Exercise 1. *Show $\partial\partial = 0$. Use formula $\varepsilon^i_{n+1}\circ\varepsilon^j_n = \varepsilon^{j+1}_{n+1}\circ\varepsilon^i_n$, where $i<j$. The definition for $\sigma\in C_n(X)$, $\sigma\colon \Delta^n\to X$, is*

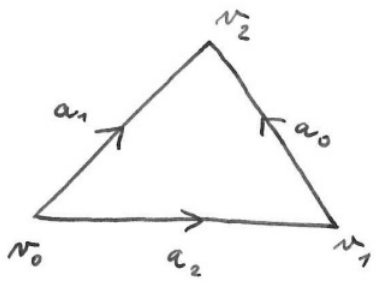
$$\partial\sigma = \sum_{i=0}^{n}(-1)^i\,\sigma\circ\varepsilon^i_n.$$

Solution. Easily workout

$$\begin{aligned}\partial(\partial\sigma) &= \partial\Big(\sum_{i=0}^{n+1}(-1)^i\,\sigma\circ\varepsilon^i_{n+1}\Big) = \sum_{i=0}^{n+1}(-1)^i\,\partial(\sigma\circ\varepsilon^i_{n+1}) = \\ &= \sum_{i=0}^{n+1}(-1)^i\Big(\sum_{j<i}(-1)^j\sigma\circ\varepsilon^i_{n+1}\circ\varepsilon^j_n + \sum_{j>i}(-1)^{j-1}\sigma\circ\varepsilon^i_{n+1}\circ\varepsilon^{j-1}_n\Big) = \\ &= \sum_{i=1}^{n+1}\sum_{j<i}(-1)^i(-1)^j\sigma\circ\varepsilon^i_{n+1}\circ\varepsilon^j_n + \sum_{i=1}^{n}\sum_{j>i}(-1)^{j+i-1}\sigma\circ\varepsilon^i_{n+1}\circ\varepsilon^{j-1}_n,\end{aligned}$$

now, with proper reindex and shift, this yields $\varepsilon^i_{n+1}\circ\varepsilon^j_n = \varepsilon^{j+1}_{n+1}\circ\varepsilon^i_n = \varepsilon^j_{n+1}\circ\varepsilon^{i-1}_n$, both sums are of the same elements but with opposite signs. Hence, $\partial\partial = 0$. □

Exercise 2. *Simplicial homology of $\partial\Delta^2$.*



Solution. Chain complex of this simplicial homology is $C_0 = \mathbb{Z}[v_0, v_1, v_2] = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, $C_1 = \mathbb{Z}[a_0, a_1, a_2] = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. So

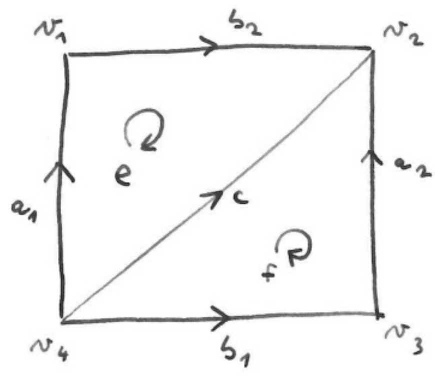
$$0\to C_1\xrightarrow{\partial} C_0\to 0,$$

where we want to determine ∂ and we know $\partial a_0 = v_2 - v_1$, $\partial a_1 = v_2 - v_0$, $\partial a_2 = v_1 - v_0$. Using simple linear algebra, we study generators $\ker\partial$ and $\mathrm{im}\partial$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 0 & 1 & -1 & 1 & 0\end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1\\ 0 & 1 & 0 & -1 & 0 & 1\\ 1 & -1 & 1 & 0 & 0 & 0\end{array}\right),$$

therefore $\ker\partial$ has a generator $a_0 - a_1 + a_2$ and $\mathrm{im}\partial$ has two generators $-v_1 + v_2$ and $-v_0 + v_2$. We get $H_0 = \dfrac{\mathbb{Z}[v_0, v_1, v_2]}{\mathbb{Z}[-v_1+v_2, -v_0+v_2]} = \dfrac{\mathbb{Z}[-v_1+v_2, v_0+v_2, v_2]}{\mathbb{Z}[-v_1+v_2, -v_0+v_2]} = \mathbb{Z}[v_0] = \mathbb{Z}$ and $H_1 = \ker\partial = \mathbb{Z}[a_0 - a_1 + a_2] = \mathbb{Z}$. □

