Exercise 1. *Prove that S^n has a nonzero vector field if and only if n is odd.*

Solution. First note, that we have $v\colon S^n \to \mathbb{R}^{n+1}$ such that $v(x) \perp x$. Consider the case of S^1 and $(x_0, x_1) \mapsto (x_1, -x_0)$. Take $(x_0, x_1, x_2, x_3, \dots, x_{2n+1}) \in S^{2n+1} \subseteq \mathbb{R}^{2n}$. Then we get $(x_1, -x_0, x_3, -x_2, \dots, x_{2n+1}, -x_{2n})$ as image and there is nothing more obvious than that the product is zero, i.e. it's perpendicular.

Note, for $S^1 \subseteq \mathbb{C}$ it is $z \mapsto ez$, where e is the complex unit, usually denoted as i.

Now we want to prove that if S^n has a nonzero vector field, then n is odd. We use the fact that $\deg(\mathrm{id}) = 1$ and $\deg(-\mathrm{id}) = (-1)^{n+1}$. Take $v\colon S^n \to S^n$. If we show $\mathrm{id} \sim -\mathrm{id}$, then $1 = (-1)^{n+1} \Rightarrow n$ is odd. The homotopy is $h(x,t)$ we are looking for is $h(x,t) = cos(t)x + sin(t)v(x)$, where $t \in [0, \pi]$. Also note $\| h(x,t) \| = cos^2 t + sin^2 t = 1$. We are done. □

Exercise 2. *Prove* $\deg(f) = \sum_{i=1}^{k} \deg(f/x_i)$.

Solution. We have $f\colon S^n \to S^n$ and $y \in S^n$ has a neighborhood (nbhd) V. Denote $f^{-1}(y) = \{x_1, \dots, x_k\}$ and U_i nbhd's of x_i, $U_i \cap U_j = \emptyset$. Let degree of f/x_i be n_i. We have $f\colon (U_i, U_i - x_i) \to (V, V - y)$ and $(f/U_i)_*\colon H_n(U_i, U_i - x_i) \cong \mathbb{Z} \to H_n(V, V - y) \cong \mathbb{Z}$, so $z \mapsto n_i z$. Consider the following diagram, its evolution (i.e. drawing order) was revealed in the tutorial class:

$$
\begin{array}{ccccc}
 & & \bigoplus H_n(U_i, U_i - x_i) & \xrightarrow{(f/U_i)_*} & H_n(V, V - y) \\
 & \overset{\cong}{\swarrow} & \downarrow k & & \downarrow h \\
H_n(S^n, S^n - x_i) & & H_n(S^n, S^n - f^{-1}(y)) & \longrightarrow & H_n(S^n, S^n - y) \\
 & \underset{\cong}{\nwarrow} & \uparrow j_* & & \uparrow j_{y*} \\
 & & H_n(S^n) & \xrightarrow{f_*} & H_n(S^n)
\end{array}
$$

We have j_{y*} iso and h is iso by excision theorem.

So, we have $H_n(S^n, S^n - f(y)) \overset{\cong}{\leftarrow} \bigoplus H_n(U_i, U_i - x_i) \cong H_n(\bigsqcup_1^k (U_i, U_i - x_i))$. Iso between $H_n(S^n, S^n - f(y))$ and $H_n(\bigsqcup_1^k (U_i, U_i - x_i))$ by excision. So k is also iso.

We can take $j_{y*} = h \circ \sum (f/U_i)_* \circ k^{-1} \circ j_*$ and for generator $1 \in H_n(S^n)$ we have $((j_y)_* \circ f_*)(1) = \deg(f) \cdot 1$. We are done.

$$
\begin{array}{ccc}
(1, \dots, 1) & \longrightarrow & \sum deg(f/x_i) \\
\uparrow & & \downarrow \\
1 & & \sum \deg(f/x_i)
\end{array}
$$

□

Exercise 3. *Compute homology groups of $\mathbb{RP}^n$ using CW-structure.*

Solution. Let X be the CW-complex, we have $C_n^{CW} = \bigoplus \mathbb{Z}[e_\alpha^n]$, $H_n(C_*^{CW}, d) = H_n(X)$. Also, $d[e_\alpha^n] = \sum_\beta d_\alpha^\beta[e_\beta^{n-1}]$, where α_α^β is the degree of the map

$$S^{n-1} \xrightarrow{f_\alpha} X^{n-1} \to X^{n-1}/(X^{n-1} - e_\beta^{n-1}) \cong S^{n-1}.$$

