Exercise 1. *(39 in total) Use theorem "$f\colon S^n \to S^n$ s.t. $f(-x) = -f(x) \Rightarrow \deg(f)$ is odd" to prove that for $f\colon S^n \to \mathbb{R}$ there exists $x \in S^n\colon f(x) = f(-x)$.*

Solution. We will use proof by contradiction. Suppose that $f(x) - f(-x)$ is always non-zero and define $g(x) = \dfrac{f(x) - f(-x)}{||f(x) - f(-x)||}$. Then obviously $g(-x) = -g(x)$. Since $g\colon S^n \to S^{n-1} \subset S^n$ is odd , the map g has odd degree, but $g(S^n) \subsetneq S^n$ which implies $\deg(g) = 0$ (g is homotopic to a constant map). Now we use well known fact that zero is even number, some say that it is the evennest number of them all (why?) and conclude desired contraction. □

Exercise 2. *(40 in total) Let $\varphi \in C^k(X; R), \psi \in C^l(Y; R)$. Prove $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$. Use $\tau = [e_0, \dots, e_{k+l+1}] \in C_{k+l+1}(X)$.*

Solution. Easily work out

$$\delta(\varphi \cup \psi)(\tau) = (\varphi \cup \psi)(\delta\tau) = (\varphi \cup \psi)\Big(\sum_{i=0}^{k+l+1} (-1)^i \tau/[e_0, \dots, \hat{e}_i, \dots, e_{k+l+1}]\Big) =$$

$$= \sum_{i=0}^{k} (-1)^i \varphi(\tau/[e_0, \dots, \hat{e}_i, \dots, e_{k+1}])\psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) +$$
$$+ \sum_{i=k+1}^{k+l+1} (-1)^i \varphi(\tau/[e_0, \dots, e_k])\psi(\tau/[e_k, \dots, \hat{e}_i, \dots, e_{k+l+1}]).$$

Now, the right hand side of the formula, the first part gives

$$\begin{aligned}(\delta\varphi \cup \psi)(\tau) &= \delta\varphi(\tau/[e_0, \dots, e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) = \\ &= \varphi(\delta\tau/[e_0, \dots, e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) = \\ &= \sum_{i=0}^{k+1} (-1)^i \varphi(\delta\tau/[e_0, \dots, \hat{e}_i, \dots, e_{k+1}]) \cdot \psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]).\end{aligned}$$

The second part is

$$\begin{aligned}(-1)^k(\varphi \cup \delta\psi)(\tau) &= (-1)^k \varphi(\tau/[e_0, \dots, e_k])\delta\psi(\tau/[e_k, \dots, e_{k+l+1}]) = \\ &= \sum_{j=0}^{l+1} (-1)^{j+k} \varphi(\delta\tau/[e_0, \dots, e_k]) \cdot \psi(\tau/[e_k, \dots, \hat{e}_{k+j}, \dots, e_{k+l+1}]).\end{aligned}$$

Now, the last summand of the first part plus the first summand of the second part yields

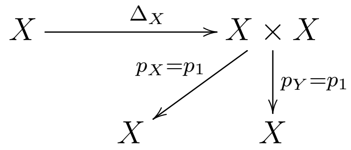
$$\begin{aligned}&(-1)^{k+1} \varphi(\tau/[e_0, \dots, e_k])\psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) + \\ &+ (-1)^k \varphi(\tau/[e_0, \dots, e_k])\psi(\tau/[e_{k+1}, \dots, e_{k+l+1}]) = 0,\end{aligned}$$

and we are done, $\mathsf{LHS} = \mathsf{RHS}$.

□

Exercise 3. *(41 in total) Prove $a \cup b = \Delta^*(a \times b)$, where $a \in H^k(X), b \in H^l(X), \Delta\colon X \to X \times X, x \mapsto (x,x)$ and $\times$ is cross product defined $\alpha \times \beta = p_X^* \alpha \cup p_Y^* \beta$ (p_X, p_Y are projections from $X \times Y$).*

Solution. By the following diagram



we have that $p_X \Delta$ is id. So, compute

$$\Delta^*(a \times b) = \Delta^*(p_X^* a \cup p_Y^* b) = \Delta^*(p_X^* a) \cup \Delta^*(p_Y^* b) = (p_X \Delta)^* a \cup (p_Y \Delta)^* b,$$

and we are done. The thing is that cup product is natural. □

Exercise 4. *(42 in total) Compute the structure of graded algebra $H^*(S^n \times S^n; \mathbb{Z})$ for n even and n odd. Use the following:*

If $H^n(Y;R)$ is free finitely generated group for all n and $(X,A), Y$ are CW-complexes, then

$$\times\colon H^*(X,A;R) \otimes H^*(Y;R) \to H^*(X \times Y, A \times Y; R)$$

is an isomorphism of graded rings.

Solution. We will omit writing the $\mathbb{Z}$ coefficients.

Now, $H^*(S^n) \otimes H^*(S^n) \to H^*(S^n \times S^n)$ and we know that for spheres $H^0 = \mathbb{Z}$ with generator 1 and $H^n = \mathbb{Z}$, denote generator a. Also, $a \cup a \in H^{2n} = 0, a \cup a = 0$, so we get $\mathbb{Z}[a]/\langle a^2 \rangle$ and $\deg(a) = n$. We can write the same for the second, so denote the other generator b and have $\deg(b) = n$ and we have $\mathbb{Z}[b]/\langle b^2 \rangle$.