Exercise 3. *Simplicial complex, model of torus, compute differentials and homology.*



Solution. Again, we get simplicial chain complex C_* formed by free abelian groups generated by equivalence classes of simplicies. Note a_1, a_2 are actually one generator, same for b_1, b_2. All the vertices are also equivalent. We choose the orientation and fix it.

Thus we get $C_0 = \mathbb{Z}[v] = \mathbb{Z}, C_1 = \mathbb{Z}[a, b, c] = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}, C_2 = \mathbb{Z}[e, f] = \mathbb{Z}\oplus\mathbb{Z}, C_3 = 0$, and the following holds: $\partial a = 0, \partial b = 0, \partial c = 0$, as well as $\partial e = a+b-c, \partial f = c-a-b, \partial(e+f) = 0$, so we get $\ker \partial = \mathbb{Z}[e + f], \mathrm{im}\partial = \mathbb{Z}[a + b - c]$.

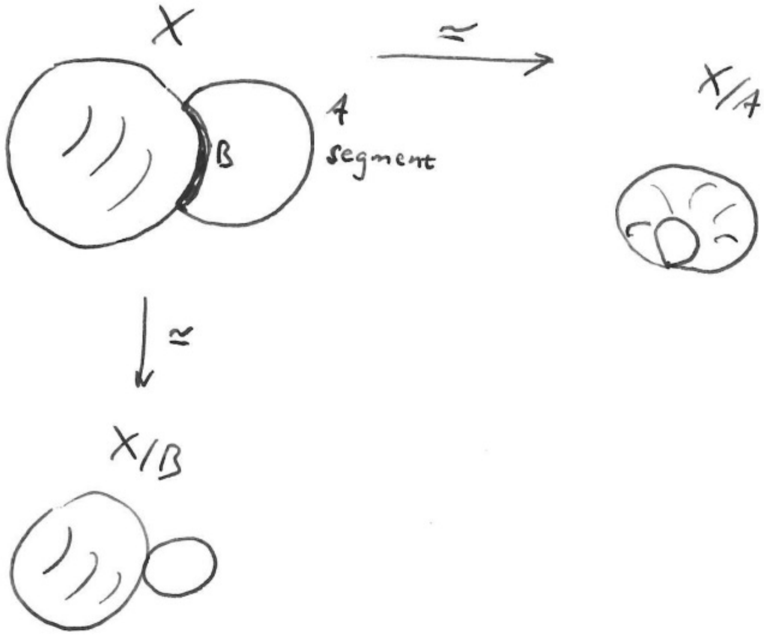
Let T be the torus. Then

$$H_2(T) = \ker \partial_2 = \mathbb{Z}[e + f] = \mathbb{Z},$$
$$H_1(T) = \mathbb{Z}[a, b, c]/\mathbb{Z}[a + b - c] = \frac{\mathbb{Z}[a, b, a + b - c]}{\mathbb{Z}[a + b - c]} = \mathbb{Z}[a, b] = \mathbb{Z} \oplus \mathbb{Z},$$
$$H_0(T) = \ker \partial_0 = \mathbb{Z}.$$

□

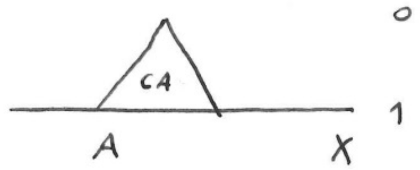
First criterion of homotopy equivalence *(Hatcher)*
Let (X, A) be a pair that satisfies HEP, i.e. $A \overset{i}{\hookrightarrow} X$ is a cofibration. Let A be contractible in itself. Then $q\colon X \to X/A$ is a homotopy equivalence.

