 0)$ with $a \in A$ has more than two connected components (three for $a > 0$, to be exact), so these points cannot lie inside a 1-cell. Therefore these points must form 0-cells, but we already know that A does not have discrete topology, a contradiction. □

Exercise 4. *Show that the Hawaiian earring given by*

$$X = \{(x, y) \in \mathbb{R}^2, (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2} \textit{ for some } n\}$$

is not a CW-complex.

Solution. Suppose that X is a CW-complex. Using similar arguments as in the previous exercise, we can see that $(0, 0)$ must be a 0-cell and that X must have either infinitely many 0-cells, or infinitely many 1-cells. But since X is compact, exercise 5 implies that X can have only finitely many cells, a contradiction. □

Exercise 5. *Prove that every compact set* A *in a CW-complex* X *can have a nonempty intersection with only finitely many cells.*

Solution. X is comprised of cells that are indexed by elemnts of some set J. Let B be a set containing exactly one point from each intersection $A \cap e^\beta, \beta \in J$. We need to show that B is closed and discrete, which will imply that B is compact (since $B \subseteq A$) and discrete, hence finite. We know that a set $C \subseteq X^n$ is closed iff both $C \cap X^{n-1}$ and $C \cap e_\alpha^n$ for each $\alpha \in J$ are closed, because $D^n \cup_f X^{n-1}$ is a pushout. Using induction, this implies that $C \subseteq X$ is closed iff $C \cap e_\alpha$ is closed for each $\alpha \in J$. Since $B \cap e_\alpha$ contains at most one point for any $\alpha \in J$ and X is T_1 (even Hausdorff), this shows that B is closed. Using the same argument, B with any one point removed is closed. Therefore B is also discrete and we are done. □

Exercise 6. *Show that for a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of abelian groups (or more generally modules over a commutative ring) the following are equivalent:*

(1) There exists $p : B \to A$ such that $pf = \mathrm{id}_A$.

(2) There exists $q : C \to B$ such that $gq = \mathrm{id}_C$.

(3) There exist $p : B \to A$ and $q : C \to B$ such that $fp + qg = \mathrm{id}_B$.

(Another equivalent condition is $B \cong A \oplus C$, with (p, g) and $f + q$ being the respective inverse isomorphisms.)

Solution.
(1) $\implies$ (2) and (3):
Since g is surjective, for any $c \in C$ there is some $b \in B$ such that $g(b) = c$. Moreover, for any other $b' \in B$ such that also $g(b') = c$, we have $b - fp(b) = b' - fp(b')$, since $b - b' \in \ker g = \mathrm{im} f$, so that $b - b' = f(a)$ and

$$fp(b - b') = fpf(a) = f(a) = b - b'.$$

This shows that we can correctly define $q(c) := b - fp(b)$ for any such b. Then we have

$$gq(c) = g(b) - gfp(b) = g(b) = c$$

(since $gf = 0$), which shows that $gq = \mathrm{id}_C$, and also $qg(b) = b - fp(b)$, hence $fp + qg = \mathrm{id}_B$.

(3) $\implies$ (1) and (2):
Applying f from the right to the equation $fp + qg = \mathrm{id}_B$ yields $fpf = f$ (since $gf = 0$), which together with the fact that f is injective implies $pf = \mathrm{id}_A$. Similarly, applying g from the left yields $gqg = g$, which together with the fact that g is surjective implies $gq = \mathrm{id}_C$. □

Exercise 7. *Let $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$ be a short exact sequence of chain modules. We have defined the connecting homomorphism $\partial_* : H_n(C) \to H_{n-1}(A)$ by the formula $\partial_*[c] = [a]$, where $\partial c = 0$, $f(a) = \partial b$ and $g(b) = c$. Show that this definition does not depend on a nor b.*

Solution. We have $\partial a = 0$ iff $f(\partial a) = 0$ (using injectivity of f) iff $0 = \partial f(a) = \partial\partial b$, and the last condition is true.
Now let $b, b' \in B$ be such that $g(b) = g(b') = c$ with $a, a' \in A$ such that $f(a) = b, f(a') = b'$. Then $b - b' \in \ker g = \mathrm{im} f$, so $b - b' = f(\overline{a})$ for some $\overline{a} \in A$. Therefore $f(\partial\overline{a}) = \partial b - \partial b' = f(a - a')$ and the injectivity of f implies $\partial\overline{a} = a - a'$, hence $[0] = [\partial a'] = [a] - [a']$ and we are done. $\square$