We know that $\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$, cell in every dimension. Attaching map is

$$f : S^i \to (\mathbb{RP})^{(i)} = \mathbb{RP}^i = S^i/(x \sim -x).$$

Then have

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ appears $n+1$ times. Now, $S^i \xrightarrow{f} \mathbb{RP}^i \to \mathbb{RP}^i/\mathbb{RP}^{i-1} \cong S^i$, hence we can apply the attaching map. Class $[y]$ has two preimages $y, -y$, there are nbhd's, $deg(f/y) = \pm 1$. Note that $f/-y = f/y \circ (-\mathrm{id})$. Now we can compute the degree: $\deg(f) = \deg(f/y) + \deg(f/-y) = \pm(1 + (-1)^{i+1})$. It is ± 2 if i is odd and 0 otherwise. So, we can add to the sequence above:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0,$$

and compute for $X = \mathbb{RP}^n$, n even: $H_0(X) = \mathbb{Z}, H_1(X) = \mathbb{Z}_2, H_2(X) = 0, H_i(X) = 0$ for i even, $H_i(X) = \mathbb{Z}_2$ for i odd. The homology is zero for $i > 0$. For n even the only difference is $H_n(X) = \mathbb{Z}$.

(Note: this is because for even they are not orientable, for odd it is oriented space.) □

Exercise 4. *Compute homology groups of oriented two dimensional surfaces. What about nonorientable?*

Solution. Denote M_g surface of genus g (it is the same as sphere with g handles, i.e. M_1 is torus and M_2 is double torus (homework 4). The CW-model is $e^0 \cup e_1^1 \cup \cdots \cup e_{2g}^1 \cup e^2$ and we have

$$0 \to \mathbb{Z} \to \bigoplus_1^{2g} \mathbb{Z} \to \mathbb{Z} \to 0.$$

Second differential is zero, $e^0 - e^0 = 0$. The first one is zero as well (glue the model, the arrows go with + and then −). Then we get $H_0 = \mathbb{Z}, H_1 = \bigoplus_1^{2g} \mathbb{Z}, H_2 = \mathbb{Z}$.

For nonorientable surfaces, N_g is modelled by one 2-dimensional disc which has boundary composed with g segments every of which repeats twice with the same orientation. So we have one cell in dimensions 2 and 0 and g cells in dimension one. We get (quite similarly)

$$0 \to \mathbb{Z} \to \bigoplus_1^{g} \mathbb{Z} \to \mathbb{Z} \to 0.$$

This equality holds: $d[e^2] = 2[e_1^1] + 2[e_2^1] + \dots$, so $H_2 = 0, H_0 = \mathbb{Z}$ and the only interesting case is

$$H_1 = \frac{\mathbb{Z}[e_1^1, \dots, e_g^1]}{\mathbb{Z}[2e_1^1 + \dots + 2e_g^1]} = \mathbb{Z}_2 + \bigoplus_1^{g-1} \mathbb{Z}.$$

□

Exercise 5. *Have $f\colon S^n \to S^n$ map of degree k. (such map always exists). Let $X = D^{n+1} \cup_f S^n$ and compute homology of X and the projection $p\colon X \to X/S^n$ in homology.*

Solution. Easy, $X = e^0 \cup e^n \cup e^{n+1}$ and $0 \to \mathbb{Z} \xrightarrow{k\cdot} \mathbb{Z} \to 0 \to \dots \to 0 \to \mathbb{Z} \to 0$, also

$$S^n \to X^{(n)} \to X^{(n)}/(X^{(n)} - e^n) = S^n.$$

We get $H_{n+1}(X) = 0, H_n = \mathbb{Z}_k, H_0(X) = \mathbb{Z}$. Note, $X/S^n \cong S^{n+1}$. So, for the p_* we have

$$p_*\colon H_{n+1}(X) = 0 \xrightarrow{0} H_{n+1}(S^{n+1}) = \mathbb{Z}$$

and

$$p_*\colon H_n(X) = \mathbb{Z}_k \xrightarrow{0} H_n(S^{n+1}) = 0$$

and at H_0 it is identity $\mathbb{Z} \to \mathbb{Z}$.

□