Now we compute tensor product $\mathbb{Z}[a]/\langle a^2 \rangle \otimes \mathbb{Z}[b]/\langle b^2 \rangle$, we have four generators: $1_a \otimes 1_b, a \otimes 1_b, 1_a \otimes b, a \otimes b$, we will denote them $1, c, d, c \cdot d$. Compute

$$(a \otimes 1_b) \cdot (1_a \otimes b) = (-1)^{0 \cdot 0}(a \cdot 1_a) \otimes (1_b \cdot b) = a \otimes b,$$

because 0 is an idempotent element, i.e. $0 \cdot 0 = 0$, and $(-1)^n = 1$ for n even, again, as in the first exercise, we use Evenness of Zero. (We refer the reader to "Principia Mathematica" *Whitehead, Russell*,(1910,1912,1913).) Continue with computation

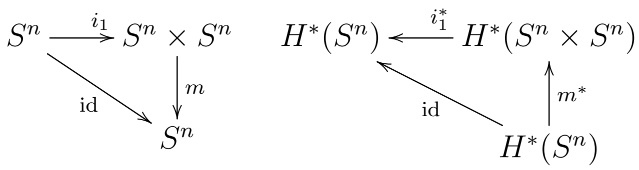
$$(1_a \otimes b) \cdot (a \otimes 1_b) = (-1)^{n \cdot n}(1_a \cdot a) \otimes (b \cdot 1_b) = (-1)^n a \otimes b,$$

so the algebra we get is $H^*(S^n \times S^n) = \mathbb{Z}[c,d]/\langle c^2, d^2, dc - (-1)^n cd \rangle$. For n even we have $dc = cd$. □

Exercise 5. *(43 in total) Prove that there is no multiplication on even dimensional spheres. Multiplication on the sphere S^n is a map $m\colon S^n \times S^n \to S^n$ such that there is an element $1 \in S^n$ satisfying $m(x,1) = x, m(1,x) = x$.*

Hint: compute $m^*\colon H^*(S^n) \to H^*(S^n \times S^n)$, describe two rings.

Solution. We have $H^*(S^n) = \mathbb{Z}[\gamma]/\langle\gamma^2\rangle$ and $H^*(S^n \times S^n) = \mathbb{Z}[\alpha,\beta]/\langle\alpha^2,\beta^2\rangle = H^*(S^n) \otimes H^*(S^n)$, because we already know, that $\alpha\beta = \beta\alpha$. Our situation can be described with two diagrams:

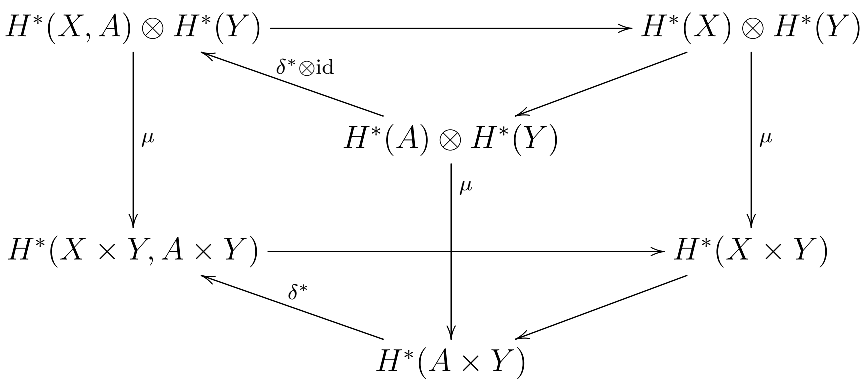


Take $m^*(\gamma) = a\alpha + b\beta$ with $a, b \in \mathbb{Z}$ and prove first, that $a = b = 1$. Use $m \circ i_1 = \mathrm{id}$, so $i_1^*(m^*\gamma) = \gamma$. This gives $i_1^*(a\alpha + b\beta) = \gamma$ and since $i_1^*(a\alpha + b\beta) = a\gamma$, we have $a = 1$, same for b. Final computation yields

$$\begin{aligned}0 &= m^*(0) = m^*(\gamma^2) = m^*(\gamma \cup \gamma) = m^*\gamma \cup m^*\gamma = \\ &= (\alpha+\beta) \cup (\alpha+\beta) = \alpha^2 + \alpha\beta + \beta\alpha + \beta^2 = 0 + 2\alpha\beta + 0 \neq 0,\end{aligned}$$

and that, my friends, is a contradiction. □

Exercise 6. *(unfinished) (44 in total) Use five lemma to prove that taking any two μ's iso's, the third μ is iso as well. (Proving five lemma might be a homework.) Also, show commutativity of the diagram.*



Solution. We will name the parts of the diagram as follows: back-square, upper-triangle, lower-triangle, left-square, right-square. The triangles come from the long exact sequence of of pairs (X, A) and $(X \times Y, A \times Y)$, the right-square commutativity comes from an inclusion, back-square commutes as well (topology knowledge). The only problematic part is the left-square and it is exercise on computation of connecting homomorphism δ^*. □