Exercise 4. $S^2 \vee S^1 \simeq S^2/S^0$ *(using First criterion)*



Solution. In the picture (hopefully) above, A is a segment as well as B, so contractible in itself. Clearly $S^2 \vee S^1 = X/B$ and $S^2/S^0 = X/A$ and $X \simeq X/A$ and $X \simeq X/B$ by criterion, therefore $X/A \simeq X/B$ and we are done. □

Exercise 5. *Let $i\colon A \hookrightarrow X$ is a cofibration, show $X/A \simeq X \cup CA = Ci$. (using First criterion)*



Solution. We know $CA \hookrightarrow X \cup CA$ is a cofibration using homework 1, exercise 2, with $Y = CA$. Then by criterion $X \cup CA \simeq X \cup CA/CA$. Also X/A is homeomorphic to $X \cup CA/CA$ (see picture above), which concludes the result.

□

Exercise 6. *Prove the first criterion of homotopy equivalence.*

Solution. We take $h\colon A \times I \to A$, on $A \times \{0\}$ it is identity on A and constant on $A \times \{1\}$.

$$\begin{array}{ccc} X \times I & \xrightarrow{f} & X \\ {\scriptstyle q\times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle q} \\ X/A \times I & \xrightarrow{\bar f} & X/A \end{array}$$

and find $g\colon X/A \to X$. Define $\bar f(x,t) = f(x,t)$, $\bar f([x],t) = [f(x,t)]$. If we define $g\colon X/A \to A, [x] \mapsto f(x,1)$, then it is well defined. Now we want to show, that the compositions are homotopy equivalent to the identities.

$g \circ q \sim \mathrm{id}_X$: $g(q(x)) = g([x]) = f(x,1)$, just the way we defined it, so f is the homotopy, as $f(-,0) = \mathrm{id}_X$ and $f(-,1) = g \circ q$,

$q \circ g \sim \mathrm{id}_{X/A}$: $q(g([x])) = q(f(x,1)) = [f(x,1)] = \bar f([x],1)$ and $\mathrm{id}_{X/A} = \bar f([x],0)$, so in this case the map $\bar f$ is homotopy. ☐

Exercise 7. *Application of the criterion: two types of suspensions, unreduced and reduced.*
Unreduced suspension: $SX = X \times I/\sim$, *where* $(x_1,0) \sim (x_2,0)$, $(x_1,1) \sim (x_2,2)$.
Reduced suspension: $\Sigma X = SX/\{x_0\} \times I = (X,x_0) \wedge (S^1,s_0)$
(this might be a homework)
The criterion says, that if $\{x_0\} \hookrightarrow X$ *is a cofibration, then* $SX \simeq \Sigma X$.

$$I \simeq \{(x_0,t), t \in I\} \subseteq SX \longrightarrow SX/\{x_0,t), t \in I\} = \Sigma X$$

Exercise 8. *There is a lemma, that says: Given the following diagram, where rows are long exact sequences and* m *is iso,*

$$\begin{array}{ccccccccccc} K_n & \xrightarrow{f} & L_n & \xrightarrow{g} & M_n & \xrightarrow{h} & K_{n-1} & \longrightarrow & L_{n-1} & \longrightarrow & M_{n-1} \\ {\scriptstyle k}\downarrow & & {\scriptstyle l}\downarrow & & {\scriptstyle m}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{K}_n & \xrightarrow{\bar f} & \overline{L}_n & \xrightarrow{\bar g} & \overline{M}_n & \xrightarrow{\bar h} & \overline{K}_{n-1} & \longrightarrow & \overline{L}_{n-1} & \longrightarrow & \overline{M}_{n-1} \end{array}$$

we get a long exact sequence

$$K_n \xrightarrow{f\oplus k} L_n \oplus \overline{K}_n \xrightarrow{l\oplus \bar f} \overline{L}_n \xrightarrow{h\circ \bar m^{-1}\circ \bar g} K_{n-1} \longrightarrow \cdots$$

This might be a homework.